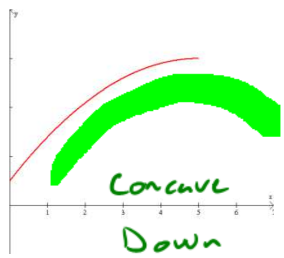
Critical Numbers $f'(x) = 0$

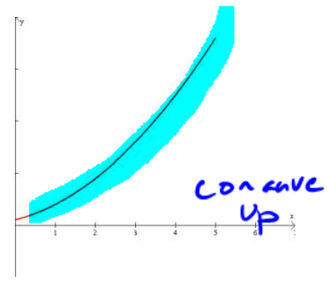
Lesson 12 – Curve Analysis (Polynomials)

**Concavity**

In business, for example, the first derivative might tell us that our sales are increasing, but the second derivative will tell us if the pace of the increase is increasing or decreasing.

CD:

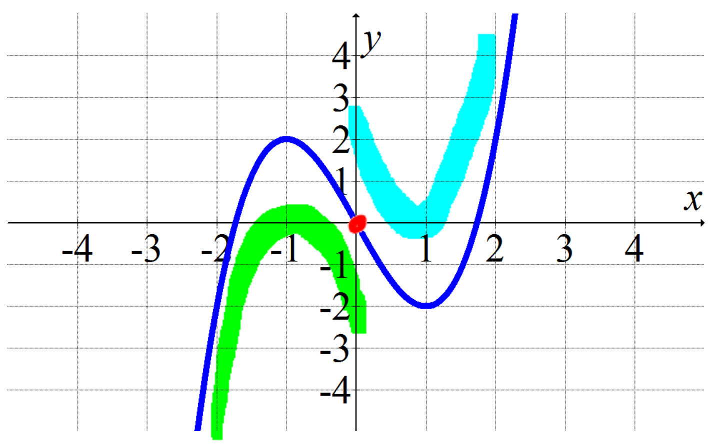



CU:

From these graphs, you can see that the shape of the curve change differs depending on whether the slopes of tangent lines are increasing or decreasing. This is the idea of **concavity**.

**Example 8:** The graph given below is the graph of a function *f*. Determine the interval(s) on which the function is concave upward and the interval(s) on which the function is concave downward.

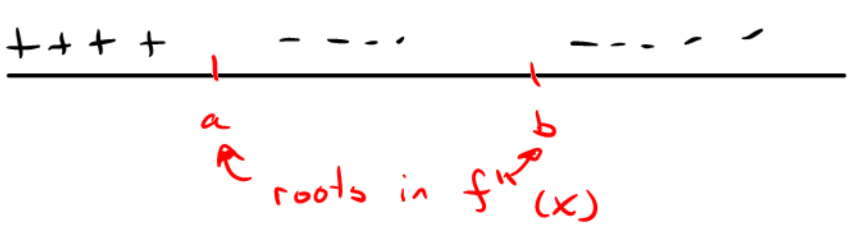



CD: $(-\infty, 0)$ CU: $(0, \infty)$

We find concavity intervals by analyzing the *second derivative* of the function. The analysis is very similar to the method we used to find increasing/decreasing intervals.

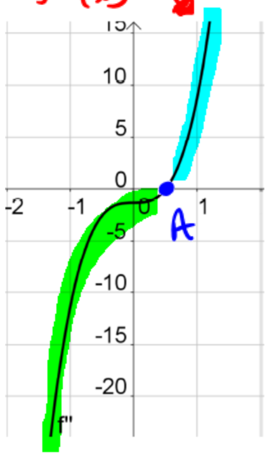
1. Use GeoGebra to graph the second derivative of the function. Then find the zero(s) of the second derivative. root( f'' )
2. Create a number line and subdivide it using the zeros of the second derivative.
3. Use the graph of the second derivative to determine the sign (positive or negative) of the *y* values of the second derivative in each interval and record this on your number line.
4. In each interval in which the second derivative is positive, the function is concave upward. In each interval in which the second derivative is negative, the function is concave downward.





**Example 9:** State intervals on which the function is concave upward and intervals on which the function is concave downward: $f(x)=\frac{1}{2}x^5-\frac{2}{3}x^2-8x-1$

$f''(x)$




Command:

$f''(x) = 10x^3 - \frac{4}{3}$

root$(f'')$

A: $(0.5109, 0)$

$f''(x)$: $- - - -$ at $a$; $0.5109$; $+ + + + +$ at $b$

CD: $(-\infty, 0.5109)$

CU: $(0.5109, \infty)$

Point of Inflection @ $x = 0.5109$

You'll also need to be able to identify the point(s) where concavity changes. A point where concavity changes is called a ***point of inflection***.

You can use the same number line that you created to determine concavity intervals to find the $x$ coordinate of any inflection points. Use the following two statements to determine if a zero of the second derivative generates an inflection point.

1. If the sign of the second derivative changes from positive to negative or from negative to positive at a number, $x=c$, then the function has an inflection point at the point $(c, f(c))$.
2. If the sign of the second derivative does not change sign at a number, $x=c$, then the function does not have an inflection point at the point $(c, f(c))$.

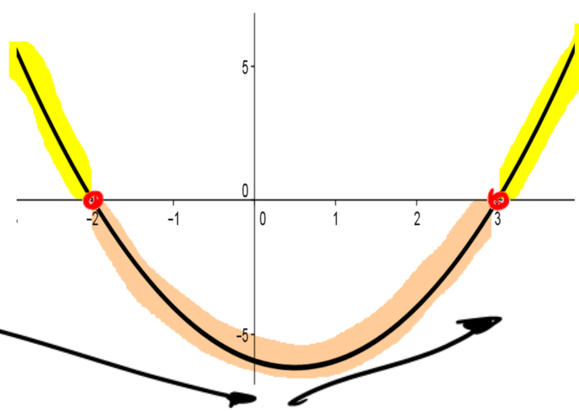
Once you find that $x=c$ generates an inflection point, you can find the $y$ coordinate of the inflection point by computing $f(c)$.

Popper Form 11

**Popper 21:** Find the interval(s) of increasing from the graph.

a. $(1.5, \infty)$
b. $(3, \infty)$
c. $(-2, 3)$
d. $(-\infty, -2) \cup (3, \infty)$

$f'(x)$




$f(x)$

$f'(x)$: $+ + + +$ | $- - -$ | $+ + +$

$a$ $(-3)$; $-2$; $b$ $(0)$; $3$; $c$ $(4)$

**Example 10:** Given $f(x)=\frac{1}{2}x^5-\frac{2}{3}x^2-8x-1$, find any inflection points.

$f''(x) = 10x^3 - \frac{4}{3}$

$10x^3 - \frac{4}{3} = 0$ $\Big\{$ root( f'')

$x = \sqrt[3]{\frac{2}{15}} \sim 0.5109$

Switch

f''(x) $-$ $-$ $-$ $+$ $+$ $+$ $+$

a (0) 0.5109 b (10)

P.o.I: $(0.5109, f(0.5109)) \rightarrow (0.5109, -5.2436)$

↳ For Polynomials : inflection point(< function >)

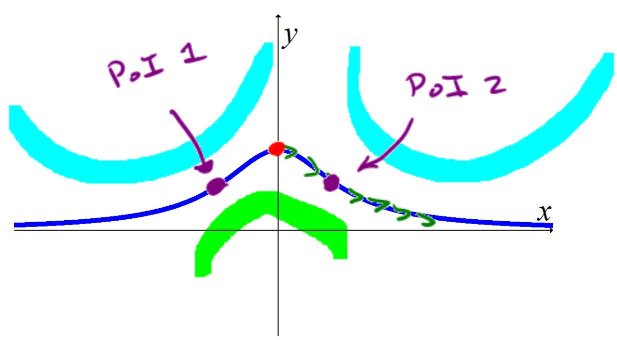
**Example 11:** Given $f(x)=x^3-3x^2-24x+32$, find any inflection points.

Command: inflection point( f )

Answer: A: (1, 6)

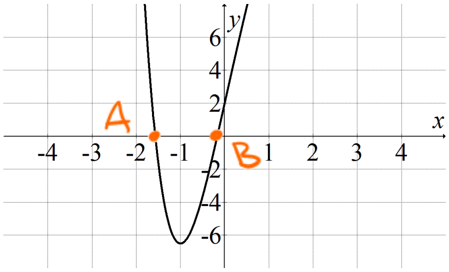
**Analyzing a Function**

**Example 12:** The graph given below is the graph of a polynomial function *f*. Which of the statements below is/are true?




1. The function has a horizontal asymptote. True
2. The function is only increasing. False ; Dec $(0, \infty)$
3. The function has one relative maximum and no relative minimum. True
4. The function has two points of inflection. True

**Example 13:** Analyze the function: $f(x) = \frac{3}{2}x^4 - 2x^3 + 12x + 2$

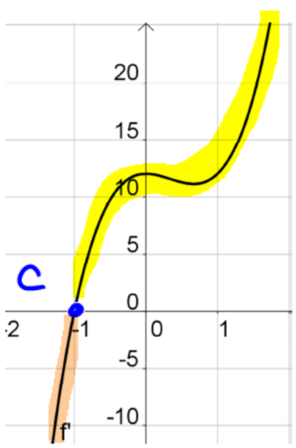



a. Domain : $(-\infty, \infty)$

b. Coordinates of any zeros.
Command: root(f)

Answer:
A: $(-1.5695, 0)$
B: $(-0.1675, 0)$

c. Interval(s) on which the function is increasing; interval(s) on which the function is decreasing.
Command: Find $f'(x)$




← $f'(x)$

root$(f')$

C: $(-1, 0)$

$- - -$ $++++$ $f'(x)$

$-\infty$ $-1$ $\infty$

Dec: $(-\infty, -1)$

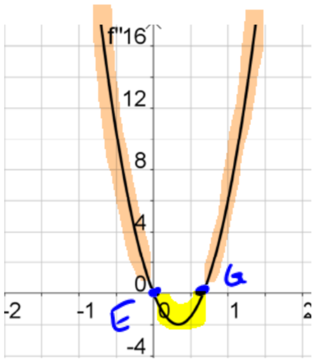
Inc: $(-1, \infty)$

d. Coordinates of any relative extrema.
Command: extremum(f)

Answer:
D: $(-1, -6.5)$

Rel. Min $(-1, f(-1))$

e. Interval(s) on which the function is concave upward; interval(s) on which the function is concave downward.




Command: Need $f''(x)$

root( f'' )

E: (0,0)

G: (0.6667, 0)

++++ | − − − | ++++ f''

0     0.6667

CU: $(-\infty, 0) \cup (0.6667, \infty)$

CD: $(0, 0.6667)$

f. Coordinates of any inflection points.

Command: inflection point( f )

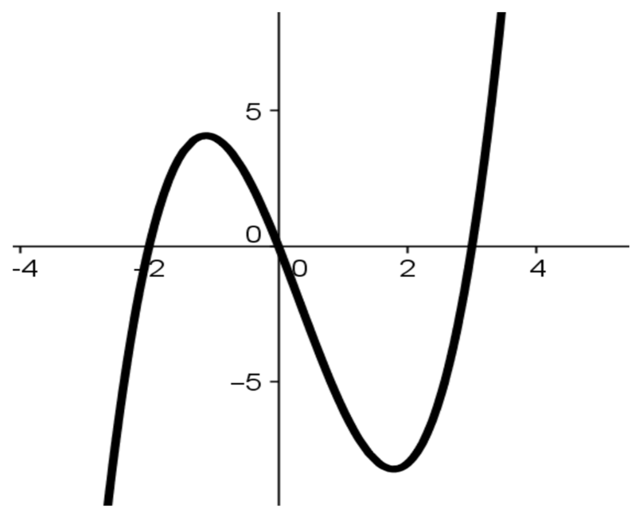
Answer:

H: (0, 2)

I: (0.6667, 9.7037)

**Popper 29:** Find the interval(s) of concave down from the graph.

a. $(-\infty, 1.5)$
b. $(1.5, \infty)$
c. $(-2,0) \cup (3, \infty)$
d. $(-\infty, -2) \cup (0, 3)$




Inc/Dec → $f'(x)$

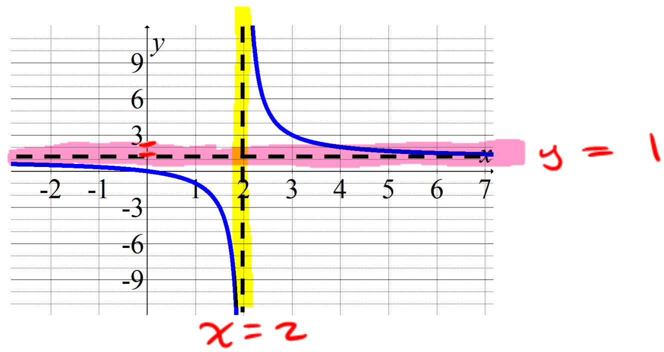
Concavity → $f''(x)$

| | | | | | | |
|---|---|---|---|---|---|---|
| $f$ | $+$ | $-$ | $\nearrow$ | $\searrow$ | $\cup$ | $\cap$ |
| $f'$ | | | $+$ | $-$ | $\nearrow$ | $\searrow$ |
| $f''$ | | | | | $+$ | $-$ |

**Math 1314**
**Lesson 13**
**Analyzing Other Types of Functions**

## Asymptotes

We will need to identify any vertical or horizontal asymptotes of the graph of a function. A **vertical asymptote** is a vertical line $x = a$ that the graph approaches as values for $x$ get closer and closer to $a$. A **horizontal asymptote** is a horizontal line $y = b$ that the graph of a function approaches as values for $x$ increase or decrease without bound. The graph of the function shown below has a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 1$.




If you are given the graph of the function, you can usually just find asymptotes visually by looking at the graph of the function.

When given a function (and not a graph), you can use these rules for finding asymptotes.

$\frac{f(x)}{g(x)}$

Rational functions may have vertical asymptote(s), horizontal asymptote(s), both or neither. To determine asymptotes of rational functions, use these rules:

Factor it out First

*To find a vertical asymptote of a function, reduce the function to lowest terms; then set the denominator equal to zero and solve for $x$.

*To find a horizontal asymptote, compare the degree of the numerator and the degree of the denominator.

- If $\deg(num) < \deg(den)$, the horizontal asymptote is $y = 0$.
- If $\deg(num) = \deg(den)$, where $p$ is the leading coefficient of the numerator and $q$ is the leading coefficient of the denominator, then the horizontal asymptote is $y = \frac{p}{q}$. Reduce the fraction to lowest terms, if possible.
- If $\deg(num) > \deg(den)$, then the graph of the function does not have a horizontal asymptote.

An exponential function may have a horizontal asymptote. An exponential function of the form $f(x) = c + a \cdot b^x$ or $g(x) = c + a \cdot e^{bx}$ will have a horizontal asymptote at $y = c$.

A logarithmic function may have a vertical asymptote. A logarithmic function of the form $f(x) = c + a \cdot \log_b(x-d)$ or $g(x) = c + a \cdot \ln(x-d)$ will have a vertical asymptote at $x = d$.

Inside Part = 0

**Example 1:** Let $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}$ find any asymptotes.

$$\frac{(x+2)(x-2)}{(x-1)(x-2)} = \frac{x+2}{x-1}$$

Command:

asymptote ( function ) $\rightarrow \{ x = 1 \;,\; y = 1 \}$

Horizontal Asymptote: Compare Degrees

$y = \frac{1}{1} = 1$

Vertical Asymptote: Denominator = 0

$x - 1 = 0$

$x = 1$

If the function you need to analyze is something other than a polynomial function, you will have some other types of information to find and some analysis techniques will be slightly different.

When working with functions different from polynomials, the critical numbers are defined as follows:

The **critical numbers** of a function are numbers in the domain of the function where $f'(x) = 0$ or where $f'(x)$ is undefined.

You must use caution when the graph of the derivative shows an asymptote, a vertical tangent line or a sharp turn in the graph of the function. If this point is in the domain of the function, *it is also a critical number and must be used in the analysis of the function*.

$$f'(x) = \frac{1}{3(x+3)^{2/3}} - 1$$

**Example 2:** The graph shown below is the derivative of $f(x) = (x+3)^{\frac{1}{3}} - x$. The domain of the function is $(-\infty, \infty)$. Find any critical numbers.

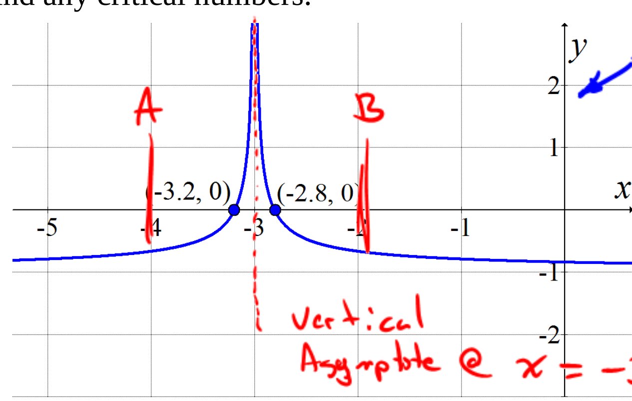
$f'(x) = 0$

roots( f', -4, -2 )

A , B

A: (-3.2, 0)

B: (-2.8, 0)




f'(x)

f'(x) is undefined @ x = -3

Vertical Asymptote @ x = -3

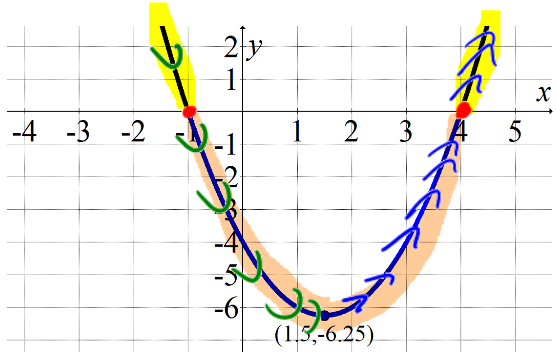


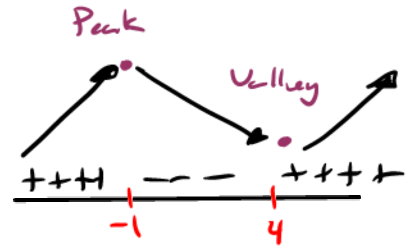
Critical Numbers: (-3.2 , -2.8 , -3 )

**Example 3:** The graph given below is the *first derivative* of a function, *f*.

roots are

$-1, 4$




Create a number line, subdividing the line at the zeros of the derivative or where the derivative is undefined.

a. Find the interval(s) on which the *f* is increasing.

$(-\infty, -1) \cup (4, \infty)$

b. Find the interval(s) on which the *f* is decreasing

$(-1, 4)$

c. Find the *x* coordinate any relative extremum of *f* (and state whether it is a relative maximum or a relative minimum) Rel. Max @ $x = -1$

Rel Min @ $x = 4$

Fact: The function *f* is concave upward on the interval(s) where *f '* is increasing. The function *f* is concave downward on the interval(s) where *f '* is decreasing.

Look back at the graph of the first derivative. Notice where the graph is increasing and where it's decreasing. Create a new number line, subdividing the line at where the change in increasing and decreasing occurs, then answer the following questions.

d. Find the interval(s) on which the function is concave upward

CU: $(1.5, \infty)$

e. Find the interval(s) on which the function is concave downward.

CD: $(-\infty, 1.5)$

f. Find the *x* coordinate of any inflection points.

$x = 1.5$

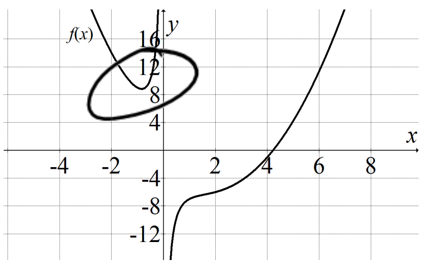
For functions different from polynomials, finding increasing/decreasing intervals, relative extrema and concavity intervals is the same as what we used in analyzing polynomial functions. When finding points of inflection for polynomials, we used the command: inflectionpoint[<polynomial>]. This command will NOT WORK FOR FUNCTIONS DIFFERENT FROM POLYNOMIALS. So we'll simply use the line test to determine any points of inflection for other types of functions.

Commands for finding zeros and extremum for functions different from polynomials:

Roots[<Function>, <Start x-Value>, <End x-Value>]
Extremum[<Function>, <Start x-Value>, <End x-Value>]

**Example 4:** Analyze $f(x)=\dfrac{x^3-4x^2-4}{x}$. *Enter the function in GGB.*

f(x)



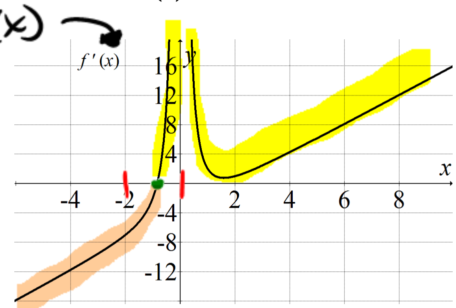

a. Domain

$x \neq 0$   $(-\infty, 0) \cup (0, \infty)$

b. Equations of any asymptotes.
Command: asymptote( f )   Answer: $\{x = 0\}$ Vertical

c. Interval(s) on which the function is increasing; interval(s) on which the function is decreasing.

f'(x)




Command: Find f'(x)

roots( f', -2, 0)

A: (-0.8393, 0)

f'(x) − − | + + + | VA | + + + +

a   -0.8393   b   0   c

Dec: $(-\infty, -0.8393)$

Inc: $(-0.8393, 0) \cup (0, \infty)$

d. Coordinates of any relative extrema.
Command: extremum( f, -2, 0)   Answer:

Rel Min $(-0.8393, f(-0.8393))$

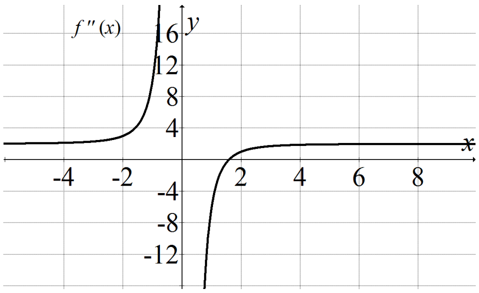
$(-0.8393, 6.8275$



1 − 4 A

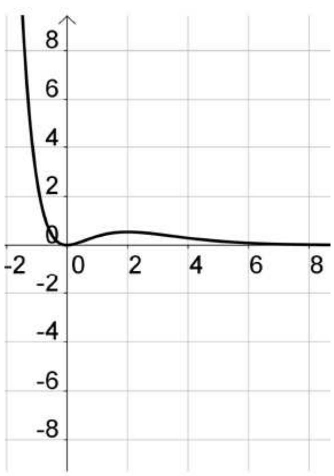
e. Interval(s) on which the function is concave upward; interval(s) on which the function is concave downward.

Command:




f. Coordinates of any inflection points.

**Example 5**: Analyze $f(x) = x^2 e^{-x}$. *Enter the function in GGB.*



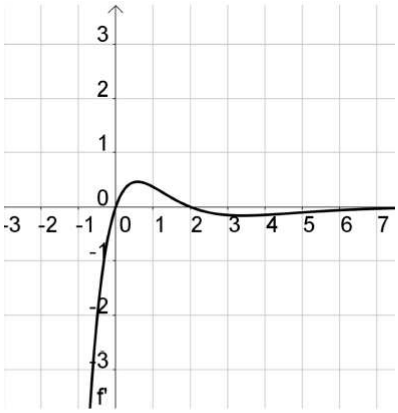

a. Domain

b. Equations of any asymptotes.

Command: Answer:

c. Interval(s) on which the function is increasing; interval(s) on which the function is decreasing.

Command:

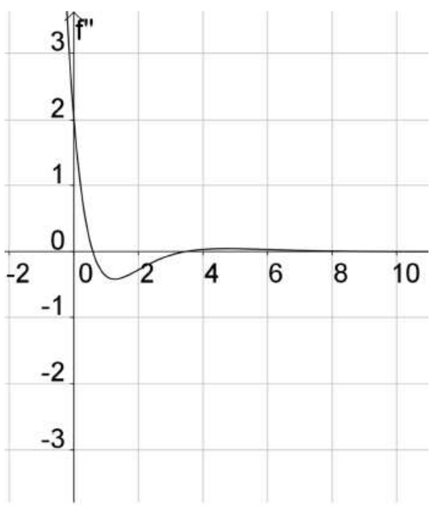



d. Coordinates of any relative extrema.

Command: Answer:

e. Interval(s) on which the function is concave upward; interval(s) on which the function is concave downward.

Command:




f. Coordinates of any inflection points.

# Math 1314
# Optimization

When you optimize something, you make it as large as possible or as small as possible under certain stated conditions. Business owners wish to make revenues and profits as large as possible, while keeping costs as small as possible. There are many other applications as well. In this unit, we'll start by looking at optimization generally; then we'll move on to some applications.

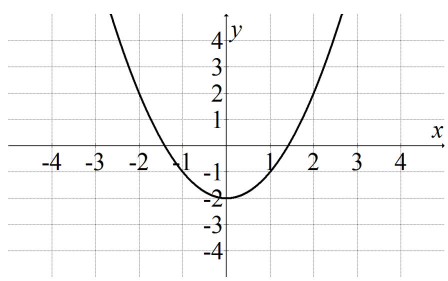
**Absolute Extrema**

**Definition:** If $f(x) \le f(c)$ for all *x* in the domain of *f*, then *f*(c) is called the **absolute maximum value** of *f*. If $f(x) \ge f(c)$ for all *x* in the domain of *f*, then *f*(c) is called the **absolute minimum value** of *f*.

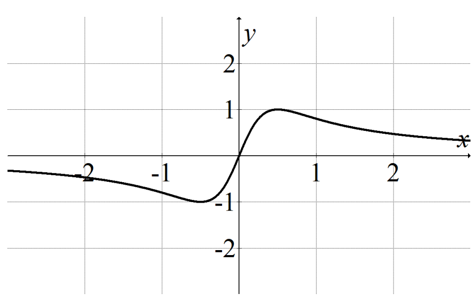
So the *y* value of the point that has the biggest *y* value in the interval of interest is the absolute maximum value. The *y* value of the point that has the smallest *y* value in the interval of interest is the absolute minimum value.

We can find absolute extrema by looking at a graph of a function.

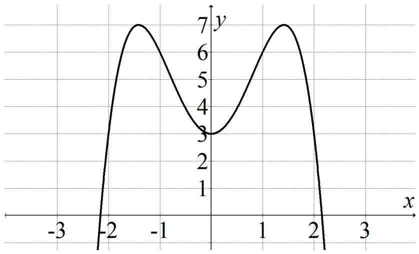
**Example 1:** State the:
a. absolute maximum value.
b. absolute minimum value.




**Example 2:** State the:
a. absolute maximum value.
b. absolute minimum value.



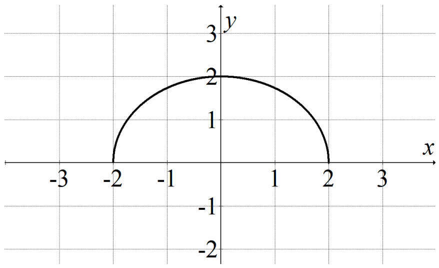

**Example 3:** Find the absolute maximum and absolute minimum values of $f(x) = 3 + 4x^2 - x^4$.
*Enter the function in GGB.*




Absolute Maximum Value

Absolute Minimum Value

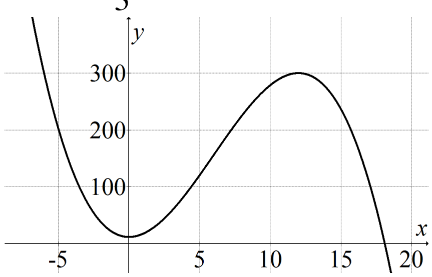
**Example 4:** Find the absolute maximum and absolute minimum values of the function $f(x)=\sqrt{4-x^2}$ . *Enter the function in GGB.*




Absolute Maximum Value

Absolute Minimum Value

**Example 5:** The height of a rocket, *t* seconds after launch, is given by the function $h(t)=-\frac{1}{3}t^3+6t^2+12$, where $h(t)$ is given in feet. *Enter the function in GGB.*




a. Find the time *t* when the rocket reaches its maximum height.
Command: Answer:

b. What is the maximum height of the rocket?

Sometimes, you'll be given a closed interval and asked to find the absolute extrema on that interval. In that case, you are only interested in the behavior of the function on that interval. If you have a continuous function on a closed interval, you are guaranteed to have both an absolute maximum and an absolute minimum value.

**Finding the Absolute Extrema of *f* on a Closed Interval**

1. Find the critical points of $f$ that lie in $(a, b)$.
2. Compute the value of the function at every critical point found in step 1 and also compute $f(a)$ and $f(b)$.
3. The absolute maximum value will be the largest value found in step 2, and the absolute minimum value will be the smallest value found in step 2.

**Example 6:** Find the absolute extrema of the function $f(x) = x^3 + 3x^2 - 1$ on [-1, 2].

**Example 7:** Find the absolute extrema of the function $f(x) = \sqrt{x}\left(x^3 - 4\right)^2$ on [0.5, 1]. *Enter the function in GGB.*

Command: Absolute Maximum Value

Command: Absolute Minimum Value