Pepper Form 02

SD is C
Lesson 4 – Limits

**Example 7:** Evaluate: \( \lim_{x \to 4} \frac{x + 8}{x - 4} \).

\[
\lim_{x \to 4} \frac{x + 8}{x - 4} = \frac{12}{0} \quad \rightarrow \quad \text{Undefined} = \text{DNE}
\]

When substitution gives you a value in the form \( \frac{k}{0} \), where \( k \) is any non-zero real number the limit DNE.

**Indeterminate Forms**

What do you do when substitution gives you the value \( \frac{0}{0} \)?

This is called an indeterminate form. It means that we are not done with the problem! We must try another method for evaluating the limit!!

Once we determine that a problem is in indeterminate form, we can use GGB to find the limit. Command: \text{limit[<Function>,<Value>]}

**Example 8:** Determine if the function given is indeterminate. If it is, use GGB to evaluate the limit.

\[
\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4}
\]

\[
= \frac{2^2 + 3(2) - 10}{2^2 - 4}
\]

\[
= \frac{4 + 6 - 10}{4 - 4}
\]

\[
= \frac{0}{0}
\]

\[
\text{More work:} \quad \lim_{x \to 2} \frac{(x + 5)(x - 2)}{(x + 2)(x - 2)}
\]

\[
= \lim_{x \to 2} \frac{x + 5}{x + 2}
\]

\[
= \frac{2 + 5}{2 + 2}
\]

\[
= \frac{7}{4}
\]

\[
= 1.75
\]

**Example 9:** Determine if the function given is indeterminate. If it is, use GGB to evaluate the limit.

\[
\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}
\]

\[
= \frac{9 - 15 + 6}{3 - 3}
\]

\[
= \frac{0}{0}
\]

\[
\rightarrow \lim_{x \to 3} \frac{(x - 3)(x - 2)}{x - 3}
\]

\[
= \lim_{x \to 3} (x - 2)
\]

\[
= 3 - 2
\]

\[
= 1
\]
Lesson 4 – Limits

**Example 10:** Determine if the function given is indeterminate. If it is, use GGB to evaluate the limit.

\[ \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{x - 4} \]

\[ \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{4} - 2}{4 - 4} = \frac{2 - 2}{0} = \frac{0}{0} \]

Question 29: Is the following function an indeterminate?

\[ \lim_{x \to 6} \frac{x^2 + 2x - 24}{x^2 + 36} \]

a. Yes
b. No

So far we have looked at problems where the target number is a specific real number. Sometimes we are interested in finding out what happens to our function as \( x \) increases (or decreases) without bound.

**Limits at Infinity**

Consider the function \( f(x) = \frac{2x^2}{x^2 + 1} \). As the value of \( x \) get larger and larger, \( f(x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>10</th>
<th>1000</th>
<th>100,000</th>
<th>1,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1.98...</td>
<td>1.99...</td>
<td>1.999...</td>
<td>1.999...</td>
</tr>
</tbody>
</table>
Lesson 4 – Limits

We say that a function \( f(x) \) has the limit \( L \) as \( x \) increases without bound (or as \( x \) approaches infinity), written \( \lim_{x \to \infty} f(x) = L \), if \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) large enough.

We say that a function \( f(x) \) has the limit \( L \) as \( x \) decreases without bound (or as \( x \) approaches negative infinity), written \( \lim_{x \to -\infty} f(x) = L \), if \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) to be negative and sufficiently large in absolute value.

We can also find a limit at infinity by looking at the graph of a function.

**Example 11:** Evaluate: \( \lim_{x \to -\infty} \frac{6x - 7}{2x} = 3 \)

We can also find limits at infinity algebraically or by recognizing the end behavior of a polynomial function.

**Example 12:** Evaluate:

A. \( \lim_{x \to -\infty} \left( -4x^3 - 7x + 5 \right) \)

B. \( \lim_{x \to -\infty} \left( 3x^4 + 1 \right) \)

C. \( \lim_{x \to -\infty} \left( -x^3 - 3x \right) \)
Lesson 4 – Limits

Limits at infinity problems often involve rational expressions (fractions). The technique we can use to evaluate limits at infinity is to divide every term in the numerator and the denominator of the rational expression by \( x^n \), where \( n \) is the highest power of \( x \) present in the denominator of the expression.

Then we can apply this theorem:

**Theorem:** Suppose \( n > 0 \). Then \( \lim_{x \to \infty} \frac{1}{x^n} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x^n} = 0 \), provided \( \frac{1}{x^n} \) is defined.

However, often students prefer to just learn some rules for finding limits at infinity.

The highest power of the variable in a polynomial is called the **degree** of the polynomial. We can compare the degree of the numerator with the degree of the denominator and come up with some generalizations.

- If the degree of the numerator is **smaller** than the degree of the denominator, then \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \).
- If the degree of the numerator is the **same** as the degree of the denominator, then you can find \( \lim_{x \to \infty} \frac{f(x)}{g(x)} \) by making a fraction from the leading coefficients of the numerator and denominator and then reducing to lowest terms.
- If the degree of the numerator is **larger** than the degree of the denominator, then it’s best to work the problem by dividing each term by the highest power of \( x \) in the denominator and simplifying. You can then decide if the function approaches \( \infty \) or \( -\infty \), depending on the relative powers and the coefficients.

The notation, \( \lim_{x \to \infty} f(x) = \infty \) indicates that, as the value of \( x \) increases, the value of the function increases without bound. This limit does not exist, but the \( \infty \) notation is more descriptive, so we will use it.

**Example 13:** Evaluate: \( \lim_{x \to \infty} \frac{5x^2 + 3x + 4}{4x^2 - 2x + 8} \)

\[
\text{Deg of Num: 2} \quad \text{Deg of Denom: 2} \quad = \quad \frac{5}{4} \quad (\text{HA: } y = \frac{5}{4})
\]

**Example 14:** Evaluate: \( \lim_{x \to \infty} \frac{4x + 5}{x^2 - 9x + 9} \)

\[
\text{Deg of Num: 1} \quad \text{Deg of Denom: 2} \quad = \quad 0
\]
Lesson 4 – Limits

Example 15: Evaluate: \( \lim_{x \to \infty} \frac{2x^4 + 4}{x^2 - x} \)

Deg of Num: 4
Deg of Denom: 2

Example 16: Evaluate: \( \lim_{x \to \infty} \frac{-5x^5 + 2x}{x^4 + 1} \)

Used GBE: \( = -\infty \)

Question 18: Evaluate:

\( \lim_{x \to -\infty} \frac{-5x^4 + 3x^3 + 8x + 1}{3x^3 - 5x} \)

a. \( -\infty \)
b. \( \infty \)
c. 0
d. \(-5/3\)
Lesson 5 – One-sided Limits and Continuity

Sometimes we are only interested in the behavior of a function when we look from one side and not from the other.

**Example 1:** Use the given graph to find \( \lim_{x \to 0} f(x) \).

![Graph showing a function with a point at (0,1) and arrows indicating one-sided limits.]

Now suppose we are *only* interested in looking at the values of \( x \) that are bigger than 0. In this case, we are looking at a one-sided limit.

We write \( \lim_{x \to 0^+} f(x) \). This is called a [right-hand limit](#), because we are looking at values on the right side of the target number.

In this case, \( \lim_{x \to 0^+} f(x) = -1 \).

If we are interested in looking only at the values of \( x \) that are smaller than 0, then we would be finding the [left-hand limit](#). The values of \( x \) that are smaller than 0 are to the left of 0 on the number line, hence the name. We write \( \lim_{x \to 0^-} f(x) \).

In this case, \( \lim_{x \to 0^-} f(x) = 1 \).

Our definition of a limit from the last lesson is consistent with this information. We say that \( \lim_{x \to a} f(x) = L \), if and only if the function approaches the same value, \( L \), from both the left side and the right side of the target number. This idea is formalized in this theorem:

**Theorem:** Let \( f \) be a function that is defined for all values of \( x \) close to the target number \( a \), except perhaps at \( a \) itself. Then \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \).

We can also find one-sided limits from piecewise defined functions.
Example 2: Consider this graph:

![Graph](image)

Find each of the following limits, if it exist.

A. \( \lim_{{x \to 0^-}} f(x) \)

B. \( \lim_{{x \to 0^+}} f(x) \)

C. \( \lim_{{x \to 0}} f(x) \)

-3 \( \neq 0 \)

Left \( \neq \) Right

Question 1: Find the limit, if it exist.

\[
\lim_{{x \to 2}} -x^2 + 4x + 4
\]

a. 0
b. -8
c. 8
d. -4
e. None of the above

Question 2 is D
Example 5: Let \( f(x) = \begin{cases} 
  x - 6, & x \leq 0 \\
  x^2 + 5x + 6, & x > 0 
\end{cases} \) is the function continuous at \( x = 0 \)?

We need to check:
1. Is \( f(0) \) defined?

2. Does \( \lim_{x \to 0} f(x) \) exist?

3. Does \( \lim_{x \to 0} f(x) = f(0) \)?