Math 1314 Lesson 7

We can also use the derivative to find the equation of a tangent line to a function at a specific point.

Recall the following graph of a tangent line to a function at a specific point.

Example 6: Let \( f(x) = \frac{1}{3}x^3 - 2x^2 + 7x \). Write an equation of the tangent line to \( f(x) \) at \((3, 12)\).

\[
\frac{df}{dx} = \frac{1}{3} \cdot 3x^2 - 2 \cdot 2x + 7
\]

\[
= x^2 - 4x + 7
\]

\[
\frac{df}{dx}(3) = (3)^2 - 4(3) + 7
\]

\[
= 9 - 12 + 7
\]

\[
= 4 \quad \text{(Tangent slope at } x = 3)\]

\[
y = mx + b
\]

\[
12 = 4(3) + b
\]

\[
b = 0
\]

\[
y = 4x \quad \text{ (Tangent line at } x = 3)\]

Question 19: Find the numerical derivative of \( f(x) = 2x^3 - 4x^2 + 8x - 10 \) when \( x = -1 \)

a. 22
b. -24
c. -4
d. 6
e. None of the above
Example 7: Let \( f(x) = \frac{5}{3} x^3 + \frac{8}{3} \). Write an equation of the tangent line to \( f(x) \) at \( x = -1 \).

\[
\begin{align*}
  f'(x) &= \frac{5}{3} \cdot 3 x^2 \\
  f'(x) &= 5 x^2 \\
  f'(-1) &= 5 (-1)^2 \\
  &\quad = 5 \text{ (slope)} \\
  f(-1) &= \frac{5}{3} (-1)^3 + \frac{8}{3} \\
  &= -\frac{5}{3} + \frac{8}{3} = \frac{3}{3} \\
  &= 1 \text{ (y-value)} \\
  y &= mx + b \\
  1 &= 5(-1) + b \\
  1 &= -5 + b \\
  b &= 6
\end{align*}
\]

Tangent line \( @ x = -1 \):
\[
y = 5x + 6
\]

For more complicated functions, we will use GGB.

Example 8: Write an equation of the line that is tangent to \( f(x) = 1.6x^3 - 2.31x^2 + 6.39x - 2.81 \) when \( x = 3.75 \).

New Command: \texttt{Tangent (<point>, <function>)}

Command: \texttt{Tangent (3.75, f )}

Answer:
\[
y = 56.565x - 139.0756
\]
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**Question 4:** Find the derivative:

\[ f(x) = -2x^3 + 4x - \frac{2}{x^2} \]

\[ f'(x) = -6x^2 + 4 - \frac{4}{x^3} \]

a. \( f'(x) = 6x^2 + 4 - \frac{4}{x^3} \)

b. \( f'(x) = -6x^2 + 4 + \frac{4}{x^2} \)

c. \( f'(x) = -6x^2 - 4 - \frac{4}{x} \)

d. \( f'(x) = -6x^2 + 4 + \frac{4}{x^3} \)

In other cases, we may want to find all values of \( x \) for which the tangent line to the graph of \( f \) is horizontal. Since the slope of any horizontal line is 0, we’ll want to find the derivative, set it equal to zero and solve the resulting equation for \( x \).

**Example 9:** Find all points on the graph of \( f(x) = 5 - 3x + 2x^2 \) where the tangent line is horizontal.

\[ f'(x) = -3 + 4x \]

\[ f'(x) = 0 \]

\[-3 + 4x = 0 \]

\[ 4x = 3 \]

\[ x = \frac{3}{4} \]

\[ \text{or} \]

\[ x = 0.75 \]

\[ f\left(\frac{3}{4}\right) = 5 - 3\left(\frac{3}{4}\right) + 2\left(\frac{3}{4}\right)^2 \]

\[ = 3.675 \]

\[ \text{Pt:} \left(0.75, 3.675\right) \]

We can also determine values of \( x \) for which the derivative is equal to a specified number. Set the derivative equal to the given number and solve for \( x \) either algebraically or by graphing.

**Question 2 is A**
Example 10: Find all values of $x$ for which $f'(x) = 3$.

$$f'(x) = x^2 + 5x - 7$$

$$f'(x) = 3$$

$x^2 + 5x - 7 = 3$

$x^2 + 5x - 10 = 0$

*Quadratic Formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

With this group of word problems, the first step will be to determine what kind of problem we have for each problem. Does it ask for a function value (FV), a rate of change (ROC) or an average rate of change (AROC). From there, we'll apply the appropriate methods.

Example 11: A country’s gross domestic product (in millions of dollars) is modeled by the function $G(t) = -2t^3 + 45t^2 + 20t + 6000$ where $0 \leq t \leq 11$ and $t = 0$ corresponds to the beginning of 1997.

a. What was the average rate of growth of the GDP over the period 1999 – 2004?

*Recall: Average Rate of Change (Difference Quotient) Formula:*

$$f(x + h) - f(x)$$

$$\frac{h}{h}$$

$$\frac{G(7) - G(2)}{7 - 2} = \frac{1455}{5} = 291 \text{ million/yr}$$

b. At what rate was GDP changing at the beginning of 2002?

Need the Derivative

$$G'(5)$$

$$= 320 \text{ million/yr}$$
Example 12: The model \( N(t) = 34.4(1 + 0.32125t)^{0.15} \) gives the number of people in the US who are between the ages of 45 and 55. Note, \( N(t) \) is given in millions and \( t = 0 \) corresponds to the **beginning of 1995**. Enter the function into GGB.

a. How large is this segment of the population projected to be at the beginning of 2011? \( t = 16 \)

Command: \[ N(16) \]

Answer: \[ 45.1631 \text{ million of people} \]

b. How fast will this segment of the population be growing at the beginning of 2011?

Command: \[ \text{Rate of Change} \rightarrow \text{Der.} \]

Answer: \[ N'(16) = 0.3544 \text{ millions of people/yr} \]

Velocity and Acceleration

A common use of rate of change is to describe the motion of an object. The function gives the position of the object with respect to time, so it is usually a function of \( t \) instead of \( x \). If the object changes position over time, we can compute its rate of change, which we refer to as velocity. We can find either the average rate of change or the instantaneous rate of change, depending on the question posed.

**Velocity** can be positive, negative or zero. If you throw a rock up in the air, its velocity will be positive while it is moving upward and will be negative while it is moving downward. We refer to the absolute value of velocity as **speed**. Velocity has two components: speed and direction.

Solving problems that involve velocity (rate of change) is a common application of the derivative. Velocity can be expressed using one of these two formulas, depending on whether units are given in feet or meters:

\[
\begin{align*}
h(t) &= -16t^2 + v_o t + h_0 \quad \text{(feet)} \\
h(t) &= -4.9t^2 + v_o t + h_0 \quad \text{(meters)}
\end{align*}
\]

In each formula, \( v_o \) is initial velocity and \( h_0 \) is initial height.

Example 13: Suppose you are standing on the top of a building that is 28 meters high. You throw a ball up into the air, with initial velocity of 10.2 meters per second. Write the equation that gives the height of the ball at time \( t \). Then use the equation to find the velocity of the ball when \( t = 2 \).

\[
\begin{align*}
h(t) &= -4.9t^2 + v_o t + h_0 \\
h(t) &= -4.9t^2 + 10.2t + 28
\end{align*}
\]

[Derivative of \( h(t) \)]
Sometimes we need to find the derivative of the derivative. Since the derivative is a function, this is something we can readily do. The derivative of the derivative is called the second derivative, and is denoted $f''(x)$. To find the second derivative, we will apply whatever rule is appropriate given the first derivative.

**Example 14:** Find the second derivative when $x = -3$: $f(x) = 4x^3 - x^2 - 7x + 5$. 

\[
\begin{align*}
\frac{d^2y}{dx^2} &= 20x^4 - 2x - 7 \\
\frac{d^3y}{dx^3} &= 80x^3 - 2 \\
\frac{d^4y}{dx^4} &= -2162
\end{align*}
\]

OR

\[
\begin{align*}
\frac{d^2y}{dx^2} &= 20x^4 - 2x - 7 \\
\frac{d^3y}{dx^3} &= 80x^3 - 2 \\
\frac{d^4y}{dx^4} &= -2162
\end{align*}
\]
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**Example 15:** Find the value of the second derivative when $x = 5$ if 
\[
f(x) = \frac{x^2 \ln x}{(x^2 + 3)^\frac{1}{3}}.
\]

*Enter the function into GGB.*

Command: Answer:

\[
f''(5) = 0.6297
\]

When you accelerate while driving, you are increasing your speed. This means that you are changing your rate of change. **Acceleration**, then, is the derivative of velocity – the rate of change of your rate of change. It follows that the second derivative of a position functions gives an acceleration function.

**Example 16:** The distance $s$ in feet covered by a car $t$ seconds after starting from rest is given by the function 
\[
s(t) = -t^3 + 12t^2 + 36t.
\] Find the acceleration when $t = 2$ seconds.

\[
s''(2) = 12 \text{ ft/sec}^2
\]
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**Question 12:** Find x-values for the horizontal tangent(s) of the following function

\[ f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 14 \]

a. \( x = -2, 3 \)
b. \( x = -3, 2 \)
c. \( x = -2, -3 \)
d. \( x = 2, 3 \)
e. None of the Above

**Question 36:** Find the second derivative when \( x = 2 \): \( f(x) = \frac{1}{5}x^5 - 4x^3 + 12x^2 - 30 \)

a. \( 8 \)
b. \( 80 \)
c. \( -88 \)
d. \( -16 \)
e. None of the Above