\[ y = \tan^{-1}(\frac{2x}{1}) = \text{Angle} \]

\[ \csc(y) \]

\[ = \frac{\sqrt{1 + 4x^2}}{2x} \]
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Half-Angle Formulas

\[
\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}
\]

\[
\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}
\]

\[
\tan\left(\frac{\theta}{2}\right) = \frac{\sin \theta}{1 + \cos \theta}
\]

Note: In the half-angle formulas the ± symbol is intended to mean either positive or negative but not both, and the sign before the radical is determined by the quadrant in which the angle \(\frac{\theta}{2}\) terminates.

When calculating trigonometric functions of multiples of \(\frac{\pi}{12}\), you have the choice of an addition formula or using half-angle formula.
When calculating trigonometric functions of multiples of \(\frac{\pi}{8}\), you have only one choice which is half-angle formula.

It is not possible to write \(\frac{\pi}{8}\) as a sum or difference of our special angles \(\frac{\pi}{3}\), \(\frac{\pi}{4}\), and \(\frac{\pi}{6}\).

Example 4: Suppose \(\cos \theta = \frac{5}{9}\) and \(\frac{3\pi}{4} < \theta < 2\pi\). Find \(\cos \frac{\theta}{2}\) and \(\sin \frac{\theta}{2}\).

\[
\frac{3\pi}{4} < \frac{\theta}{2} < \pi \quad \text{in} \quad \text{Q2}
\]

\[
\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1 + \cos \theta}{2}} = -\sqrt{\frac{1 + \frac{-5}{9}}{2}} = -\sqrt{\frac{\frac{4}{9}}{2}} = -\frac{\sqrt{2}}{3}
\]

\[
\sin\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1 - \cos \theta}{2}} = +\sqrt{\frac{1 - \frac{-5}{9}}{2}} = +\sqrt{\frac{\frac{4}{9}}{2}} = +\frac{\sqrt{2}}{3}
\]
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Example 5: Calculate

a. \[
\cos \left( \frac{\pi}{8} \right) = \frac{\sqrt{2 + \sqrt{2}}}{2}
\]

b. \[
\cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}
\]

c. \[
\cos \left( \frac{\pi}{8} \right) = \frac{\sqrt{2 + \sqrt{2}}}{2}
\]

d. \[
\cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}
\]

Popper 2:

Simplify \(1 + \sin^2 x - \cos^2 x \tan^2 x\)

a. \(\sin^2 x\)

b. \(\sec^2 x\)

c. \(1\)

d. \(\cos^2 x\)

Popper 3:

Suppose \(\sin t = \frac{1}{3}\) and \(\frac{\pi}{2} < t < \pi\). Find \(\sin 2t\)

a. \(\frac{4\sqrt{2}}{9}\)

b. \(\frac{2\sqrt{2}}{9}\)

c. \(-2\sqrt{2}\)

d. \(-4\sqrt{2}\)

\(\sin 2t = 2\sin t \cos t\)

a. \(\frac{1}{3}\)

b. \(-\frac{\sqrt{8}}{3}\)

c. \(-2\sqrt{2}\)

d. \(-4\sqrt{2}\)
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b. \( \sin\left(\frac{x}{12}\right) \)

\[ \frac{x}{12} = \frac{\pi}{12} \]
\[ \beta = \frac{\pi}{12} \]

\[ \text{It is in QI.} \]
\[ \sin \text{ is pos in QI.} \]
\[ \frac{\alpha}{2} = \frac{\pi}{12} \]
\[ \alpha = \frac{\pi}{6} \]

\[ = + \sqrt{1 - \cos^2 \frac{x}{12}} = \sqrt{1 - \cos^2 \frac{\pi}{12}} = \sqrt{1 - \frac{\sqrt{3}}{2}} \]
\[ = \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2} \]
\[ = \frac{1}{2} \sqrt{2 - \sqrt{3}} \]

\[ \boxed{c. \tan(105^\circ).} \]

\[ \frac{\theta}{2} = 105^\circ \]
\[ \theta = 210^\circ \]

\[ \tan\left(\frac{\theta}{2}\right) = \frac{\sin \theta}{1 + \cos \theta} \]
\[ = \frac{-\sqrt{2}}{1 + (-\sqrt{3}/2)} = \frac{-\sqrt{2}}{\frac{2 - \sqrt{3}}{2}} \]
\[ = \frac{-\sqrt{2}}{2 - \sqrt{3}} \cdot \frac{2}{2} = \frac{-\sqrt{2}}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{-2 - \sqrt{3}}{4 - 3} \]
\[ = -2 - \sqrt{3} \]
\[ \boxed{} \]
Section 6.3: Trigonometric Equations

An equation that contains a trigonometric expression is called a **trigonometric equation**. Since trigonometric functions are periodic, there may be infinitely solutions to some trigonometric equations.

**Example 1:** Find all solutions in the interval \([0, \pi]\) of the equation \(\sin x = \frac{\sqrt{3}}{2}\).

Knowing the unit circle:

\[
x = \frac{\pi}{3}, \quad \frac{2\pi}{3}
\]

**Example 2:** Find all solutions of the equation. \(\sin x = \frac{\sqrt{3}}{2}\)

\[
x = \frac{\pi}{3} + 2\pi k \quad \text{or} \quad \frac{2\pi}{3} + 2\pi k
\]

**Example 3:** Find all solutions of the equation. \(2\sin^2 x - 3\sin x + 1 = 0\)

\[
x = \frac{\pi}{6} + 2\pi k
\]

**Example 4:** Find all solutions of the equation \(\sin x + 2\csc x = -3\).

\[
x = \frac{3\pi}{2} + 2\pi k
\]

**Example 5:** Find all solutions of the equation. \(\cot x = \frac{1}{\sqrt{3}}\)

\[
x = \frac{\pi}{3} + \pi k
\]
Example 6: Find all solutions of the equation $-2\cos^2 x - \sin x = -1$.

$$-2 \left[ 1 - \sin^2 x \right] - \sin x + 1 = 0$$

$$-2 + 2\sin^2 x - \sin x + 1 = 0$$

$$2\sin^2 x - \sin x - 1 = 0$$

$$(2\sin x + 1)(\sin x - 1) = 0$$

$$2\sin x + 1 = 0 \quad \sin x - 1 = 0$$

$$\sin x = -\frac{1}{2} \quad \sin x = 1$$

$$x = \frac{7\pi}{6} + 2\pi k \quad x = \frac{\pi}{2} + 2\pi k$$

Example 7: Solve the equation $4\cos \theta - 3\sec \theta = 0$ in the interval $[0, 2\pi)$.

$$4\cos^2 \theta - 3 = 0$$

$$4\cos^2 \theta = 3$$

$$\cos^2 \theta = \frac{3}{4}$$

$$\cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Example 8: Solve the equation $\cos^2 \theta + \frac{1}{2}\cos \theta - \frac{1}{2} = 0$ in the interval $[0, 2\pi]$.

$$2\left( \cos^2 \theta + \frac{1}{2}\cos \theta - \frac{1}{2} \right) = 2 \cdot 0$$

$$2\cos^2 \theta + \cos \theta - 1 = 0$$

$$(2\cos \theta - 1)(\cos \theta + 1) = 0$$

$$2\cos \theta - 1 = 0 \quad \cos \theta + 1 = 0$$

$$\cos \theta = \frac{1}{2} \quad \cos \theta = -1$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \quad \theta = \pi$$

Example 9: Find all solutions of the equation $\cos 2\theta + 3 = 5\cos \theta$ in the interval $[0, 2\pi]$.

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= 1 - 2\sin^2 \theta$$

$$= 2\cos^2 \theta - 1$$

$$2\cos^2 \theta - 1 + 3 = 5\cos \theta$$

$$2\cos^2 \theta - 5\cos \theta + 2 = 0$$

$$(2\cos \theta - 1)(\cos \theta - 2) = 0$$

$$\cos \theta = \frac{1}{2} \quad \cos \theta = 2$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \quad \text{Undefined}$$

$$\text{No Answer}$$
Example 10: Solve the equation \( \cos x + \sin x \tan x = 2 \) in the interval \([0, 2\pi]\).

\[
\cos x + \sin x \cdot \frac{\sin x}{\cos x} = 2
\]

\[
\cos x = \frac{1}{2}
\]

\[
x = \frac{\pi}{3}, \frac{5\pi}{3}
\]

Equations Involving Multiple Angles

Example 11: Find all solutions of \( \tan \left( \frac{\pi}{9} x \right) = -1 \) in the interval \([0, 2\pi]\).

\[
\tan \theta = -1 \quad \text{between } [0, \pi]
\]

\[
\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}
\]

\[
\theta = \frac{3x}{9}
\]

\[
x = \frac{\pi}{3}
\]

Divide these \( \pi \) by \( 3 \)

Example 12: Find all solutions of \( \sin (9x) = 1 \) in the interval \([0, \frac{2\pi}{9}]\).

\[
\sin \theta = 1 \quad \text{between } [0, 2\pi]
\]

\[
\theta = \frac{\pi}{2}
\]

\[
x = \frac{\pi}{9}
\]

Example 13: Find all the solutions of \( \csc (9x) = -2 \) \([0, 2\pi]\).

\[
\csc \theta = -2 \quad \theta \in [0, \frac{\pi}{2})
\]

\[
\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}
\]

\[
2x = \theta
\]

\[
x = \frac{\theta}{2} = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}
\]
Example 14: Find all the solutions of \(-\tan(7\pi x) = 1\) in the interval \(\left(-\frac{1}{14}, \frac{1}{14}\right)\).

\[-\tan \theta = 1 \quad \iff \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(-\frac{7\pi}{14}, \frac{7\pi}{14}\right)\]

\[\theta = -\frac{\pi}{4}\]

\[-7\pi x = \theta\]

\[x = \frac{\theta}{-7\pi} = \frac{-\frac{\pi}{4}}{-7\pi} = \frac{1}{28}\]

Example 15: Find all the solutions of \(\sin\left(\frac{x}{2} - \frac{\pi}{3}\right) = \frac{1}{2}\) in the interval \([0, 2\pi)\).

\[\sin \theta = \frac{1}{2} \quad \iff \quad \theta \in \left\{ -\frac{\pi}{6}, \frac{\pi}{6} \right\}\]

\[\frac{x}{2} - \frac{\pi}{3} = -\frac{\pi}{6}\]

\[\frac{x}{2} = \frac{\pi}{6}\]

\[x = \frac{\pi}{3}\]

\[\frac{x}{2} - \frac{\pi}{3} = \frac{\pi}{6}\]

\[\frac{x}{2} = \frac{\pi}{2}\]

\[x = \pi\]

\[\text{Not in Domain}\]