The Circle

Definition: A circle is the set of all points that are equidistant from a fixed point. The fixed point is called the center and the distance from the center to any point on the circle is called the radius.

An equation of a circle whose center is at the origin will be $x^2 + y^2 = r^2$, where $r$ is the radius of the circle.

So $x^2 + y^2 = 25$ is an equation of a circle with center $(0, 0)$ and radius 5.

Here’s the graph of this circle:
Example 1: State the center and the radius of the circle and then graph it:  \( x^2 + y^2 - 16 = 0 \).

\[ x^2 + y^2 = 16 \]

Center: \((0, 0)\)

Radius: \( r^2 = 16 \)

\( r = 4 \)

The standard form of the equation of a circle is \((x-h)^2 + (y-k)^2 = r^2\), where the center of the circle is the point \((h,k)\) and the radius is \(r\). Notice that if the center of the circle is \((0, 0)\) you’ll get the equation we saw earlier.

Example 2: State the center and the radius of the circle and then graph it:

\((x-2)^2 + (y+3)^2 = 4\)

Center \((2, -3)\)

Radius: \( r^2 = 4 \)

\( r = 2 \)
Sometimes the equation will be given in the general form, and your first step will be to rewrite the equation in the standard form. You’ll need to complete the square to do this.

**Example 3:** Complete the square for each.

a. \( x^2 + 20x + (10)^2 \)

b. \( x^2 - 5x + \left(-\frac{5}{2}\right)^2 \)

c. \( 3x^2 + 12x + \frac{12}{3} \)

\( 3\left(x^2 + 4x + (2)^2\right) \)

**Example 4:** Write the equation in standard form, find the center and the radius and then graph the circle: \( x^2 + y^2 + 6x - 10y + 44 = 26 \)

\[
(x + 3)^2 + (y - 5)^2 = 100
\]

**Center:** \((-3, 5)\)

**Radius:** 4
Example 5: Write the equation in standard form, find the center and the radius and then graph the circle: \(5x^2 + 5y^2 - 20x + 10y = 20\)

\[
5x^2 - 20x + 5y^2 + 10y = 20
\]

\[
5(x^2 - 4x) + 5(y^2 + 2y) = 20
\]

\[
s(x^2 - 4x + 4) + 5(y^2 + 2y + 1) = 20 + 5(4)
\]

\[
s(x - 2)^2 + 5(y + 1)^2 = 45
\]

\[
7
\]

\[
(x - 2)^2 + (y + 1)^2 = 9
\]

- **Center:** (2, -1)
- **Radius:** 3

We can also write the equation of a circle, given appropriate information.

Example 6: Write the equation of a circle with center (2, 5) and radius 9.

\[
(x - h)^2 + (y - k)^2 = r^2
\]

\[
(x - 2)^2 + (y - 5)^2 = 9^2
\]

\[
(x - 2)^2 + (y - 5)^2 = 81
\]

Example 7: Write an equation of a circle with center (-1, 3) which passes through the point (4, -7).

\[
(x - h)^2 + (y - k)^2 = r^2
\]

\[
(x + 1)^2 + (y - 3)^2 = r^2
\]

\[
(4 + 1)^2 + (-7 - 3)^2 = r^2
\]

\[
(5)^2 + (-10)^2 = r^2
\]

\[
25 + 100 = r^2
\]

\[
r^2 = 125
\]

\[
(x + 1)^2 + (y - 3)^2 = 125
\]
Example 8: Write an equation of a circle if the endpoints of the diameter of the circle are (6, -3) and (-4, 7).

Find Midpoint:

\[ M = \left( \frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right) \]

\[ = \left( \frac{6 + (-4)}{2}, \frac{-3 + 7}{2} \right) \]

\[ = \left( 1, 2 \right) \]

Center

\[ (x - 1)^2 + (y - 2)^2 = r^2 \]

\[ (x - 1)^2 + (y - 2)^2 = r^2 \]

\[ (6 - 1)^2 + (-3 - 2)^2 = r^2 \]

\[ 5^2 + (-5)^2 = r^2 \]

\[ 50 = r^2 \]

\[ (x - 1)^2 + (y - 2)^2 = 50 \]

Sometimes, you’ll need to be able to manipulate an equation of a circle:

Example 9: Suppose \((x - 2)^2 + (y + 1)^2 = 9\). Solve the equation for \(x\). Then solve the equation for \(y\).

\[ (x - 2)^2 = 9 - (y + 1)^2 \]

\[ \sqrt{(x - 2)^2} = \sqrt{9 - (y + 1)^2} \]

\[ x - 2 = \pm \sqrt{9 - (y + 1)^2} \]

\[ x = 2 \pm \sqrt{9 - (y + 1)^2} \]

\[ \begin{align*}
2 + \sqrt{9 - (y + 1)^2} & \quad \text{Right Half} \\
2 - \sqrt{9 - (y + 1)^2} & \quad \text{Left Half}
\end{align*} \]

\[ \sqrt{(y + 1)^2} = \sqrt{9 - (x - 2)^2} \]

\[ y + 1 = \pm \sqrt{9 - (x - 2)^2} \]

\[ y = -1 \pm \sqrt{9 - (x - 2)^2} \]

\[ \begin{align*}
-1 + \sqrt{9 - (x - 2)^2} & \quad \text{Top} \\
-1 - \sqrt{9 - (x - 2)^2} & \quad \text{Bottom}
\end{align*} \]
Math 1330  Section 8.2
Section 8.2: Ellipses

An **ellipse** is the set of all points, the sum of whose distances from two fixed points is constant. Each fixed point is called a **focus** (plural = **foci**).

Basic Ellipses centered at the origin

**Basic “vertical” ellipse:**

Equation: \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \ a > b \)

- **Foci:** \((0, \pm c)\), where \( c^2 = a^2 - b^2 \)
- **Vertices:** \((0, \pm a)\)
- **Eccentricity:** \( e = \frac{c}{a} \)

The **eccentricity** provides a numerical measure of how much the ellipse deviates from being a **circle**. The \( e \) is a number between 0 and 1.

**Basic “horizontal” ellipse:**

Equation: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a > b \)

- **Foci:** \((\pm c, 0)\), where \( c^2 = a^2 - b^2 \)
- **Vertices:** \((\pm a, 0)\)
- **Eccentricity:** \( e = \frac{c}{a} \)

For ellipses, the line segment joining the vertices is called the **Major Axis** (length 2a) and the line segment through the center and perpendicular to the major axis with endpoints on the ellipse is called the **Minor Axis** (length 2b).
Graphing Ellipses:

To graph an ellipse with center at the origin:

- Rearrange into the form \( \frac{x^2}{\text{number}} + \frac{y^2}{\text{number}} = 1 \).
- Decide if it’s a “horizontal” or “vertical” ellipse.
  - if the bigger number is under \( x^2 \), it’s horizontal (longer in \( x \)-direction).
  - if the bigger number is under \( y^2 \), it’s vertical (longer in \( y \)-direction).
- Use the square root of the number under \( x^2 \) to determine how far to measure in \( x \)-direction.
- Use the square root of the number under \( y^2 \) to determine how far to measure in \( y \)-direction.
- Draw the ellipse with these measurements. Be sure it is smooth with no sharp corners.
- \( c^2 = a^2 - b^2 \) where \( a^2 \) and \( b^2 \) are the denominators. (Subtract the small denominator from the large denominator to get \( c^2 \).)
- The foci are located \( c \) units from the center on the long axis.

To graph an ellipse with center not at the origin:

- Rearrange (complete the square if necessary) to look like \( \frac{(x-h)^2}{\text{number}} + \frac{(y-k)^2}{\text{number}} = 1 \).
- Start at the center \( (h, k) \) and then graph it as before.

Example 1: Graph \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \).

Center: \((0, 0)\)  
Equation: \(\text{Horizontal}\)  
Vertices: \((-4, 0), (4, 0)\)  
\(16 - 9 = 7 \pm \sqrt{7}\)  
Foci: \((-\sqrt{7}, 0), (\sqrt{7}, 0)\)  
Major Axis: 8 units  
Minor Axis: 6 units  
\(x\)-direction: 4 units  
\(y\)-direction: 3 units
Example 2: Graph $4x^2 - 8x + 9y^2 - 54y = -49$

Center: $(1, 3)$

Equation: Horizontal

Vertices: $(1 + \frac{\sqrt{3}}{3}, 3) \quad (1 - \frac{\sqrt{3}}{3}, 3)$

Foci: $(1 + \sqrt{5}, 3) \quad (1 - \sqrt{5}, 3)$

Major Axis: $2a = 6\sqrt{3}$

Minor Axis: $2b = 4\sqrt{3}$

$c^2 = 9 - 4 = 5 \quad c = \sqrt{5}$
Example 3: Use the given features of each of the following ellipses to write an equation for the ellipse in standard.

Foci (2, 5) and (2, -5) \[ a = 9 \]

Midpoint: (2, 0) \[ C = 5 \text{ units} \]

\[ c^2 = b^2 - a^2 \]

\[ (5)^2 = 9^2 - b^2 \]

\[ 25 = 81 - b^2 \]

\[ b^2 = 81 - 25 = 56 \]

Equation: \[ \frac{(x-2)^2}{56} + \frac{y^2}{51} = 1 \]
Section 8.3: Hyperbolas

A hyperbola is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a focus (plural = foci).

The focal axis is the line passing through the foci.

(0,−c)

Basic “Vertical” Hyperbola:

Equation: \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \)

Asymptotes: \( y = \pm \frac{a}{b} x \)

Foci: (0,±c), where \( c^2 = a^2 + b^2 \)

Vertices: (0,±a)

Basic “Horizontal” Hyperbola:

Equation: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \)

Asymptotes: \( y = \pm \frac{b}{a} x \)

Foci: (±c,0), where \( c^2 = a^2 + b^2 \)

Vertices: (±a,0)

Note: The transverse axis is the line segment joining the two vertices. The conjugate axis is the line segment perpendicular to the transverse axis, passing through the center and extending a distance b on either side of the center. (These terms will make more sense after we do the graphing examples.)
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Graphing Hyperbolas:

To graph a hyperbola with center at the origin:

- Rearrange into the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) or \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \).

- Decide if it’s a “horizontal” or “vertical” hyperbola.
  - if \( x^2 \) comes first, it’s horizontal (vertices are on x-axis).
  - If \( y^2 \) comes first, it’s vertical (vertices are on y-axis).

- Use the square root of the number under \( x^2 \) to determine how far to measure in x-direction.
- Use the square root of the number under \( y^2 \) to determine how far to measure in y-direction.

- Draw a box with these measurements.

- Draw diagonals through the box. These are the asymptotes. Use the dimensions of the box to determine the slope and write the equations of the asymptotes.

- Put the vertices at the edge of the box on the correct axis. Then draw a hyperbola, making sure it approaches the asymptotes smoothly.

- \( c^2 = a^2 + b^2 \) where \( a^2 \) and \( b^2 \) are the denominators.

- The foci are located \( c \) units from the center, on the same axis as the vertices.

To graph a hyperbola with center not at the origin:

- Rearrange (complete the square if necessary) to look like \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \) or \( \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \).

- Start at the center \((h, k)\) and then graph it as before.

- To write down the equations of the asymptotes, start with the equations of the asymptotes for the similar hyperbola with center at the origin. Then replace \( x \) with \( x - h \) and replace \( y \) with \( y - k \).
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Example 1: Graph \( \frac{x^2}{36} - \frac{y^2}{4} = 1 \).

Center: \((0,0)\)

Equation: Horizontal

Vertices: \((-6,0), (6,0)\)

Foci: \((-\sqrt{10},0), (\sqrt{10},0)\)

Asymptotes: \(y = \pm \frac{2}{3} x\)

Example 2: Graph \( \frac{y^2}{4} - \frac{x^2}{9} = 1 \).

Center: ______________

Equation: ______________

Vertices: ______________

Foci: ______________

Asymptotes: ______________
Example 3: Write the hyperbola in standard form. Then find the vertex, focus and asymptotes:
\[ 9y^2 - 54y - 25x^2 - 200x - 544 = 0 \]

Example 4: Identify the type of conic section represented by each of the following equations.

a. \[ 16x^2 + 64x = 9y^2 + 54y - 1 \]

b. \[ x^2 - 4x + y^2 - 4y = -4 \]

c. \[ x^2 + 6x - 2y + 33 = 0 \]
Example 5: Write an equation of the hyperbola with center (2, -5), one vertex at (2, -6) and one focus at (2, -8).
Popper 1: Identify the type of conic section.

\[ 4x^2 - 9y^2 = 36 \]

a. Circle  
b. Ellipse  
c. Parabola  
d. Hyperbola

Popper 2: Rewrite the equation by completing the square.

\[ x^2 - 20x + 6 = 0 \]

a. \((x + 10)^2 = -6\)  
b. \((x - 10)^2 = 94\)  
c. \((x - 10)^2 = -6\)  
d. \((x - 20)^2 = 394\)