$x = r\cos\theta$

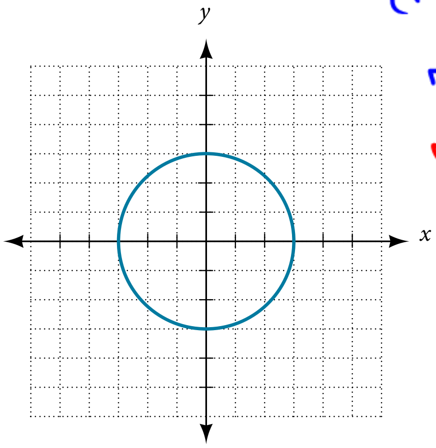
$y = r\sin\theta$

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be benefic al to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems.

Example 6 Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

Solution $x = r\cos\theta \quad y = r\sin\theta$



$(r\cos\theta)^2 + (r\sin\theta)^2 = 9$

$r^2\cos^2\theta + r^2\sin^2\theta = 9$

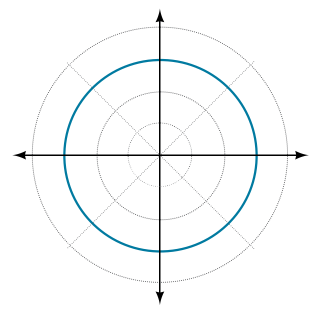
$r^2[\cos^2\theta + \sin^2\theta] = 9$

$r^2(1) = 9$

$r^2 = 9$

$r = \pm 3$

$r = 3, \quad r = -3$



Example 7 Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

Solution

$(r\cos\theta)^2 + (r\sin\theta)^2 = 6r\sin\theta$

$r^2\cos^2\theta + r^2\sin\theta = 6r\sin\theta$

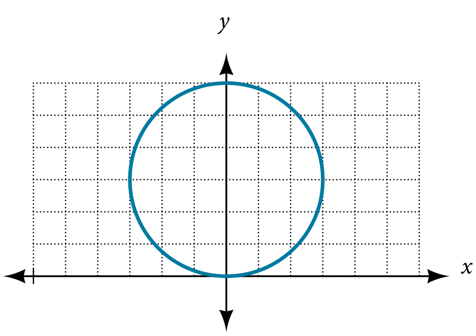
$r^2(\cos^2\theta + \sin^2\theta) = 6r\sin\theta$

$\frac{r^2}{r} = \frac{6r\sin\theta}{r} \rightarrow r^2 - 6r\sin\theta = 0$

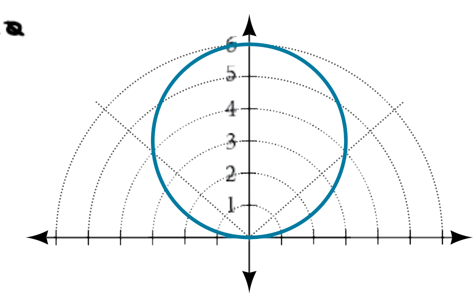
$r(r - 6\sin\theta) = 0$

$r = 0 \; ; \; r - 6\sin\theta = 0$

$r = 6\sin\theta$



$r = 6\sin\theta$



Example 8 a) Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

b) Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form. Circle

$x = r\cos\theta \qquad y = r\sin\theta$

Solution

$y = 3x + 2$

$r\sin\theta = 3r\cos\theta + 2$

Solve for r

$r\sin\theta - 3r\cos\theta = 2$

$r(\sin\theta - 3\cos\theta) = 2$

$$r = \frac{2}{\sin\theta - 3\cos\theta}$$

b.

$(r\sin\theta)^2 = 3 - (r\cos\theta)^2$

$r^2\sin^2\theta = 3 - r^2\cos^2\theta$

$r^2\sin^2\theta + r^2\cos^2\theta = 3$

$r^2(\sin^2\theta + \cos^2\theta) = 3$

$r^2 = 3$

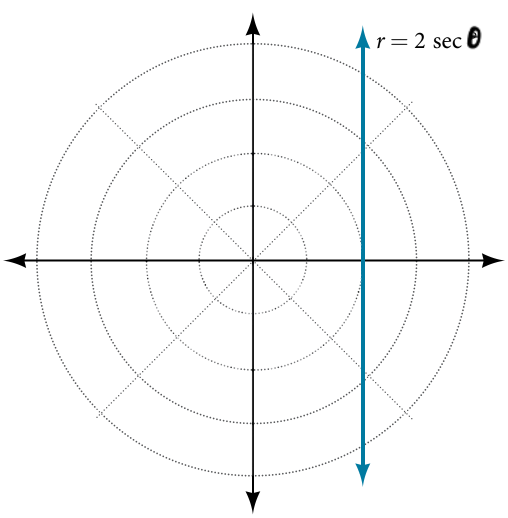
$r = \sqrt{3} \qquad r = -\sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 9 Convert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution

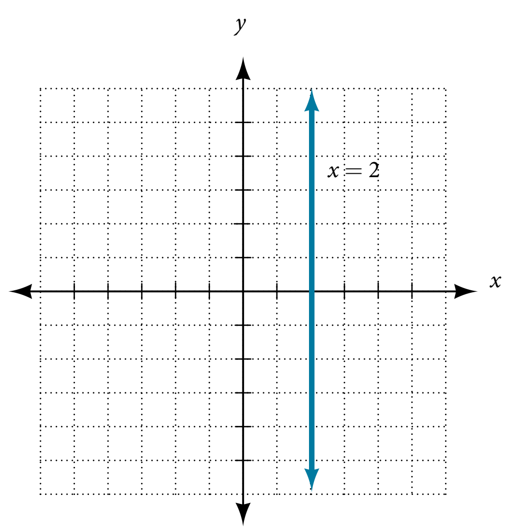


$r = 2\sec\theta$

$r = \dfrac{2}{\cos\theta}$

$r \cdot \cos\theta = 2$

$$x = 2$$



Exercise: Convert the polar equation $r = 2\csc\theta$ to a rectangular equation, and draw its corresponding graph.

$r = 2\csc\theta$

$r = \dfrac{2}{\sin\theta}$

$r \cdot \sin\theta = 2$

$$y = 2$$

$x^2+y^2=r^2$ $\quad r = +\sqrt{y^2+x^2}$ or $-\sqrt{y^2+x^2}$

Example 10 Rewrite the polar equation $r = \dfrac{3}{1 - 2\cos\theta}$ as a Cartesian equation.

Solution $r(1-2\cos\theta)=3$

$r - 2r\cos\theta = 3$

$r - 2x = 3$

$(r)^2 = (3+2x)^2$

$r^2 = (3+2x)^2$

$$x^2+y^2 = (3+2x)^2$$

$x^2+y^2 = 9+12x+4x^2$

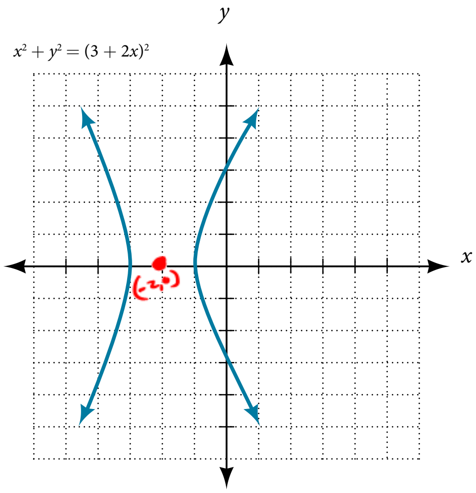
$y^2 - 3x^2 - 12x = 9$

$y^2 - 3(x^2+4x) = 9$ (Hyperbola)

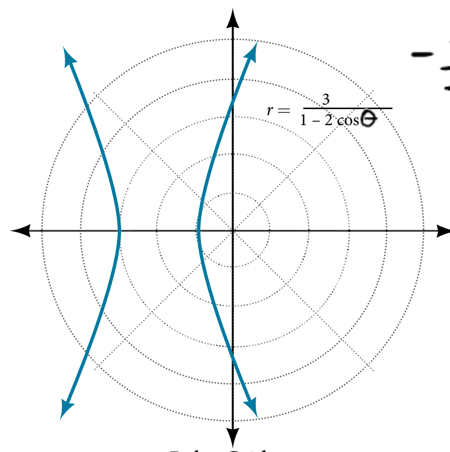
$y^2 - 3(x+2)^2 = 9 - 12 \rightarrow \dfrac{y^2 - 3(x+2)^2}{-3} = \dfrac{-3}{-3}$

$-\dfrac{y^2}{3} + (x+2)^2 = 1$

$(x+2)^2 - \dfrac{y^2}{3} = 1$



Coordinate Grid



Polar Grid

Exercise: Rewrite the polar equation $r = 2\sin\theta$ in Cartesian form.

Example 11 Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

$x = r\cos\theta \qquad y = r\sin\theta$

$\dfrac{x}{r} = \cos\theta \qquad \dfrac{y}{r} = \sin\theta$

Solution $r = 2\sin\theta\cos\theta$

$r = 2\left(\dfrac{x}{r}\right)\left(\dfrac{y}{r}\right)$

$r = \dfrac{2xy}{r^2}$

$r = \dfrac{2xy}{x^2+y^2}$

$(r)^2 = \left(\dfrac{2xy}{x^2+y^2}\right)^2$

$r^2 = \dfrac{4x^2y^2}{(x^2+y^2)^2}$

$x^2+y^2 = \dfrac{4x^2y^2}{(x^2+y^2)^2}$

$(x^2+y^2)^3 = 4x^2y^2$

$(x^2+y^2)^3 - 4x^2y^2 = 0$

MORE EXERCISES

Popper 24 1–5 A–E

For the following exercises, convert the given Cartesian equation to a polar equation.

$x = 3$

$x^2 + y^2 = 4y$

$x^2 + y^2 = 9$

$y = 4$

$x^2 + y^2 = 3x$

$x^2 = 9y$

$y = 4x^2$

$x^2 - y^2 = x$

$y^2 = 9x$

$y = 2x^4$

$x^2 - y^2 = 3y$

$9xy = 1$

Wednesday 10pm

Subject: 1330 Final Class

46–50
E – A

BYE

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

$r = 3\sin\theta$

$\dfrac{\bar{u} \cdot \bar{v}}{|u|\,|v|} = \cos\theta$

$r = 2\sec\theta$

$r = 4$

$r = 4\cos\theta$

$r = 3\csc\theta$

$r = -10\sin\theta$

$r = \dfrac{4}{\sin\theta + 7\cos\theta}$

$r = \sqrt{r\cos\theta + 2}$

$r = \dfrac{1}{4\cos\theta - 3\sin\theta}$

$r = \dfrac{6}{\cos\theta + 3\sin\theta}$

$r^2 = 4\sec\theta\csc\theta$

$r = \dfrac{3}{\cos\theta - 5\sin\theta}$

$\theta = -\dfrac{2\pi}{3}$

$\theta = \dfrac{\pi}{4}$

LEARNING OBJECTIVES

In this section, you will:

- Identify a conic in polar form.
- Graph the polar equations of conics.
- Define conics in terms of a focus and a directrix.

9.3 CONIC SECTIONS IN POLAR COORDINATES

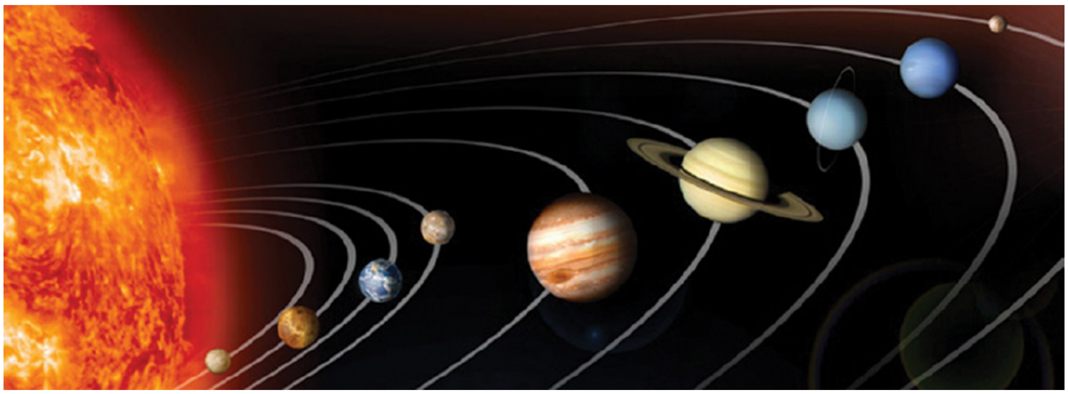


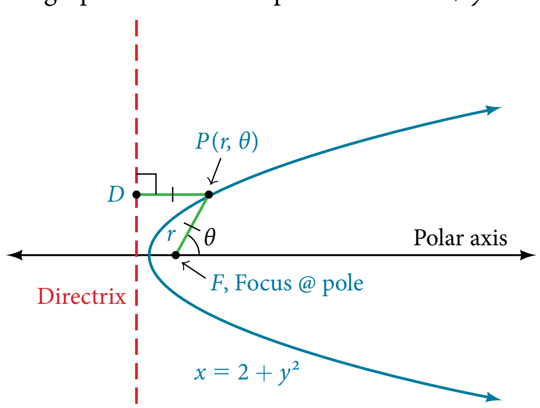
Figure 1 Planets orbiting the sun follow elliptical paths. (credit: NASA Blueshift, Flickr)

Most of us are familiar with orbital motion, such as the motion of a planet around the sun or an electron around an atomic nucleus. Within the planetary system, orbits of planets, asteroids, and comets around a larger celestial body are often elliptical. Comets, however, may take on a parabolic or hyperbolic orbit instead. And, in reality, the characteristics of the planets' orbits may vary over time. Each orbit is tied to the location of the celestial body being orbited and the distance and direction of the planet or other object from that body. As a result, we tend to use polar coordinates to represent these orbits.

In an elliptical orbit, the periapsis is the point at which the two objects are closest, and the apoapsis is the point at which they are farthest apart. Generally, the velocity of the orbiting body tends to increase as it approaches the periapsis and decrease as it approaches the apoapsis. Some objects reach an escape velocity, which results in an infinite orbit. These bodies exhibit either a parabolic or a hyperbolic orbit about a body; the orbiting body breaks free of the celestial body's gravitational pull and fires off into space. Each of these orbits can be modeled by a conic section in the polar coordinate system.

Identifying a Conic in Polar Form

Any conic may be determined by three characteristics: a single focus, a fixed line called the directrix, and the ratio of the distances of each to a point on the graph. Consider the parabola $x = 2 + y^2$ shown in **Figure below**.



In **The Parabola**, we learned how a parabola is defined by the focus (a fixed point) and the directrix (a fixed line). In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus $P(r, \theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.

If F is a fixed point, the focus, and D is a fixed line, the directrix, then we can let e be a fixed positive number, called the **eccentricity**, which we can define as the ratio of the distances from a point on the graph to the focus and the point on the graph to the directrix. Then the set of all points P such that $e = \frac{PF}{PD}$ is a conic. In other words, we can define a conic as the set of all points P with the property that the ratio of the distance from P to F to the distance from P to D is equal to the constant e.

For a conic with eccentricity e,

- if $0 \leq e < 1$, the conic is an ellipse
- if $e = 1$, the conic is a parabola
- if $e > 1$, the conic is an hyperbola

With this definition, we may now define a conic in terms of the directrix, $x = \pm\, p$, the eccentricity e, and the angle θ. Thus, each conic may be written as a **polar equation**, an equation written in terms of r and θ.

The Polar Equation for a Conic

For a conic with a focus at the origin, if the directrix is $x = \pm\, p$, where p is a positive real number, and the **eccentricity** is a positive real number e, the conic has a **polar equation**

$$r = \frac{e.p}{1 \pm e \cos\theta}$$

For a conic with a focus at the origin, if the directrix is $y = \pm\, p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{e.p}{1 \pm e \sin\theta}$$

Rewriting the eccentricity ***e =m/n as a fraction*** where m and n are two positive integers, we have the polar equation for each conic to written as:

$$r = \frac{m.p}{n \pm m \cos\theta} \qquad \text{or} \qquad r = \frac{m.p}{n \pm m \sin\theta}$$

- if $m <$ n, the conic is an ellipse
- if $m =$ n, the conic is a parabola
- if $m >$ n, the conic is an hyperbola

Example 1 Identifying a Conic Given the Polar Form

For each of the following equations, identify the conic with focus at the origin, the directrix, and the eccentricity.

a. $r = \dfrac{6}{3 + 2\sin\theta}$ **b.** $r = \dfrac{12}{4 + 5\cos\theta}$ **c.** $r = \dfrac{7}{2 - 2\sin\theta}$ **d.** $r(2 - \cos\theta) = 1$

Solution

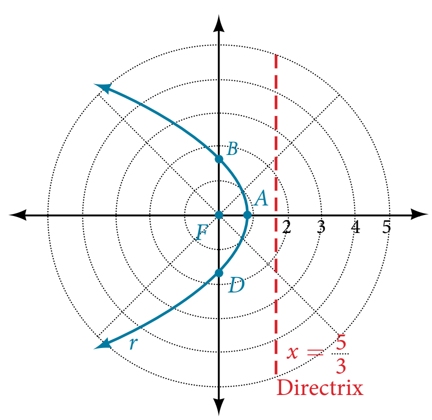
Graphing the Polar Equations of Conics

When graphing in Cartesian coordinates, each conic section has a unique equation. This is not the case when graphing in polar coordinates. We must use the eccentricity of a conic section to determine which type of curve to graph, and then determine its specific characteristics. The first step is to rewrite the conic in standard form as we have done in the previous example. In other words, we need to rewrite the equation so that the denominator begins with 1. This enables us to determine e and, therefore, the shape of the curve. The next step is to substitute values for θ and solve for r to plot a few key points. Setting θ equal to 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ provides the vertices so we can create a rough sketch of the graph.

Example 2 **Graphing a Parabola in Polar Form:** Graph $r = \dfrac{5}{3 + 3\cos\theta}$.

Solution

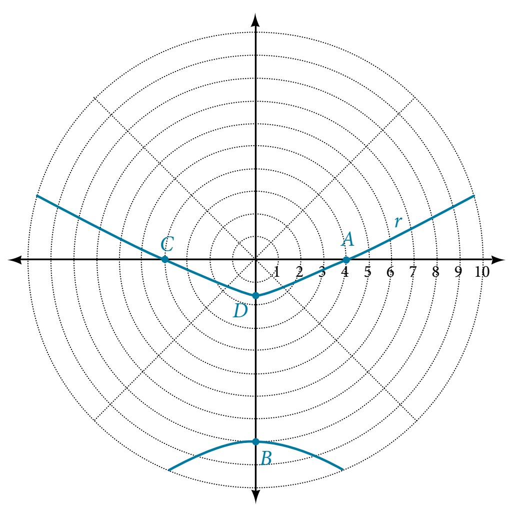
	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \dfrac{5}{3 + 3\cos\theta}$	$\frac{5}{6} \approx 0.83$	$\frac{5}{3} \approx 1.67$	undefined	$\frac{5}{3} \approx 1.67$



Example 3 **Graphing a Hyperbola in Polar Form** Graph $r = \dfrac{8}{2 - 3\sin\theta}$.

Solution

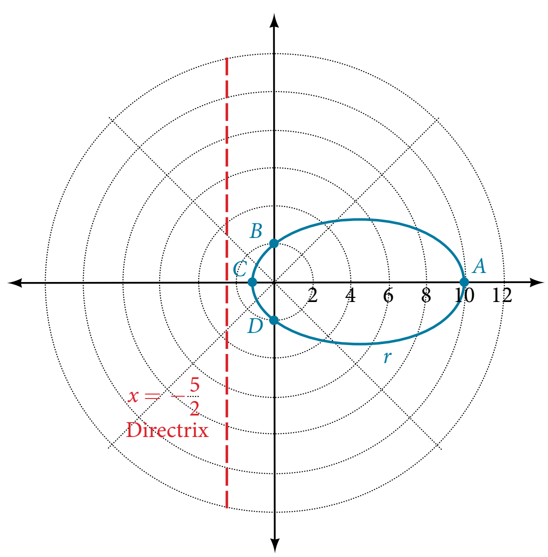
	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{8}{2 - 3\sin\theta}$	4	-8	4	$\frac{8}{5} = 1.6$



Example 4 **Graphing an Ellipse in Polar Form** Graph $r = \dfrac{10}{5 - 4\cos\theta}$.

Solution

	A	B	C	D
θ	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$
$r = \dfrac{10}{5 - 4\cos\theta}$	10	2	$\dfrac{10}{9} \approx 1.1$	2



Graph $r = \dfrac{2}{4 - \cos\theta}$.

Defining Conics in Terms of a Focus and a Directrix

So far we have been using polar equations of conics to describe and graph the curve. Now we will work in reverse; we will use information about the origin, eccentricity, and directrix to determine the polar equation.

Example 5 **Finding the Polar Form of a Vertical Conic Given a Focus at the Origin and the Eccentricity and Directrix**

Find the polar form of the conic given a focus at the origin, $e = 3$ and directrix $y = -2$.

Solution

Example 6 **Finding the Polar Form of a Horizontal Conic Given a Focus at the Origin and the Eccentricity and Directrix**

Find the polar form of a conic given a focus at the origin, $e = \frac{3}{5}$, and directrix $x = 4$.

Solution

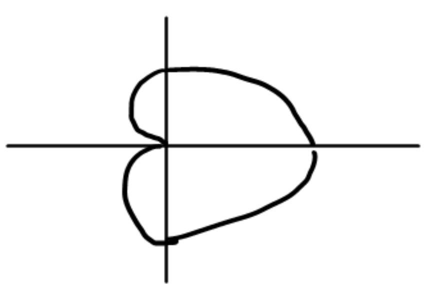
Example 7 Converting a Conic in Polar Form to Rectangular Form

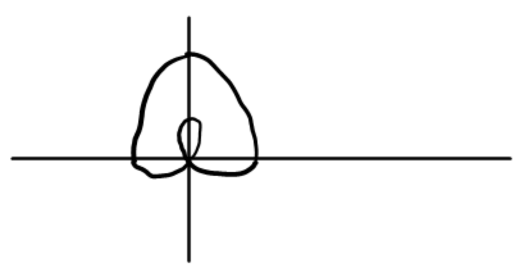
Convert the conic $r = \dfrac{1}{5 - 5\sin\theta}$ to rectangular form.

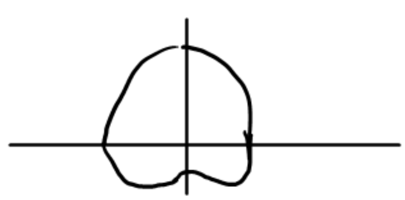
Solution

Find the polar form of the conic given a focus at the origin, $e = 1$, and directrix $x = -1$.

Convert the conic $r = \dfrac{2}{1 + 2\cos\theta}$ to rectangular form.







$r = a \pm b\sin\theta$

$a \pm b\cos\theta$

Petals $r = s$

MORE EXERCISES

For the following exercises, identify the conic with a focus at the origin, and then give the directrix and eccentricity.

$r = \dfrac{6}{1 - 2\cos\theta}$

$r = \dfrac{3}{4 - 4\sin\theta}$

$r = \dfrac{8}{4 - 3\cos\theta}$

$r = \dfrac{5}{1 + 2\sin\theta}$

$r = \dfrac{16}{4 + 3\cos\theta}$

$r = \dfrac{3}{10 + 10\cos\theta}$

$r = \dfrac{2}{1 - \cos\theta}$

$r = \dfrac{4}{7 + 2\cos\theta}$

$r(1 - \cos\theta) = 3$

$r(3 + 5\sin\theta) = 11$

$r(4 - 5\sin\theta) = 1$

$r(7 + 8\cos\theta) = 7$

For the following exercises, convert the polar equation of a conic section to a rectangular equation.

$r = \dfrac{4}{1 + 3\sin\theta}$

$r = \dfrac{2}{5 - 3\sin\theta}$

$r = \dfrac{8}{3 - 2\cos\theta}$

$r = \dfrac{3}{2 + 5\cos\theta}$

$r = \dfrac{4}{2 + 2\sin\theta}$

$r = \dfrac{3}{8 - 8\cos\theta}$

$r = \dfrac{2}{6 + 7\cos\theta}$

$r = \dfrac{5}{5 - 11\sin\theta}$

$r(5 + 2\cos\theta) = 6$

$r(2 - \cos\theta) = 1$

$r(2.5 - 2.5\sin\theta) = 5$

$r = \dfrac{6\sec\theta}{-2 + 3\sec\theta}$

$r = \dfrac{6\csc\theta}{3 + 2\csc\theta}$

For the following exercises, graph the given conic section. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse, label the vertices and foci. If it is a hyperbola, label the vertices and foci.

$r = \dfrac{5}{2 + \cos\theta}$

$r = \dfrac{2}{3 + 3\sin\theta}$

$r = \dfrac{10}{5 - 4\sin\theta}$

$r = \dfrac{3}{1 + 2\cos\theta}$

$r = \dfrac{8}{4 - 5\cos\theta}$

$r = \dfrac{3}{4 - 4\cos\theta}$

$r = \dfrac{2}{1 - \sin\theta}$

$r = \dfrac{6}{3 + 2\sin\theta}$

$r(1 + \cos\theta) = 5$

$r(3 - 4\sin\theta) = 9$

$r(3 - 2\sin\theta) = 6$

$r(6 - 4\cos\theta) = 5$

For the following exercises, find the polar equation of the conic with focus at the origin and the given eccentricity and directrix.

Directrix: $x = 4;\ e = \dfrac{1}{5}$

Directrix: $x = -4;\ e = 5$

Directrix: $y = 2;\ e = 2$

Directrix: $y = -2;\ e = \dfrac{1}{2}$

Directrix: $x = 1;\ e = 1$

Directrix: $x = -1;\ e = 1$

Directrix: $x = -\dfrac{1}{4};\ e = \dfrac{7}{2}$

Directrix: $y = \dfrac{2}{5};\ e = \dfrac{7}{2}$

Directrix: $y = 4;\ e = \dfrac{3}{2}$

Directrix: $x = -2;\ e = \dfrac{8}{3}$

Directrix: $x = -5;\ e = \dfrac{3}{4}$

Directrix: $y = 2;\ e = 2.5$

Directrix: $x = -3;\ e = \dfrac{1}{3}$