An Introduction to Functions

Most of this course will deal with **functions**. Suppose we start with two sets, $A$ and $B$. A function is a rule which assigns one and only one element of set $B$ to each element in set $A$. Set $A$ is called the **domain** of the function and set $B$ is called the **range**.

We’ll start by looking at **mappings**.

A mapping relates each element in the oval on the left with an element in the oval on the right. You need to be able to state whether or not the mapping defines a function, and, if it defines a function, you should be able to state the domain and range of the function.

**Example 1:** State whether or not the mapping represents a function. If it does, identify its domain and range.

Functions are usually written using function notation. If an equation is solved for $y$, such as $y = mx + b$, we would write this using function notation as $f(x) = mx + b$, read “$f$ of $x$,” denoting the value of the function at $x$. We can also use other lower case letters to denote a function, such as $g, h, j, k$, etc.

Often, you will be asked to state the domain of a stated function. Domain is a subset of the set of real numbers.
Math 1330 Algebra Review

Reminder: Interval Notation

(-3, 5) all \(x\) such that \(-3 < x < 5\)

[-3, 5] all \(x\) such that \(-3 \leq x \leq 5\)

[-3, 5) all \(x\) such that \(-3 \leq x < 5\)

[-3, \(\infty\)] all \(x\) such that \(x \geq -3\)

(-\(\infty\), 5) all \(x\) such that \(x < 5\)

(-\(\infty\), \(\infty\)) all real numbers

Next, we’ll state the domain of several types of functions.

The domain of any \textbf{polynomial function} is \((-\infty, \infty)\), or all real numbers.

The domain of any \textbf{rational function}, where both the numerator and the denominator are polynomials, is all real numbers except the values of \(x\) for which the denominator equals 0.

\textbf{Example 2:} State the domain of the function. Write your answer using interval notation.

\[ f(x) = \frac{x - 3}{x + 7} \]

\[
\begin{align*}
x + 7 &= 0 \\
x &= -7
\end{align*}
\]

\textit{Exclude this}

\(x \neq -7\)

\[ D: \ (-\infty, -7) \cup (-7, \infty) \]

The domain of any radical function with \textbf{even index} is the set of real numbers for which the radicand is \textit{greater than or equal to 0}. The domain of any radical function with \textbf{odd index} is \((-\infty, \infty)\).

\textbf{Example 3:} State the domain of the function. Write your answer using interval notation.

\[ h(x) = \sqrt{x + 4} \]

\[
\begin{align*}
x + 4 &\geq 0 \\
x &\geq -4
\end{align*}
\]

\[ D: [-4, \infty) \]
Example 4: State the domain of the function. Write your answer using interval notation.

\[ g(x) = \frac{\sqrt{x-1}}{x-5} \]

\[ x - 5 \neq 0 \quad x - 1 \geq 0 \]
\[ x \neq 5 \quad x \geq 1 \]

\[ D: [1, 5) \cup (5, \infty) \]

Increasing/Decreasing

A function is **increasing** on an interval \((a, b)\) if \(f(x_1) < f(x_2)\) for each \(x_1, x_2 \in (a, b)\). You can think: a function is increasing if the \(y\) values are getting bigger as we look from left to right.

A function is **decreasing** on an interval \((a, b)\) if \(f(x_1) > f(x_2)\) for each \(x_1, x_2 \in (a, b)\). You can think: a function is decreasing if the \(y\) values are getting smaller as we look from left to right.

A **turning point** is a point where the graph of a function changes from increasing to decreasing or where it changes from decreasing to increasing. 

Example 5: Identify any turning points, and intervals of increasing and decreasing on the graph of this function.

\[ \text{Inc: } (-\infty, -2) \cup (3, \infty) \]
\[ \text{Dec: } (-2, 3) \]
Math 1330  Algebra Review

**Even and Odd Functions**

A function $f$ is **even** if $f(-x) = f(x)$ for all $x$ in the domain of $f$. Since an even function is symmetric with respect to the $y$-axis, the points $(-x, y)$ and $(x, y)$ are on the same graph.

\[ f(x) = x^2 \]

A function is **odd** if $f(-x) = -f(x)$ for all $x$ in the domain of the function. Odd functions have symmetry with respect to the origin. Here is an example of an odd function.

**Example 6:** Determine if $f(x) = 5x^4 - 3x^2 + 2$, is odd, even or neither.

\[
\begin{align*}
\ f(-x) & = f(x) \\
\ f(-x) & = 5(-x)^4 - 3(-x)^2 + 2 \\
\ & = 5x^4 - 3x^2 + 2 \quad \text{(Original Function)}
\end{align*}
\]

Even Function
Example 7: Determine if \( f(x) = x^3 - x + 1 \), is odd, even or neither.

Test for odd

\[
\frac{f(-x)}{-f(x)} = \frac{(-x)^3 - (-x) + 1}{-\left( x^3 - x + 1 \right)}
\]

\[
\frac{f(-x)}{-f(x)} = \frac{-x^3 + x + 1}{x^3 - x + 1} = -1
\]

\[\text{Neither}\]

Operations on Functions

Let \( f \) and \( g \) be two functions. The following are functions whose domains are the set of real numbers common to the domain of \( f \) and \( g \), defined as follows:

**Sum**: \( (f + g)(x) = f(x) + g(x) \)

**Difference**: \( (f - g)(x) = f(x) - g(x) \)

**Product**: \( (fg)(x) = f(x)g(x) \)

**Quotient**: \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0 \)

**Composition**: \( (f \circ g)(x) = f(g(x)) \)

The domain of the composition \( f \circ g \) is the set of all \( x \) such that

1. \( x \) is in the domain of \( g \) (the "inside" function)
2. \( g(x) \) is in the domain of \( f \) (the "outside" function)

**Example 8**: Suppose \( f(x) = x^2 - 4x + 3 \) and \( g(x) = 3x - 1 \).

a. \( f(g(2)) \)

\[
f \left( g(2) \right) = f \left( 3 \cdot 2 - 1 \right) = f(5) = 5^2 - 4(5) + 3
\]

\[
= 25 - 20 + 3
\]

\[
= 8
\]
Math 1330 Algebra Review

b. \( g(f(x)) = g(x^2 - 4x + 3) = 3(x^2 - 4x + 3) - 1 \)

\[
= 3x^2 - 12x + 9 - 1
\]

\[
= 3x^2 - 12x + 8
\]

Example 9: The graphs of two functions, \( f \) and \( g \), are shown below. Find the following

\[
(a \circ g)(2) = f(g(z)) = f(-1) \]

\[
g(z) = -1 \quad = 2
\]

b. \( g(f(-3)) = (g \circ f)(-3) \quad \therefore \quad g(0) = -5 \)

\[
f(-3) = 0
\]
Next, you will need to be able to form a difference quotient. You will need to compute the following in a few steps.

\[
\frac{f(x + h) - f(x)}{h}, \quad h \neq 0
\]

**Example 10:** Find the difference quotient: \( f(x) = x^2 - 2x - 9 \)

\[
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 2(x+h) - 9 - (x^2 - 2x - 9)}{h} \]

\[
= \frac{2xh + h^2 - 2h}{h} = \frac{h(2x + h - 2)}{h} = 2x + h - 2
\]

**Example 11:** Find the difference quotient:

\[
\frac{f(x+h) - f(x)}{h} = \frac{\frac{2(x+h)+3}{(x+h-4)} - \frac{2x+3}{(x-4)}}{h} \cdot \frac{(x+h-4)(x-4)}{(x+h-4)(x-4)} = \frac{(x-4)(2x+2h+3) - (x+h-4)(2x+3)}{h \cdot (x+h-4)(x-4)}
\]

\[
= \frac{2x^2 + 2xh + 3x - 9h - 12 - [2x^2 + 3x + 2xh + 3h - 8x - 12]}{h \cdot (x+h-4)(x-4)} = \frac{-11h}{h \cdot (x+h-4)(x-4)} = \frac{-11}{(x+h-4)(x-4)}
\]
Inverse Function

The inverse function of a one-to-one function is a function $f^{-1}(x)$ such that $(f \circ f^{-1}) = (f^{-1} \circ f) = x$.

To determine if two functions are inverses of one another, you need to compose the functions in both orders. Your result should be $x$ in both cases. That is, given two functions $f$ and $g$, the functions are inverses of one another if and one if $f(g(x)) = g(f(x)) = x$.

Note that if $A$ is the domain of $f$ and $B$ is the range of $f$, then the domain of $f^{-1}$ is $B$ and the range of $f^{-1}$ is $A$.

You need to be able to find the inverse of a function. Follow this procedure to find an inverse function:

1. Rewrite the function as $y = f(x)$.
2. Interchange $x$ and $y$.
3. Solve the equation you wrote in step 2 for $y$.
4. Rewrite the inverse using inverse notation, $f^{-1}(x)$

Example 12: Find the inverse of the following function.

$$f(x) = \frac{3 + x}{2 - x}$$

$$y = \frac{3 + x}{2 - x}$$

$$x = \frac{3 + y}{2 - y}$$

$$x \cdot (2 - y) = 3 + y$$

$$2x - xy = 3 + y$$

$$-y - xy = 3 - 2x$$

$$y(-1 - x) = 3 - 2x$$

$$y = \frac{3 - 2x}{-1 - x}$$

$$f^{-1}(x) = \frac{3 - 2x}{-1 - x}$$

$$\circ$$

$$\frac{3 - 2x}{-1(1 + x)} = - \frac{3 - 2x}{1 + x}$$

$$\circ$$

$$= \frac{2x - 3}{1 + x}$$
Math 1330   Algebra Review

Example 13: Find the inverse of the following function.

\[ f(x) = \frac{3}{x+4} \]

\[ y = \frac{3}{\sqrt{y+4}} \]
\[ x = \frac{3}{\sqrt{y+4}} \]
\[ (x)^3 = (\frac{3}{\sqrt{y+4}})^3 \]
\[ x^3 = y + 4 \]
\[ x^3 - 4 = y \]

\[ f^{-1}(x) = x^3 - 4 \]

Example 14: If \( f(-1) = 2, f^{-1}(-1) = 0 \) and \( f(2) = 5 \), find \( f(0) \) and \( f^{-1}(5) \).

\[ \begin{array}{c|c|c|c}
   x & y & f(x) & f^{-1}(x) \\
\hline
   -1 & 0 & 2 & 1 \\
   2 & -1 & 5 & -1 \\
   5 & 2 & - & 2 \\
\end{array} \]

Polynomial Function

A polynomial function is a function which can be written in the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \]

The numbers \( a_0, a_1, \ldots, a_n \) are called the coefficients of the polynomial function and \( a_n \neq 0 \).

\( x^n \) is the largest variable term of the polynomial function of \( x \) of degree \( n \). The number \( a_n \), is the coefficient of the variable to the highest power, is called the leading coefficient.

Note: The variable is only raised to positive integer powers—no negative or fractional exponents. However, the coefficients may be any real numbers, including fractions or irrational numbers like \( \pi \). The domain of any polynomial function is all real numbers.

Example 15: Given \( f(x) = -3x^6 - 5x^4 + 4x^3 - 2x^2 - 9x + 11 \). What is the degree and leading coefficient?

Degree: 6
LC: -3
Math 1330 Algebra Review

Facts about polynomials:
- They are smooth curves, with no jumps or sharp points.
- A polynomial has at most \( n - 1 \) turning points.
- A polynomial has at most \( n \) \( x \)-intercepts.
- A polynomial has exactly one \( y \)-intercept.

**End Behavior of Polynomial Functions**

The behavior of a graph of a function to the far left or far right is called its **end behavior**.

- **Even Degree**
  - Positive Leading Coefficient: \( \lim_{x \to -\infty} f(x) = -\infty \)
  - Negative Leading Coefficient: \( \lim_{x \to -\infty} f(x) = \infty \)

- **Odd Degree**
  - Positive Leading Coefficient: \( \lim_{x \to -\infty} f(x) = -\infty \)
  - Negative Leading Coefficient: \( \lim_{x \to -\infty} f(x) = \infty \)

**Zeros of polynomials**

If \( f \) is a polynomial and \( c \) is a real number for which \( f(c) = 0 \), then \( c \) is called a **zero** of \( f \), or a **root** of \( f \).

If \( c \) is a zero of \( f \), then
- \( c \) is an \( x \)-intercept of the graph of \( f \).
- \( (x - c) \) is a factor of \( f \).
So if we have a polynomial in factored form, we know all of its \( x \)-intercepts.

- Every factor gives us an \( x \)-intercept.
- Every \( x \)-intercept gives us a factor.

Note: In factoring the equation for the polynomial function \( f \), if the same factor \( x - c \) occurs \( k \) times, we call \( c \) a repeated zero with multiplicity \( k \).

**Description of the Behavior at Each \( x \)-intercept**

1. **Even Multiplicity**: The graph touches the \( x \)-axis, but does not cross it (looks like a parabola there).
2. **Odd Multiplicity of 1**: The graph crosses the \( x \)-axis (looks like a line there).
3. **Odd Multiplicity greater than or equal to 3**: The graph crosses the \( x \)-axis and it looks like a cubic there.

**Rational Functions**

Definition: A **rational function** is a function that can be written in the form \( f(x) = \frac{P(x)}{Q(x)} \), where \( P(x) \) and \( Q(x) \) are polynomials, consists of all real numbers \( x \) such that \( Q(x) \neq 0 \).

You will need to be able to find the following:

- Domain
- Intercepts
- Holes
- Vertical asymptotes
- Horizontal asymptotes
- Slant asymptotes
- Behavior near the vertical asymptotes

**Domain**: The domain of \( f \) is all real numbers except those values for which \( Q(x) = 0 \).

**\( x \)-intercept(s)**: All values of \( x \) for which \( P(x) = 0 \).

**\( y \)-intercept**: The \( y \) intercept of the function is \( f(0) \).

**Holes**: The graph of the function will have a hole if there is a common factor in the numerator and denominator.

**Vertical asymptotes**: The graph of the function has a vertical asymptote at any value of \( x \) for which \( Q(x) = 0 \) and \( P(x) \neq 0 \).
Example 16: Given the following function. Find the domain, x and y intercepts, hole(s) if any and vertical asymptotes if any.

\[ f(x) = \frac{x^2 + x - 6}{x^3 + 2x^2 - 3x} = \frac{(x+3)(x-2)}{x(x-3)(x-1)} \]

Hole: \( x + 3 = 0 \)
\( x = -3 \)

Domain: \( x \neq -3, 0, 1 \)

x-int: Numerator = 0
\( x - 2 = 0 \)
\( x = 2 \)

Y-int: \( f(0) = \text{undefined} \)
(0, 0)

VA: Denominator = 0
\( x = 0 \)
\( x - 1 = 0 \)
\( x = 1 \)

Horizontal asymptotes: You can determine if a graph of a function has a horizontal asymptote by comparing the degree of the numerator with the degree of the denominator.

Shorthand: degree of \( f = \text{deg}(f) \), numerator = \( N \), denominator = \( D \)
1. If \( \text{deg}(N) > \text{deg}(D) \) then there is no horizontal asymptote.
2. If \( \text{deg}(N) < \text{deg}(D) \) then there is a horizontal asymptote and it is \( y = 0 \) (x-axis).
3. If \( \text{deg}(N) = \text{deg}(D) \) then there is a horizontal asymptote and it is \( y = \frac{a}{b} \), where \( a \) is the leading coefficient of the numerator and \( b \) is the leading coefficient of the denominator.

Slant asymptote: The graph of the function may have a slant asymptote if the degree of the numerator is one more than the degree of the denominator. To find the equation of the slant asymptote, use long division to divide the denominator into the numerator. The quotient is the equation of the slant asymptote.

Behavior near the vertical asymptotes: The graph of the function will approach either \( \infty \) or \( -\infty \) on each side of the vertical asymptotes. To determine if the function values are positive or negative in each region, find the sign of a test value close to each side of the vertical asymptotes.

Increasing w/o Bounds \( (\infty) \)

Decreasing w/o Bounds \( (-\infty) \)
Example 17: Does the function cross the horizontal asymptote?

a. \( f(x) = \frac{2x^2 - 2x - 3}{3x^2 + x - 6} \)

\[ \lim_{x \to \infty} f(x) = \text{HA} \]

HA: \( y = \frac{2}{3} \)

\[ f(x) = \frac{2}{3} \]

\[ \frac{2x^2 - 2x - 3}{3x^2 + x - 6} = \frac{2}{3} \]

\[ 6x^2 - 6x - 9 = 6x^2 + 2x - 12 \]

b. \( f(x) = \frac{1 - 2x}{x + 1} \)

HA: \( y = -2 \)  \( \text{HA} : \frac{-2}{1} = -2, \ y = -2 \)

\[ \frac{1 - 2x}{x + 1} = -2 \]

\[ 1 - 2x = -2x - 2 \]

\[ 1 = -2 \ (\text{False}) \]

will never cross HA
Exponential Functions

Functions whose equations contain a variable in the exponent are called exponential functions.

The function \( f(x) = a^x \) is the exponential function with base \( a > 0 \) and \( a \neq 1 \).

We'll look at two cases of the exponential function, \( a > 1 \) and \( 0 < a < 1 \).

For \( a > 1 \):
- Domain: \((-\infty, \infty)\)
- Range: \((0, \infty)\)
- Key point: \((0, 1)\)
- Horizontal asymptote: \( y = 0 \) since \( y \to 0 \) as \( x \to -\infty \)

The graph of \( f(x) = a^x \) with \( a > 1 \) has this shape (larger \( a \) results in a steeper graph):

![Graph of \( f(x) = a^x \) with \( a > 1 \).](image1)

The graph of \( f(x) = a^x \) with \( 0 < a < 1 \) has this shape (smaller \( a \) results in a steeper graph):

![Graph of \( f(x) = a^x \) with \( 0 < a < 1 \).](image2)
The Number “e”

More on transformations of the exponential function $f(x) = a^x$, but with $a = e$ (the natural base).

**Definition:** $e$ is the “limiting value” of $\left( 1 + \frac{1}{x} \right)^x$ as $x$ grows to infinity. 

$e \approx 2.718281282459$. It is an irrational number, like π. This means it cannot be written as a fraction nor as a terminating or repeating decimal.

**Example 1:** Solve for $x$.

a.) $5^{3x-1} = 125$

$5^{3x-1} = 5^3$

$3x - 1 = 3$

$x = \frac{4}{3}$

b.) $2^{5x-2} = 32$

$(2^3)^{5x-2} = 2^5$

$2^{15x-6} = 2^5$

$x = \frac{4}{15}$

$15x - 6 = 5$

$15x = 11$
Logarithmic Functions

The exponential function is 1-1; therefore, it has an inverse function. The inverse function of the exponential function with base \( a \) is called the \textit{logarithmic function with base} \( a \).

For \( x > 0 \) and \( a > 0 \) and \( a \) not equal to 1, \( y = \log_a x \) is equivalent to \( a^y = x \).

The function \( f(x) = \log_a x \) is the \textit{logarithmic function with base} \( a \).

The \textit{common logarithm} is the logarithm with base 10. We denote this as \( \log_{10} x = \log x \).

The \textit{natural logarithm} is the logarithm with base \( e \). We denote this as \( \log_e x = \ln x \).

Laws of Logarithms

If \( m, n \) and \( a \) are positive numbers, \( a \neq 1 \), then

1. \( \log_a mn = \log_a m + \log_a n \)
2. \( \log_a \frac{m}{n} = \log_a m - \log_a n \)
3. \( \log_a m^n = n \log_a m \)
4. \( \log_a 1 = 0 \) (Key PRO)
5. \( \log_a a = 1 \)
6. \( \log_a a^x = x \) (Inverse Properties)
7. \( \log_a m = \frac{\log m}{\log a} \) (change of bases formula)

Characteristics of the Graphs of Logarithmic Functions of the Form \( f(x) = \log_a x \)

1. The \( x \)-intercept is \((1, 0)\) and there is no \( y \)-intercept.
2. The \( y \)-axis is a vertical asymptote.
3. The domain is all positive real numbers.
4. The range is all real numbers.

If \( a > 1 \), the graph of \( f(x) = \log_a x \) looks like:
If $0 < a < 1$, the graph of $f(x) = \log_a x$ looks like:

Note: If a logarithmic function is translated to the left or to the right, the vertical asymptote is shifted by the amount of the horizontal shift.

**Example 2:** Evaluate, if possible.

\[
\begin{align*}
\log_6 36 &= 2 \\
\log_2 \frac{1}{8} &= -3 \\
\log_5 125 &= 3 \\
\log_{10} 100 &= 2 \\
\log_4 2 &= \frac{1}{2} \cdot \sqrt{2} = 2 \\
\log_{0.001} 1 &= 0 \\
\log_3 \sqrt[3]{81} &= \frac{1}{3} \log_3 81 = 3 \\
\log_5 \sqrt[5]{125} &= \frac{1}{5} \log_5 125 = 3 \\
\frac{1}{3} \cdot 4 &= \frac{4}{3} \\
\frac{1}{4} \cdot 3 &= \frac{3}{4}
\end{align*}
\]

**Example 3:** Solve for $x$: $5^{3x} = 9$. (a) Give the exact value using natural logarithms.

\[
\begin{align*}
\ln 5^{3x} &= \ln 9 \\
\frac{3x}{\ln 5} \cdot \ln 5 &= \ln 9 \\
x &= \frac{\ln 9}{3 \ln 5} = \frac{\ln 3^2}{3 \ln 5} = \frac{2 \cdot \ln 3}{3 \ln 5}
\end{align*}
\]

or
\[
\frac{2 \cdot \ln 3}{3 \ln 5}
\]
Math 1330   Algebra Review 2

Example 4: Solve for \( x \): \( 4e^{x+5} + 5 = 7 \). (a) Give the exact value using natural logarithms.

\[
4e^{x+5} + 5 = 7
\]
\[
\frac{4e^{x+5}}{4} = \frac{2}{4}
\]
\[
e^{x+5} = \frac{1}{2}
\]
\[
\ln(e^{x+5}) = \ln\left(\frac{1}{2}\right)
\]
\[
x + 5 = \ln\left(\frac{1}{2}\right)
\]

\[
x = \ln\left(\frac{1}{2}\right) - 5
\]
\[
= \ln(1) - \ln(2) - 5
\]
\[
= -\ln(2) - 5
\]
\[
-\ln 2 - 5
\]

Example 5: Solve for \( x \): \( e^{2x} - 9e^x + 20 = 0 \). (a) Give the exact value using natural logarithms.

\[
e^{2x} - 9e^x + 20 = 0
\]
\[
(e^x)^2 - 9(e^x) + 20 = 0
\]
\[
\text{"y"}^2 - 9\text{"y"} + 20 = 0
\]
\[
(y - 4)(y - 5) = 0
\]
\[
(e^x - 4)(e^x - 5) = 0
\]
\[
e^x - 4 = 0 \quad e^x - 5 = 0
\]
\[
e^x = 4 \quad e^x = 5
\]
\[
\ln e^x = \ln 4 \quad \ln e^x = \ln 5
\]
\[
x = \ln 4 \quad x = \ln 5
\]

\[
x = \ln 4 = \ln 2^2 = 2\ln 2
\]
\[
x = \ln 5
\]
Example 6: Solve for $x$: $\frac{25^{3x-2}}{\sqrt[125]{x}} = \frac{1}{x}$ (a) Give the exact value using natural logarithms.

\[
5^{6x-4} = \left(\left(\frac{5}{3}\right)^x\right)^{-1}
\]

\[
5 \cdot 2^x = -\frac{3}{2}x
\]

\[
\frac{15}{2}x = 4
\]

\[
x = \frac{46}{15}
\]

Example 7: Solve for $x$: $\log_6(x) + \log_6(x + 1) = \log_6 2$

\[
\log_6(x(x+1)) = \log_6(2)
\]

\[
\log_6(x^2 + x) = \log_6(2)
\]

\[
x^2 + x = 2
\]

\[
x^2 + x - 2 = 0
\]

\[
(x+2)(x-1) = 0
\]

\[
x = -2, 1
\]

Check in original:

\[
\log_6(-2) = \text{undefined}
\]

Example 7: Solve for $x$: $\log_6 x + \log_6(5 - x) = 2$

\[
\log_6(x(5-x)) = 2
\]

\[
\log_6(5x - x^2) = 2
\]

\[
x^2 - 5x + 36 = 0
\]

\[
\frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(36)}}{2(1)}
\]

\[
5 \pm \sqrt{25 - 144} > 5 \pm \sqrt{-119}
\]

No Solution
Math 1330   Section 4.1
Section 4.1: Special Triangles and Trigonometric Ratios

In this section, we’ll work with some special triangles before moving on to defining the six trigonometric functions.
Let’s look at right angles and talk about the Pythagorean Theorem.

Important Triangles

30-60-90 Triangles

In a 30° – 60° – 90° triangle, the length of the hypotenuse is two times the length of the shorter leg.
The length of the longer leg is \(\sqrt{3}\) times the length of the shorter leg.

45-45-90 Triangles

In a 45° – 45° – 90° triangle, the legs have the same length. The length of the hypotenuse is \(\sqrt{2}\) times
the length of either leg.
Example 1: Find $x$.

$$\triangle 12\sqrt{2}$$

1. $45^\circ - 45^\circ - 90^\circ$
2. $1 : 1 : \sqrt{2}$
3. $12\sqrt{2} : 12\sqrt{2} : 12\sqrt{2}\sqrt{2}$

$$x = 12\sqrt{2} \cdot \sqrt{2} = 24$$

Example 2: Find $x$ and $y$.

$$\triangle 28$$

1. $30^\circ : 60^\circ : 90^\circ$
2. $1 : \sqrt{3} : 2$
3. $x : y = 28$

$$x = \frac{28 \cdot \sqrt{3}}{2} = 14$$

$$y = 2\sqrt{3} \cdot 14 = 14\sqrt{3}$$

Example 3: Find $x$.

$$\triangle 4\sqrt{2}$$

1. $30^\circ : 60^\circ : 90^\circ$
2. $1 : \sqrt{3} : 2$
3. $y = 4\sqrt{2} : x$

$$y = \frac{4\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{4\sqrt{6}}{3}$$

$$x = 2 \cdot y = 2 \cdot \frac{4\sqrt{6}}{3} = \frac{8\sqrt{6}}{3}$$

Example 4: Given $\Delta ABC$ with sides $a = 1$ and $b = 3$ find the length of the hypotenuse side $c$.

$$a^2 + b^2 = c^2$$

$$1^2 + (3)^2 = c^2$$

$$10 = c^2$$

$$c = \sqrt{10}$$
Math 1330    Section 4.1
Trigonometric Functions:

A trigonometric function is a ratio of the lengths of the sides of a triangle. If we fix an angle, then as to that angle, there are three sides, the adjacent side, the opposite side, and the hypotenuse. We have six different combinations of these three sides, so there are a total of six trigonometric functions. The inputs for the trigonometric functions are angles and the outputs are real numbers.

![Diagram of a right triangle with sides a, b, and c, and angle θ with a positive c.]  

<table>
<thead>
<tr>
<th>Trig Function</th>
<th>Symbol</th>
<th>Ratio of lengths</th>
<th>Value in above picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cosine</td>
<td>$\cos \theta$</td>
<td>adjacent leg hypotenuse</td>
<td>$\frac{b}{c}$</td>
</tr>
<tr>
<td>Sine</td>
<td>$\sin \theta$</td>
<td>opposite leg hypotenuse</td>
<td>$\frac{a}{c}$</td>
</tr>
<tr>
<td>Tangent</td>
<td>$\tan \theta$</td>
<td>opposite leg adjacent leg</td>
<td>$\frac{a}{b}$</td>
</tr>
<tr>
<td>Secant</td>
<td>$\sec \theta$</td>
<td>hypotenuse adjacent leg</td>
<td>$\frac{c}{b}$</td>
</tr>
<tr>
<td>Cosecant</td>
<td>$\csc \theta$</td>
<td>hypotenuse opposite leg</td>
<td>$\frac{c}{a}$</td>
</tr>
<tr>
<td>Cotangent</td>
<td>$\cot \theta$</td>
<td>adjacent leg opposite leg</td>
<td>$\frac{b}{a}$</td>
</tr>
</tbody>
</table>

You need to memorize the six trigonometric functions and how to find them. A useful mnemonic device:

SOH-CAH-TOA

$S = \frac{O}{H}$  
$C = \frac{A}{H}$  
$T = \frac{O}{A}$

$$
\sin \theta \quad \tan \theta \quad \frac{1}{\sin \theta}
$$

$$
\csc \theta \quad \cot \theta = \frac{1}{\tan \theta}
$$
Example 5: Find the values of all six trigonometric functions for the angle \( \theta \) in the figure below.

\[
\text{Pyth. Thm. } a^2 + b^2 = c^2 \\
(3\sqrt{6})^2 + b^2 = (3)^2 \\
9 + b^2 = 9 \\
b^2 = 3 \\
b = \sqrt{3}
\]

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{6}}{3} \\
\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{3} \\
\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2} \\
\csc \theta = \frac{1}{\sin \theta} \\
\sec \theta = \frac{1}{\cos \theta} \\
\cot \theta = \frac{1}{\tan \theta}
\]

Example 6: Suppose that \( \triangle DEF \) is a right triangle and \( D \) is an acute angle. If \( \sin D = \frac{4}{5} \), find \( \cos D \) and \( \tan D \).

\[
\sin D = \frac{4}{5} = \frac{\text{opp}}{\text{hyp}} \\
\cos D = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \\
\tan D = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \\
\cos F = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}
\]

Basic Trigonometric Identities:

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \\
\cot \theta = \frac{\cos \theta}{\sin \theta} \\
\csc \theta = \frac{1}{\sin \theta} \\
\sec \theta = \frac{1}{\cos \theta} \\
\tan \theta = \frac{1}{\cot \theta} \\
\cot \theta = \frac{1}{\tan \theta} \\
\sec \theta = \frac{1}{\cos \theta}
\]
Example 7: Suppose that $\theta$ is an acute angle in a right triangle and $\sec \theta = \frac{5\sqrt{3}}{4}$. Find cosine $\theta$.

\[
\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5\sqrt{3}}{4}
\]

\[
\sec^2 \theta = \frac{1}{\cos^2 \theta} \implies \cos^2 \theta = \frac{1}{\sec^2 \theta}
\]

\[
\cos \theta = \frac{4}{5\sqrt{3}} = \frac{4\sqrt{3}}{15}
\]
Section 4.2: Radians, Arc Length, and Area of a Sector

An angle is formed by two rays that have a common endpoint (vertex). One ray is the initial side and the other is the terminal side. We typically will draw angles in the coordinate plane with the initial side along the positive x axis.

$\angle B, \angle ABC, \angle CBA$, and $\theta$ are all notations for this angle. When using the notation $\angle ABC$ and $\angle CBA$, the vertex is always the middle letter.

We measure angles in two different ways, both of which rely on the idea of a complete revolution in a circle. The first is degree measure. In this system of angle measure one complete revolution is $360^\circ$.

So one degree is $\frac{1}{360}$ of the circle.

The second method is called radian measure. One complete revolution is $2\pi$. The problems in this section are worked in radians. Radians is a unit free measurement.

Let $c$ be a circle with radius $r$. Let $s$ denote the length of the arc intercepted by a central angle. The radian measure, $\theta$, of the central angle is the ratio of $s$ to $r$ or $\theta = \frac{s}{r}$. Note that $s = r \cdot \theta$.

$\theta$ has to be in radian measure.

Note: There is a little more than 3 radian lengths from $0^\circ$ to $180^\circ$ ($180^\circ = \pi$). This gives us the conversion factor to go from degrees to radian measure and back.
The Radian Measure of an Angle

If an angle has a measure of 2.5 radians, we write \( \theta = 2.5 \) radians or \( \theta = 2.5 \). There should be no confusion as to whether radian or degree measure is being used. If \( \theta \) has a degree measure of, say, 2.5 we must write \( \theta = 2.5^\circ \) and not \( \theta = 2.5 \).

Converting degrees to radians and radians to degrees:

1. To convert degrees to radians, multiply degrees by \( \frac{\pi}{180^\circ} \).
2. To convert radians to degrees, multiply radians by \( \frac{180^\circ}{\pi} \).

Let’s do the conversions:

Example 1:

a. \( \frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{900^\circ}{6} = 150^\circ \)

b. \( \frac{3\pi}{2} \cdot \frac{180^\circ}{\pi} = \frac{540^\circ}{\pi} = 270^\circ \)

c. \( \frac{315^\circ}{180^\circ} = \frac{315\pi}{180} = \frac{63\pi}{36} = \frac{21\pi}{12} = \frac{7\pi}{4} \)

Common Angles (Memorize these!)

\[
\begin{align*}
360^\circ &= 2\pi \\
180^\circ &= \pi \\
90^\circ &= \frac{\pi}{2} \\
60^\circ &= \frac{\pi}{3} \\
45^\circ &= \frac{\pi}{4} \\
30^\circ &= \frac{\pi}{6}
\end{align*}
\]
**Example 2:** If two angles of a triangle have radian measures $\frac{\pi}{12}$ and $\frac{2\pi}{5}$, find the radian measure of the third angle.

Triangle $\rightarrow 140^\circ \rightarrow \pi$

$$\frac{\pi}{12} + \frac{2\pi}{5} + x = \pi$$

$$\frac{29\pi}{60} + x = \pi$$

$$x = \frac{31\pi}{60}$$

**Arc length and sector area of a circle:**

![Diagram of a circle with arc and angle labels]

Arc length: $s = r\theta$  
Remember that $s$ is the length from A to B.

Sector area formula: $A = \frac{1}{2} r^2 \theta$

**Example 3:** Find the arc length and area of the sector formed by the $40^\circ$ angle.

$40^\circ \rightarrow \text{Radians}$

$$\frac{40^\circ \cdot \pi}{180^\circ} = \frac{40\pi}{180} = \frac{2\pi}{9}$$

Arc length: $s = r \cdot \theta$

$$s = 3 \cdot \frac{2\pi}{9} = \frac{6\pi}{9} = \frac{2\pi}{3}$$

Area of sector: $A = \frac{1}{2} r^2 \theta$

$$A = \frac{1}{2} \cdot 3^2 \cdot \frac{2\pi}{9} = \frac{16\pi}{9} = \frac{16\pi}{9}$$
Math 1330  Section 4.2

Example 4: Given a circle with radius 6 inches and a sector length of 15 inches find $\theta$.

\[
S = r \cdot \theta \\
15 = 6 \cdot \theta \\
\theta = \frac{15}{6} = \frac{5}{2} \text{ radians}
\]

Example 5: Given a circle the area of sector is $\frac{\pi}{3}$ cm$^2$ and the central angle is $\frac{\pi}{6}$. Find the radius

\[
A = \frac{1}{2} r^2 \theta \\
\frac{\pi}{3} = \frac{1}{2} r^2 \cdot \frac{\pi}{6} \\
\frac{\pi}{3} \cdot \frac{12}{\pi} = r^2 \\
4 = r^2 \\
2 = r
\]

Example 6: Find the length of the sector with central angle 60° and radius 3 m.

\[
S = r \cdot \theta \\
= 3 \cdot 60^\circ \\
= 180^\circ \\
\theta \rightarrow \text{Radians}
\]

\[
60^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{3} \\
S = r \theta \\
= 3 \cdot \frac{\pi}{3} = \pi
\]

Remember: that $\theta$ (the central angle) has to be in radians when you compute the area sector and arc length.
Linear Speed and Angular Velocity

Suppose you are riding on a merry-go-round.

The ride travels in a circular motion, and the horses usually move up and down. Some of the horses are right along the edge of the merry-go-round, and some are closer to the center. If you are on one of the horses at the edge, you will travel farther than someone who is on a horse near the center. But the length of time that both people will be on the ride is the same. If you were on the edge, not only did you travel farther, you also traveled faster.

However, everyone on the merry-go-round travels through the same number of degrees (or radians).

There are two quantities we can measure from this, angular velocity and linear velocity. These are sometimes referred to as angular speed and linear speed.

The angular velocity of a point on a rotating object is the number of degrees (or radians or revolutions) per unit of time through with the point turns. This will be the same for all points on the rotating object.

We let the Greek letter $\omega$ (omega) represent angular velocity. Using the definition above, $\omega = \frac{\theta}{t}$

The linear velocity of a point on the rotating object is the distance per unit of time that the point travels along its circular path. This distance will depend on how far the point is from the axis of rotation (the center of the merry-go-round).

We denote linear velocity by $v$. Using the definition above, $v = \frac{s}{t}$, where $s$ is the arc length.

Linear Speed in Terms of Angular Speed

The linear velocity, $v$, of a point a distance $r$ from the center of rotation is given by $v = r\omega$, where $\omega$ is the angular velocity in radians per unit of time.
Example 7: If the speed of a revolving gear is 20 rpm (revolutions per minute),
   a. find the number of degrees per minute through which the gear turns.

   b. Find the number of radians per minute through which the gear turns.

Example 8: A car has wheels with a 12 inch radius. If each wheel’s rate of turn is 6 revolutions per second,
   a. Find the angular speed in units of radians/second.

   b. How fast (linear speed) is the car moving in units of inches/second?