Section 4.2: Radians, Arc Length, and Area of a Sector

An angle is formed by two rays that have a common endpoint (vertex). One ray is the initial side and the other is the terminal side. We typically will draw angles in the coordinate plane with the initial side along the positive x axis.

\[ \theta \]

\( \angle B, \angle ABC, \angle CBA \), and \( \theta \) are all notations for this angle. When using the notation \( \angle ABC \) and \( \angle CBA \), the vertex is always the middle letter.

We measure angles in two different ways, both of which rely on the idea of a complete revolution in a circle. The first is degree measure. In this system of angle measure one complete revolution is \( 360^\circ \).

So one degree is \( \frac{1}{360} \) of the circle.

The second method is called radian measure. One complete revolution is \( 2\pi \). The problems in this section are worked in radians. Radians is a unit free measurement.

Let \( c \) be a circle with radius \( r \). Let \( s \) denote the length of the arc intercepted by a central angle. The radian measure, \( \theta \) of the central angle is the ratio of \( s \) to \( r \) or \( \theta = \frac{s}{r} \). Note that \( s = r \cdot \theta \).

\( \theta \) has to be in radian measure.

Note: There is a little more than 3 radian lengths form \( 0^\circ \) to \( 180^\circ \) \((180^\circ = \pi)\). This gives us the conversion factor to go from degrees to radian measure and back.
The Radian Measure of an Angle

If an angle has a measure of 2.5 radians, we write \( \theta = 2.5 \) radians or \( \theta = 2.5 \). There should be no confusion as to whether radian or degree measure is being used. If \( \theta \) has a degree measure of, say, 2.5 we must write \( \theta = 2.5^\circ \) and not \( \theta = 2.5 \).

Converting degrees to radians and radians to degrees:

1. To convert degrees to radians, multiply degrees by \( \frac{\pi}{180^\circ} \).
2. To convert radians to degrees, multiply radians by \( \frac{180^\circ}{\pi} \).

Let’s do the conversions:

Example 1:

a. \( \frac{5\pi}{6} \)

b. \( \frac{3\pi}{2} \)

c. \( 315^\circ \)

Common Angles (Memorize these!)

\[
\begin{align*}
360^\circ &= 2\pi \\
180^\circ &= \pi \\
90^\circ &= \frac{\pi}{2} \\
60^\circ &= \frac{\pi}{3} \\
45^\circ &= \frac{\pi}{4} \\
30^\circ &= \frac{\pi}{6}
\end{align*}
\]
Example 2: If two angles of a triangles have radian measures $\frac{\pi}{12}$ and $\frac{2\pi}{5}$, find the radian measure of the third angle.

Arc length and sector area of a circle:

![Diagram of a circle with arc and sector]

Arc length: $s = r\theta$  
Remember that $s$ is the length form A to B.

Sector area formula: $A = \frac{1}{2} r^2 \theta$

Example 3: Find the arc length and area of the sector formed by the $40^\circ$ angle.
Example 4: Given a circle with radius 6 inches and a sector length of 15 inches find $\theta$.

Example 5: Given a circle the area of sector is $\frac{\pi}{3}$ cm$^2$ and the central angle is $\frac{\pi}{6}$. Find the radius.

Example 6: Find the length of the sector with central angle $60^\circ$ and radius 3 m.

Remember: that $\theta$ (the central angle) has to be in radians when you compute the area sector and arc length.
Linear Speed and Angular Velocity

Suppose you are riding on a merry-go-round.

The ride travels in a circular motion, and the horses usually move up and down. Some of the horses are right along the edge of the merry-go-round, and some are closer to the center. If you are on one of the horses at the edge, you will travel farther than someone who is on a horse near the center. But the length of time that both people will be on the ride is the same. If you were on the edge, not only did you travel farther, you also traveled faster.

However, everyone on the merry-go-round travels through the same number of degrees (or radians).

There are two quantities we can measure from this, angular velocity and linear velocity. These are sometimes referred to as angular speed and linear speed.

The angular velocity of a point on a rotating object is the number of degrees (or radians or revolutions) per unit of time through which the point turns. This will be the same for all points on the rotating object.

We let the Greek letter $\omega$ (omega) represent angular velocity. Using the definition above, $\omega = \frac{\theta}{t}$

The linear velocity of a point on the rotating object is the distance per unit of time that the point travels along its circular path. This distance will depend on how far the point is from the axis of rotation (the center of the merry-go-round).

We denote linear velocity by $v$. Using the definition above, $v = \frac{s}{t}$, where $s$ is the arc length.

$$\frac{s}{t} = \frac{r\theta}{t} = r\frac{\theta}{t}$$

**Linear Speed in Terms of Angular Speed**

The linear velocity, $v$, of a point a distance $r$ from the center of rotation is given by $v = r\omega$, where $\omega$ is the angular velocity in radians per unit of time.
Example 7: If the speed of a revolving gear is 20 rpm (revolutions per minute),
a. find the number of degrees per minute through which the gear turns.

b. Find the number of radians per minute through which the gear turns.

Example 8: A car has wheels with a 12 inch radius. If each wheel’s rate of turn is 6 revolutions per second,
a. Find the angular speed in units of radians/second.

b. How fast (linear speed) is the car moving in units of inches/second?