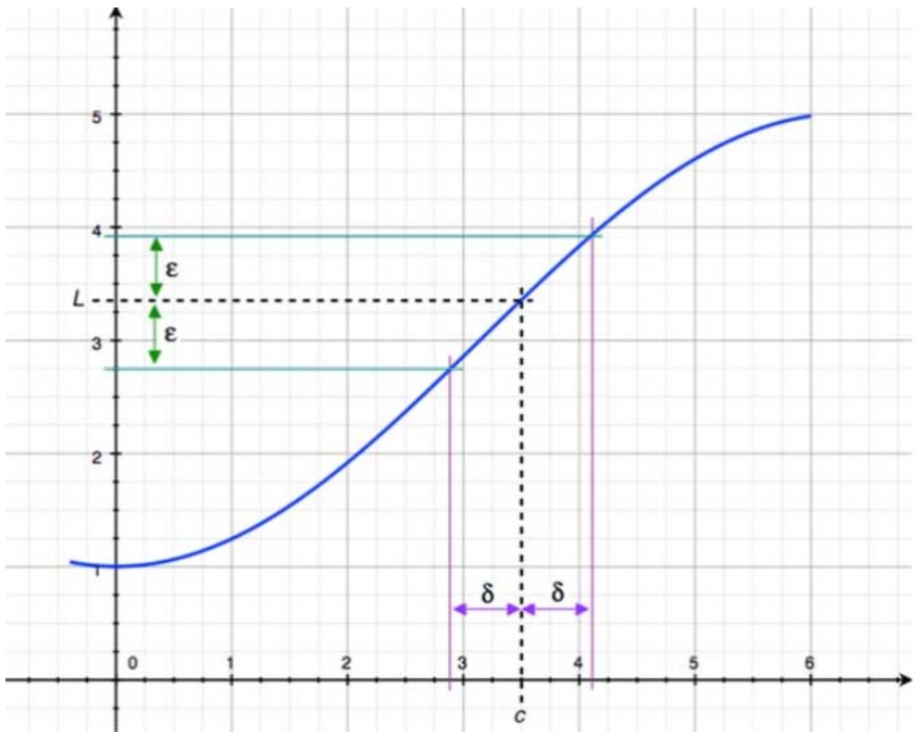
Section 1.3: The Definition of a Limit

The Limit of a Function

If we let the limit of a function be equal to L and c be the fixed value that x approaches, then we can say $\lim_{x \to c} f(x) = L$ if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.



Example 1: Show that $\lim_{x \to 2}(5x - 2) = 8$ using the definition of a limit.

Identify the following pieces

$c =$ $f(x) =$ $L =$

Example 2: Give the largest δ that works with ε = 0.1 for the limit, $\lim_{x \to -1} (1 - 2x) = 3$

Arithmetic Rules for Limits

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then:

1. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$
2. $\lim_{x \to c} [k \cdot f(x)] = kL$
3. $\lim_{x \to c} [f(x) \cdot g(x)] = LM$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \ M \neq 0$

Example 3: Let $\lim_{x \to c} f(x) = 3$, $\lim_{x \to c} g(x) = -2$ and $\lim_{x \to c} h(x) = -5$. Evaluate the following

$\lim_{x \to c} (2f(x) - g(x) \cdot h(x) + [h(x)])$

Example 4: Evaluate the following limits:

a. $\lim_{x \to 2} \dfrac{3x^2 + 2x}{2x + 1}$

b. $\lim_{x \to -1} 2x^3 - 3x^2 + 4x + 12$

c. $\lim_{x \to -1} \dfrac{3x^2 + 2x}{2x^2 + 3x + 1}$

d. $\lim_{x \to 1} \dfrac{2x^3 - 2x}{x - 1}$

So if a simple plug in didn't give us an actual answer, which means we have to approach the limit a different way.

1. Factoring
2. Distribution, common denominators
3. Using the conjugate

Example 5: Back to 4d

$$\lim_{x \to 1} \frac{2x^3 - 2x}{x - 1}$$

Example 6: Evaluate: $\lim_{x \to 0} x\left(4 - \frac{7}{x}\right)$

Example 7: Evaluate: $\lim_{x \to -3} \frac{2x^3 + 54}{x + 3}$

Example 8: Evaluate $\lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5}$

Example 9: Evaluate $\lim_{x\to -1} \frac{|x+1|}{x+1}$

Example 10: Evaluate $\lim_{h\to 0} \frac{\frac{1}{4}-\frac{1}{h+4}}{h}$

Limit of Piecewise Functions

Example 11: Let $f(x)=\begin{cases} \sqrt{5-x}, & x<0 \\ x^2, & 0\le x<2 \\ -4x+16, & x\ge 2 \end{cases}$, find $\lim_{x\to 0} f(x)$

Example 12: $f(x)=\begin{cases}\dfrac{x^2-16}{x-4}, & x\neq 4\\ 9, & x=4\end{cases}$, find $\lim\limits_{x\to 4} f(x)$

Limits at Infinity

If a limit at infinity exists and it's equal to a single real number L then they are written as $\lim\limits_{x\to\infty} f(x)=L$ or $\lim\limits_{x\to-\infty} f(x)=L$. These limits deal with what is happening to the y-values to the far left or right side of the graph (function).

Limits at infinity problems often involve rational expressions (fractions). The technique we can use to evaluate limits at infinity is to recall some rules from Algebra used to find horizontal asymptotes. These rules came from "limits at infinity" so they'll surely work for us here.

The highest power of the variable in a polynomial is called the **degree** of the polynomial.

We can compare the degree of the numerator with the degree of the denominator and come up with some generalizations.

- If the degree of the numerator is smaller than the degree of the denominator, then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=0$

- If the degree of the numerator is the same as the degree of the denominator, then you can find $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}$ by making a fraction from the leading coefficients of the numerator and denominator and then reducing to lowest terms.

- If the degree of the numerator is larger than the degree of the denominator, then the limit does not exist.

Let's see how these generalizations came to be.

Example 13: Evaluate the following limits

$$\lim_{x \to \infty} \frac{1}{x}$$

$$\lim_{x \to \infty} \frac{1}{x^2}$$

$$\lim_{x \to \infty} \frac{1}{x^n}$$

Example 14: Evaluate $\lim_{x \to \infty} \frac{2x+7}{x^4+5x^2+6}$

Example 15: Evaluate $\lim_{x\to\infty}\frac{2x^2-3x+1}{4x-x^2}$

Example 15: Evaluate $\lim_{x\to\infty}\frac{x^4-3x^3+1}{4x-x^2}$