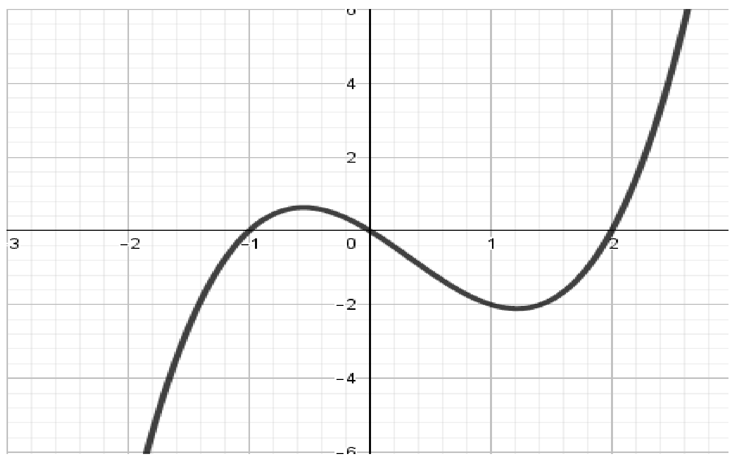
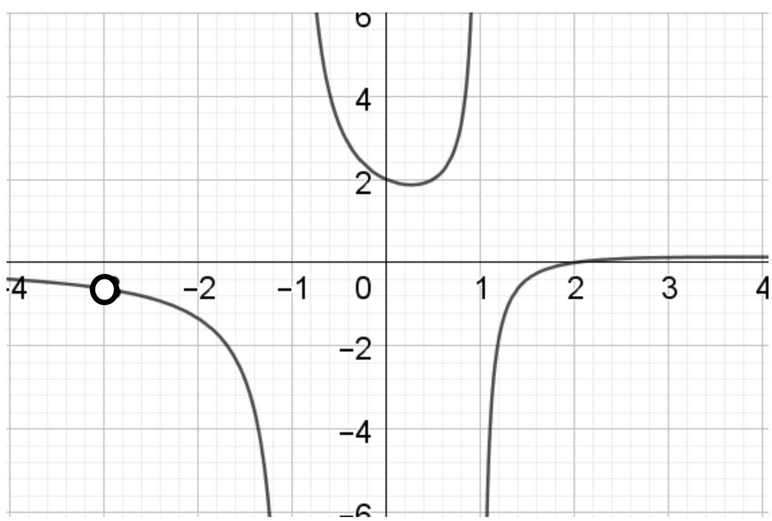
Section 1.4: Continuity

A function is a **continuous** at a point if its graph has no gaps, holes, breaks or jumps at that point. If a function is not continuous at a point, then we say it is **discontinuous** at that point.

The function $f(x) = x(x+1)(x-2)$ graphed below is continuous everywhere.



The function below is NOT continuous everywhere, it is discontinuous at $x = -3$ and $x = \pm 1$



We'll focus on the classifying them in a moment.

The following types of functions are continuous over their domain.

- Polynomials, Rational Functions, Root Functions, Trigonometric Functions, Inverse Trigonometric Functions, Exponential Functions, Logarithmic Functions

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Theorem: If f and g are continuous at c, a is a real number then each are also continuous at c.

I. $f \pm g$
II. af
III. fg
IV. $\frac{f}{g}$ provided $g(c) \neq 0$

Theorem: If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Example 1: Discuss the continuity for:

a. $g(x) = 4x^7 + x$ Discontinuous Continuous

b. $f(x) = \dfrac{x-3}{x^2 - x - 6}$ Discontinuous Continuous

c. $h(x) = \dfrac{2x}{1 - \cos x}$ Discontinuous Continuous

Types of Discontinuity

Removable Discontinuity occurs when:

- $\lim_{x \to c} f(x) \neq f(c)$ (or the limit exist, but $f(c)$ is undefined.

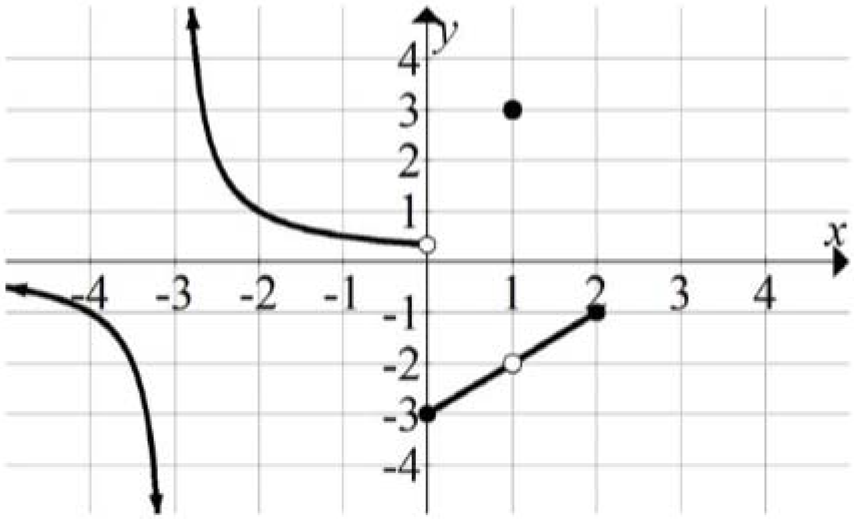
Jump Discontinuity occurs when:

- $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^-} f(x)$ exists, but are not equal.

Infinite Discontinuity occurs when:

- $\lim_{x \to c^\pm} f(x) \to \pm\infty$ on at least one side of c. Infinite discontinuities are generally associated with a vertical asymptote

Example 2: Identify and state the discontinuity.



One-Sided Continuity

A function *f* is called **continuous from the left at *c*** if $\lim_{x \to c^-} f(x) = f(c)$ and **continuous from the right at *c*** if $\lim_{x \to c^+} f(x) = f(c)$

In the example 2, we have continuity from the right at x = 0 and continuity on the left at x = 2

Continuity Stated a Bit More Formally

A function *f* is said to be **continuous at the point *x* = *c*** if the following three conditions are met:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

To check if a function is continuous at a point, we'll use the three steps above. This process is called the **three step method.**

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Example 3: Let $f(x) = \begin{cases} x^3 & x < 1 \\ \sqrt{x} & x \geq 1 \end{cases}$ is the function continuous at $x = 1$?

So let's go through the process

1. Is $f(1)$ defined?

2. Check to see if $\lim_{x \to 1} f(x)$ exist.

So check: $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$

Does $\lim_{x \to 1} f(x)$ exist?

3. $\lim_{x \to 1} f(x) = f(1)$?

If at least one of the three steps fails, identify the type of discontinuity

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Example 4: Let $f(x)=\begin{cases}2x^2+9 & x<3\\ 2 & x=3\\ x^3 & x>3\end{cases}$ is the function continuous at $x = 3$?

So let's go through the process

1. Is $f(3)$ defined?

2. Check to see if $\lim_{x\to 3} f(x)$ exist.

So check: $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$

Does $\lim_{x\to 3} f(x)$ exist?

3. $\lim_{x\to 3} f(x) = f(3)$?

Example 5: Let $f(x)=\begin{cases}2x-3 & x<2\\ cx-x^2 & x\geq 2\end{cases}$ is the function continuous at $x = 2$?

So let's go through the process

1. Is $f(2)$ defined?

2. $\lim_{x\to 2} f(x)$ must exist, so we need to make sure $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.

3. Set $\lim_{x\to 2} f(x) = f(2)$ to find *c*.

Example 6: Find A and B so that $f(x)=\begin{cases} 2x^2-1 & x<-2 \\ A & x=-2 \\ Bx-3 & x>-2 \end{cases}$ is continuous.

1. Find $f(-2)$.

2. $\lim_{x\to -2} f(x)$ must exist, so we need to make sure $\lim_{x\to -2^-} f(x)=\lim_{x\to -2^+} f(x)$.

3. Since $\lim_{x\to -2} f(x)=f(-2)$ then

Example 7: The function $f(x)=\dfrac{x-25}{\sqrt{x}-5}$ is defined everywhere except at $x=25$. If possible define $f(x)$ at 25 so that it becomes continuous at 25.

Example 8: Given $f(x)$, find the points where the function is discontinuous and classify these points.

$$f(x) = \begin{cases} \dfrac{1}{x} & x < 2 \\ \dfrac{1}{x} & 2 < x \le 4 \\ x & 4 < x < 6 \\ 4 & x = 6 \\ x & x > 6 \end{cases}$$