A function is a continuous at a point if its graph has no gaps, holes, breaks or jumps at that point. If a function is not continuous at a point, then we say it is discontinuous at that point.

The function \( f(x) = x(x + 1)(x - 2) \) graphed below is continuous everywhere.

The function below is NOT continuous everywhere, it is discontinuous at \( x = -3 \) and \( x = \pm 1 \).

We’ll focus on the classifying them in a moment.

The following types of functions are continuous over their domain.

- Polynomials, Rational Functions, Root Functions, Trigonometric Functions, Inverse Trigonometric Functions, Exponential Functions, Logarithmic Functions
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**Theorem:** If \( f \) and \( g \) are continuous at \( c \), \( a \) is a real number then each are also continuous at \( c \).

I. \( f \pm g \)
II. \( af \)
III. \( fg \)
IV. \( \frac{f}{g} \) provided \( g(c) \neq 0 \)

**Theorem:** If \( g \) is continuous at \( c \) and \( f \) is continuous at \( g(c) \), then \( f \circ g \) is continuous at \( c \).

**Example 1:** Discuss the continuity for:

a. \( g(x) = 4x^2 + x \)  
   \[ \text{Discontinuous} \quad \text{Continuous} \]

b. \( f(x) = \frac{x - 3}{x^2 - x - 6} \)  
   \[ \text{Discontinuous} \quad \text{Continuous} \]

c. \( h(x) = \frac{2x}{1 - \cos x} \)  
   \[ \text{Discontinuous} \quad \text{Continuous} \]

**Types of Discontinuity**

**Removable Discontinuity** occurs when:
- \( \lim_{x \to c} f(x) \neq f(c) \) (or the limit exist, but \( f(c) \) is undefined.)

**Jump Discontinuity** occurs when:
- \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) exists, but are not equal.

**Infinite Discontinuity** occurs when:
- \( \lim_{x \to c^-} f(x) \to \pm\infty \) on at least one side of \( c \). Infinite discontinuities are generally associated with a vertical asymptote.
Example 2: Identify and state the discontinuity.

One-Sided Continuity

A function $f$ is called continuous from the left at $c$ if $\lim_{x \to c^-} f(x) = f(c)$ and continuous from the right at $c$ if $\lim_{x \to c^+} f(x) = f(c)$.

In the example 2, we have continuity from the right at $x = 0$ and continuity on the left at $x = 2$.

Continuity Stated a Bit More Formally

A function $f$ is said to be continuous at the point $x = c$ if the following three conditions are met:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

To check if a function is continuous at a point, we’ll use the three steps above. This process is called the three step method.
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**Example 3:** Let $f(x) = \begin{cases} \frac{x^3}{\sqrt{x}} & x < 1 \\ \frac{x}{\sqrt{x}} & x \geq 1 \end{cases}$ is the function continuous at $x = 1$?

So let's go through the process

1. Is $f(1)$ defined?

2. Check to see if $\lim_{x \to 1} f(x)$ exist.

So check: $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$

Does $\lim_{x \to 1} f(x)$ exist?

3. $\lim_{x \to 1} f(x) = f(1)$?

If at least one of the three steps fails, identify the type of discontinuity
Example 4: Let \( f(x) = \begin{cases} 2x^2 + 9 & x < 3 \\ 2 & x = 3 \\ x^3 & x > 3 \end{cases} \) is the function continuous at \( x = 3 \)?

So let’s go through the process

1. Is \( f(3) \) defined?
2. Check to see if \( \lim_{x \to 3} f(x) \) exist.

So check: \( \lim_{x \to 3^-} f(x) \) and \( \lim_{x \to 3^+} f(x) \)

Does \( \lim_{x \to 3} f(x) \) exist?

3. \( \lim_{x \to 3} f(x) = f(3) \)?

Example 5: Let \( f(x) = \begin{cases} 2x - 3 & x < 2 \\ cx - x^2 & x \geq 2 \end{cases} \) is the function continuous at \( x = 2 \)?

So let’s go through the process

1. Is \( f(2) \) defined?
2. \( \lim_{x \to 2} f(x) \) must exist, so we need to make sure \( \lim_{x \to 2} f(x) = \lim_{x \to 2^+} f(x) \).

3. Set \( \lim_{x \to 2} f(x) = f(2) \) to find \( c \).
Example 6: Find A and B so that 
\[ f(x) = \begin{cases} 
2x^2 - 1 & x < -2 \\
A & x = -2 \\
Bx - 3 & x > -2 
\end{cases} 
\]
is continuous.

1. Find \( f(-2) \).

2. \( \lim_{x \to -2} f(x) \) must exist, so we need to make sure \( \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) \).

3. Since \( \lim_{x \to -2} f(x) = f(-2) \) then

Example 7: The function 
\[ f(x) = \frac{x - 25}{\sqrt{x} - 5} \]
is defined everywhere except at \( x = 25 \). If possible define \( f(x) \) at 25 so that it becomes continuous at 25.
Example 8: Given $f(x)$, find the points where the function is discontinuous and classify these points.

$$f(x) = \begin{cases} 
\frac{1}{x} & x < 2 \\
\frac{1}{x} & 2 < x \leq 4 \\
x & 4 < x < 6 \\
4 & x = 6 \\
4 & x > 6 
\end{cases}$$