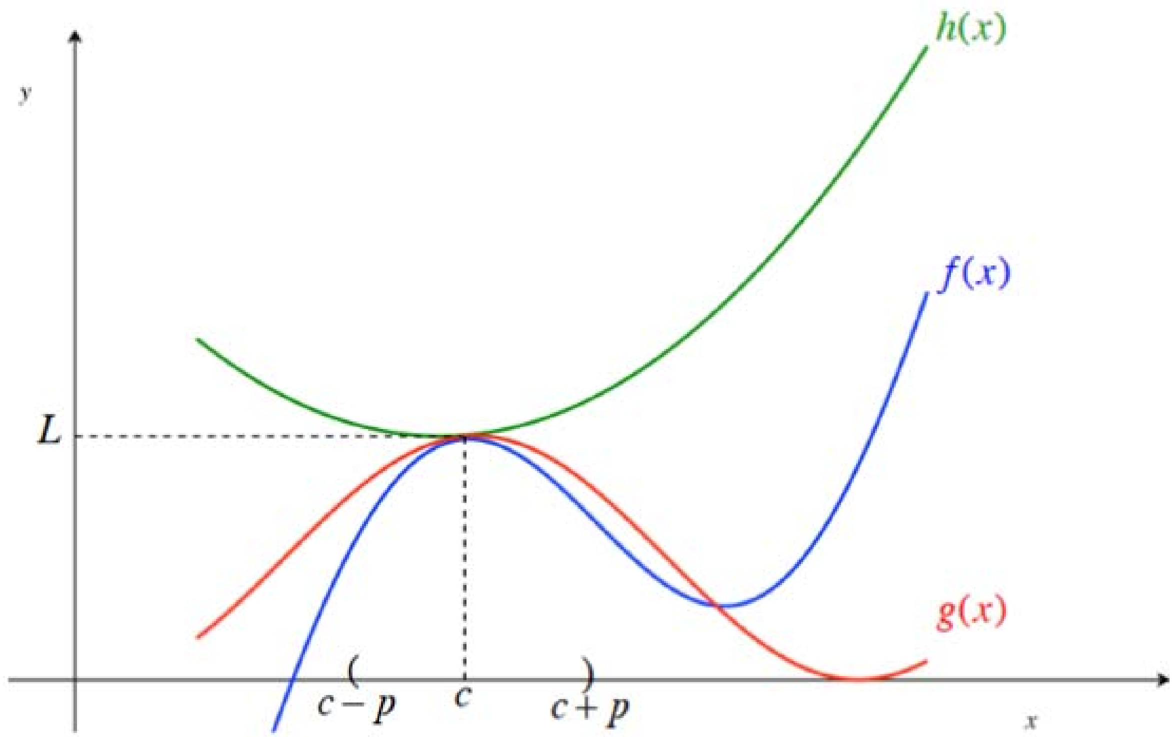
Section 1.6: The Pinching Theorem; Trigonometric Limits

Theorem: Let p > 0 and *c* be a real number. Suppose *f (x), g (x) and h (x)* are defined in an open interval $(c-p, c+p)$ (except possibly at *x = c*). If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$.



Before we move on, we should note that:

$$\lim_{x\to 0} \sin(x) = 0 \quad \text{and} \quad \lim_{x\to 0} \cos(x) = 1$$

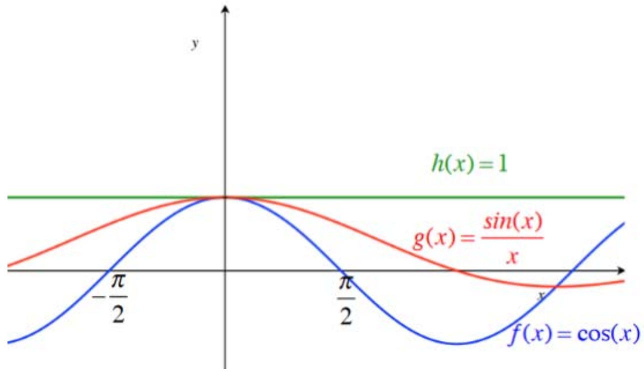
These limits can be found by direct substitution or by simply recalling their graphs.

Example 1: Evaluate.

a. $\lim_{x\to 0} \frac{\sin(3x)}{4x-1}$

b. $\lim_{x\to 0} \frac{1-5\cos(3x)}{12}$

Take $\lim_{x\to 0}\frac{\sin(x)}{x}$. If we use direct substitution we get an indeterminate form $\frac{0}{0}$, but the Pinching Theorem allows us to prove that the $\lim_{x\to 0}\frac{\sin(x)}{x}=1$



Example 2: Evaluate.

a. $\lim_{x\to 0}\frac{\sin(7x)}{x}$

b. $\lim_{x\to 0}\frac{\sin(2x)}{5x}$

In fact: $\lim_{x\to 0}\frac{\sin(ax)}{bx}=\frac{a}{b}$

Example 3: Evaluate

a. $\lim_{x\to 0}\frac{\sin(7x)}{3x}$

b. $\lim_{x \to 0} \frac{\sin(8x)}{2x}$

c. $\lim_{x \to 0} \frac{x}{\sin x}$

d. $\lim_{x \to 0} \frac{5x}{\sin(3x)}$

e. $\lim_{x \to 0} \frac{\sin(6x)+3}{4x}$

Take $\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$. If we use direct substitution we get an indeterminate form $\frac{0}{0}$, but the Pinching Theorem allows us to prove that the $\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$

In fact: $\lim_{x \to 0} \frac{1-\cos(ax)}{bx} = 0$

Example 4: Evaluate.

a. $\lim_{x \to 0} \frac{1-\cos(6x)}{8x}$

b. $\lim_{x \to 0} \frac{\cos(2x)-1}{3x}$

And in some cases you'll need to use older information

$\sin^2(\theta)+\cos^2(\theta)=1$ $\tan^2(\theta)+1=\sec^2(\theta)$ $1+\cot^2(\theta)=\csc^2(\theta)$

$\sin(2\theta)=2\sin(\theta)\cos(\theta)$

$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$

$\cos(2\theta)=2\cos^2(\theta)-1$

$\cos(2\theta)=1-2\sin^2(\theta)$

Example 5:

a. $\lim_{x\to 0}\frac{\cos x\tan x}{7x}$

b. $\lim_{x\to 0}\frac{1-\cos^2(2x)}{5x}$

c. $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$

d. $\lim_{x\to 0}\frac{3x^2}{1-\cos(3x)}$