Section 2.1
The Definition of the Derivative

We are interested in finding the slope of the tangent line at a specific point.

We need a way to find the slope of the tangent line analytically for every problem that will be exact every time.

We can draw a secant line across the curve, then take the coordinates of the two points on the curve, \( P \) and \( Q \), and use the slope formula to approximate the slope of the tangent line.

Now suppose we move point \( Q \) closer to point \( P \). When we do this, we’ll get a better approximation of the slope of the tangent line.

When we continue to move point \( Q \) even closer to point \( P \), we get an even better approximation. We are letting the distance between \( P \) and \( Q \) get smaller and smaller.
Now let’s give these two points names. We’ll express them as ordered pairs.

Now we’ll apply the slope formula to these two points.

\[ m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h} \]

This expression is called a difference quotient also called the average rate of change.

The last thing that we want to do is to let the distance between \( P \) and \( Q \) get arbitrarily small, so we’ll take a limit.

This gives us the definition of the slope of the tangent line.

The slope of the tangent line to the graph of \( f \) at the point \( P(x, f(x)) \) is given by

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

provided the limit exists.

We find the instantaneous rate of change when we take the limit of the difference quotient.

The derivative of \( f \) with respect to \( x \) is the function \( f'(x) \) defined by

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]. The domain of \( f'(x) \) is the set of all \( x \) for which the limit exists.

Note that: \( \frac{dy}{dx} = \frac{d}{dx} f(x) = y' \)

Section 2.1 – The Definition of a Derivative
Example 1: Use the limit definition of the derivative to find $f'(x)$ for $f(x) = 3x^2 - x$.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

Then find $f'(c)$ when $c = 1$. 
Example 2: Use the limit definition of the derivative to find \( f'(x) \) for \( f(x) = -\frac{2}{x-1} \).

Recall: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)
Try this one: Find the derivative of \( f(x) = \sqrt{x} + 2 \)

Recall: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)
Since the derivative is a “formula” for finding the slope of a tangent line, then given a certain \(x\)-value, we can find its slope AND its equation.

We’ll may use the point-slope equation of a line: \(y - y_1 = m(x - x_1)\)

Example 3: Find the equation of the line tangent to the function \(f(x) = x^2 + x\) at the point \((2, 6)\).

Recall: \(f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\)

Example 4: Given \(\lim_{h \to 0} \left[ \frac{((5 + h)^2 - (5 + h)) - (5^2 - 5)}{h} \right]\), give the function \(f\) and the value \(c\)

Try this one: Given \(\lim_{h \to 0} \left[ \frac{\tan\left(\frac{\pi}{6} + h\right) - \sqrt{3}}{h} \right]\), give the function \(f\) and the value \(c\).
Example 5: If $f(1) = 5$ and $f'(1) = 6$, give the equation of the tangent line at $x = 1$.

**Differentiability**

A function $f$ is **differentiable at an x-value $c$** if \[ \lim_{{h \to 0}} \frac{f(c+h) - f(c)}{h} \] exists.

A function $f$ is not differentiable where a function has:

- **Cusp**
- **Vertical Tangent**
- **Hole/Removable Discontinuity**
- **Vertical Asymptote**
- **Jump Discontinuity**

If the limit fails to exist, we say that the function is not differentiable at $c$. 

Section 2.1 – The Definition of a Derivative
If \( f \) is differentiable at \( c \), then it is continuous at \( c \).

However, if a function \( f \) is continuous at \( c \), then it may or may not be differentiable at \( c \).

For example, take \( f(x) = |x| \). This function is \textbf{continuous everywhere, but it’s not differentiable} at \( x = 0 \), since the one-sided limits do not agree there.

Example 6: Use the graph below to answer the following questions.

a. Give any \( x \)-values where the function is not differentiable.

b. Give any \( x \)-values where the function is continuous by not differentiable.
Example 7: Given that \( f(x) = |9x^2 - 64| \), determine any \( x \)-values where \( f \) is not differentiable.