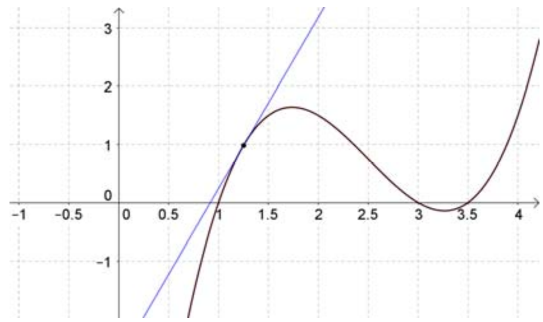
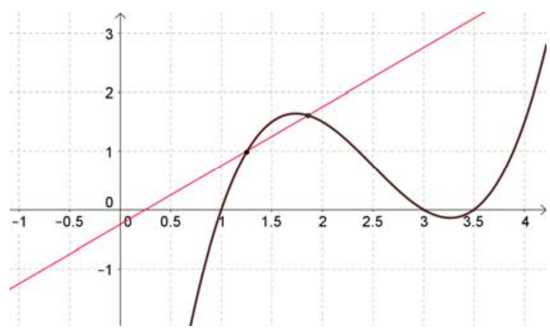
Section 2.1
The Definition of the Derivative

We are interested in finding the slope of the tangent line at a specific point.

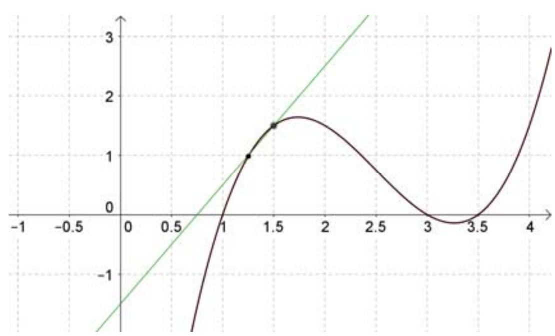


We need a way to find the slope of the tangent line analytically for every problem that will be exact every time.

We can draw a secant line across the curve, then take the coordinates of the two points on the curve, *P* and *Q*, and use the slope formula to approximate the slope of the tangent line.

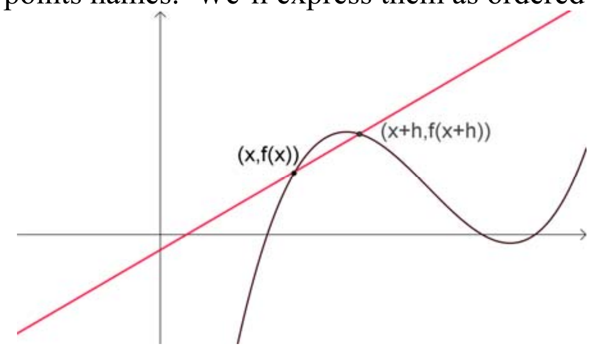


Now suppose we move point *Q* closer to point *P*. When we do this, we'll get a better approximation of the slope of the tangent line.



When we continue to move point *Q* even closer to point *P*, we get an even better approximation. We are letting the distance between *P* and *Q* get smaller and smaller.

Now let's give these two points names. We'll express them as ordered pairs.

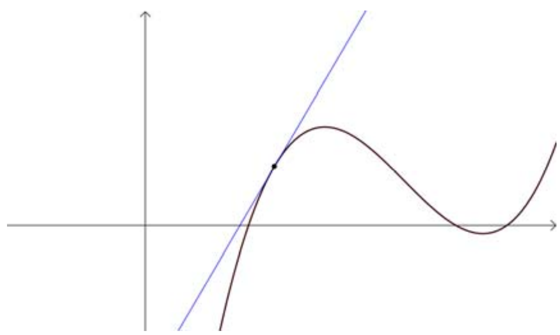


Now we'll apply the slope formula to these two points.

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

This expression is called a **difference quotient** also called the **average rate of change.**

The last thing that we want to do is to let the distance between *P* and *Q* get arbitrarily small, so we'll take a limit.



This gives us the definition of the **slope of the tangent line**.

The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

We find the **instantaneous rate of change** when we take the limit of the difference quotient.

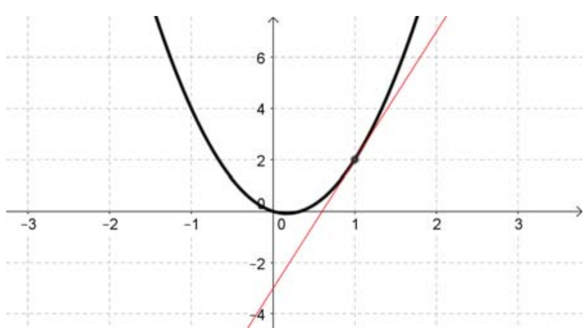
The **derivative of *f* with respect to *x*** is the function f' (read "*f* prime") defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The domain of $f'(x)$ is the set of all *x* for which the limit exists.

Note that: $\frac{d}{dx} f(x) = y' = \frac{dy}{dx}$

Example 1: Use the limit definition of the derivative to find $f'(x)$ for $f(x) = 3x^2 - x$.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Then find $f'(c)$ when $c = 1$.



Example 2: Use the limit definition of the derivative to find $f'(x)$ for $f(x) = -\frac{2}{x-1}$.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Try this one: Find the derivative of $f(x)=\sqrt{x+2}$

Recall: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Since the derivative is a "formula" for finding the slope of a tangent line, then given a certain *x*-value, we can find its slope AND its equation.

We'll may use the point-slope equation of a line: $y - y_1 = m(x - x_1)$

Example 3: Find the equation of the line tangent to the function $f(x) = x^2 + x$ at the point (2, 6).

Recall: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Example 4: Given $\lim_{h \to 0} \left[\dfrac{[(5+h)^2 - (5+h)] - (5^2 - 5)}{h} \right]$, give the function *f* and the value *c*

Try this one: Given $\lim_{h \to 0} \left[\dfrac{\tan\left(\dfrac{\pi}{6} + h\right) - \dfrac{\sqrt{3}}{3}}{h} \right]$, give the function *f* and the value *c*.

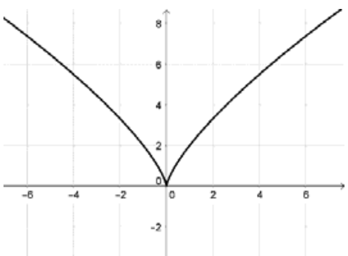
Example 5: If $f(1) = 5$ and $f'(1) = 6$, give the equation of the tangent line at x = 1.

Differentiability

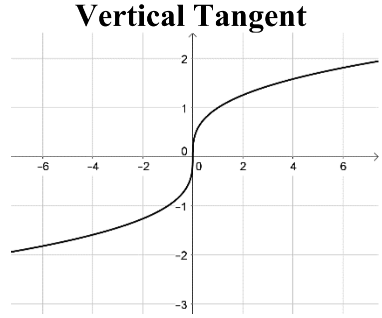
A function f is **differentiable at an x-value *c*** if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists.

A function f is not differentiable where a function has:

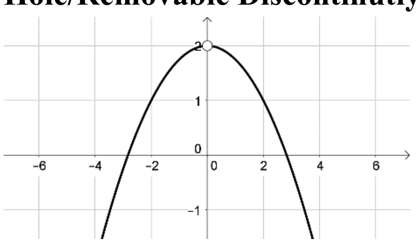
Cusp



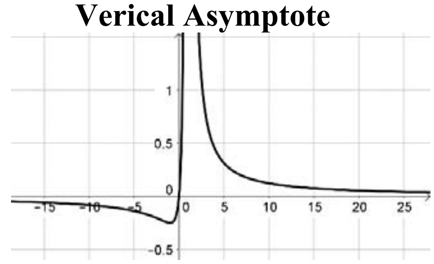
Vertical Tangent



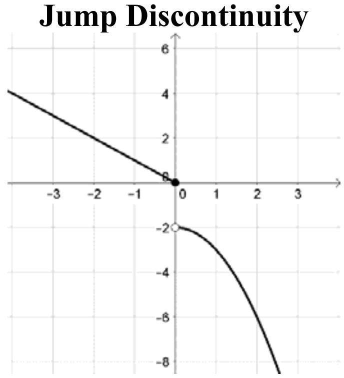
Hole/Removable Discontinutiy



Verical Asymptote



Jump Discontinuity

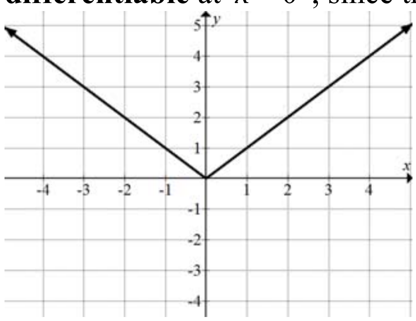


If the limit fails to exist, we say that the function is not differentiable at *c*.

If f is differentiable at c, then it is continuous at c.

However, if a function f is continuous at c, then it may or may not be differentiable at c.

For example, take $f(x) = |x|$. This function is **continuous everywhere, but it's not differentiable** at $x = 0$, since the one-sided limits do not agree there.

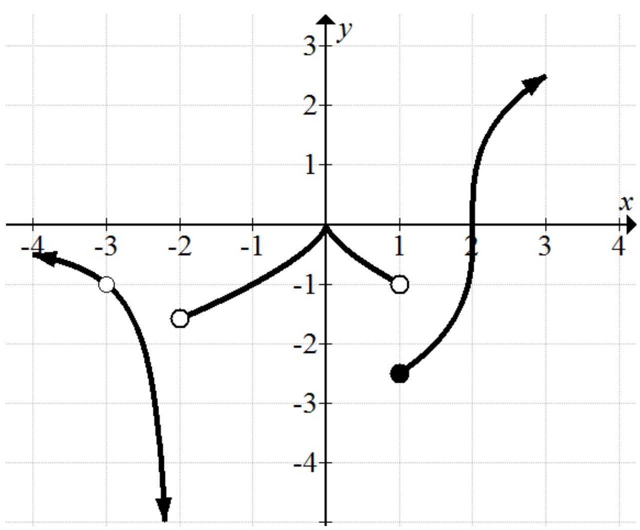


Recall:

$$f(x) = |x| = \begin{cases} x, & \text{for x>0} \\ -x, & \text{for x<0} \end{cases}$$

Example 6: Use the graph below to answer the following questions.
a. Give any x-values where the function is not differentiable.

b. Give any x-values where the function is continuous by not differentiable.



Example 7: Given that $f(x) = \left|9x^2 - 64\right|$, determine any x-values where f is not differentiable.