Section 2.3: Differentiation Rules

The Product Rule

If f and g are differentiable at x, then so is the product fg. Moreover,

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

This rule can be extended to the product of more functions:

$$(uvw)' = u'vw + uv'w + uvw'$$

Example 1: Find the derivative of $f(x) = (3x+5)(2x^4 - x)$

Example 2: Find the derivative of $h(x) = x^3 \sin(x)$.

The Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then the quotient f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Example 3: Find the derivative of $h(x) = \dfrac{x}{2x^2+1}$.

Example 4: Find the derivative of $f(x) = \dfrac{\tan x}{4x+2}$.

What if we wanted the derivative of something like, $f(x)=(5x+1)^7$. Well, to be able to use the rules we've learned so far, we'd have to expand the expression. This would be too much work. We have a rule for finding derivatives involving functions of these type.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

This rule is one of the most important rules of differentiation. It helps us with many complicated functions.

Example 5: Let $f(x)=x^{10}$ and $g(x)=x^3+x+1$. If $h(x)=(f \circ g)(x)$, find the derivative of $h(x)$.

Example 6: Find the derivative of $f(x)=sin\left(-x^2+1\right)$.

Example 7: Find the derivative of $f(x)=\sqrt[4]{6x^4+8x}$.

Example 8: Find the derivative of $f(x)=cos^4\left(\sqrt{x}\right)$

Example 9: Let $g(x)=\dfrac{1}{x^3-x}$, find $\left.\dfrac{dy}{dx}\right|_{x=2}$.

Example 10: Find an equation for the tangent line to the graph of $f(x)=x^2\cos(2x)$ at $x=\dfrac{\pi}{2}$

Example 11: The following information is given about two functions f and g.

$$f(2)=7,\ f'(2)=1,\ f(5)=4,\ f'(5)=4,$$

$$g(2)=5,\ g'(2)=3,\ g(5)=10,\ g'(5)=6.$$

a. If $h(x)=(f/g)(x)$, find $h'(2)$.

b. If $h(x)=\left[f(x)\right]^3$, find $h'(2)$.

If $h(x)=(f \circ g)(x)$, find $h'(1)$.

The Chain Rule in Leibniz Notation

Let *y* be a function in terms of *u* and *u* be a function in terms of *x*. As a result, *y* may be expressed as a function in terms of *x*. How do we differentiate *y* with respect to *x*? Expressing *y* as a function in terms of *x* and then differentiating it is an option, but we are looking for a better way.

The chain rule can be written as: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

That is, the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x.

This formula can be extended to more variables; each new variable adds a new link to the chain.

Example 12: If $y = u^3 + 5u$ and $u = x - 2\sqrt{x}$, evaluate $\left.\frac{dy}{dx}\right|_{x=4}$.

Example 13: Find the second derivative of $g(x) = \dfrac{x}{x^2 - 1}$.

Example 14: Let $f(x) = \left(-4x^2 + 25\right)^2$. Determine the value(s) of x for when $f'(x) > 0$.

TRY THESE:

Find the slope of the tangent line to the curve $f(x)=\dfrac{x^2+x}{x+5}$ at $x=1$. Then find its equation. What is the normal equation?

Find the derivative of each of the following functions:

a. $f(x)=x\left(-x+3x^2\right)^{1/3}$

b. $h(x)=\left(x^2+\dfrac{1}{x^2}\right)^3$

c. $y=\sqrt{x}\left(3x^2+5x\right)$

d. $f(x)=\dfrac{10x+40}{(x-4)^3}$

Find the second derivative of $f(x)=(x^2+2)^5$