

Section 3.1: Related Rates

Position, Velocity and Acceleration

One-dimensional motion: Left and Right. Suppose an object moves along a straight line, left and right, and at each time t its position is $s(t)$. This is called the **position function**.

Assuming its derivative exists, the derivative of the position function is the **velocity function**, usually denoted by $v(t) = s'(t)$. This will give the rate of change at time t (how fast the position is changing or the instantaneous rate of change).

- If the velocity is positive, the object is moving right (positive direction). *Here the position is increasing.*
- If the velocity is negative, the object is moving left (negative direction). *Here the position is decreasing.*
- If the velocity is zero, the object has stopped.
- *Speed* is the absolute value of velocity (how fast the object is going).

Assuming the velocity function is differentiable, **acceleration** is defined to be the rate of change of velocity per unit time (how fast is the velocity changing). Usually denoted by $a(t) = v'(t)$.

- Positive acceleration corresponds to accelerating (increasing velocity).
- Negative acceleration corresponds to decelerating (decreasing velocity).

One more connection...

- The object is speeding up if the velocity and acceleration have the same sign.
- The object is slowing down if the velocity and acceleration have the opposite signs.

Note: Positive acceleration DOES NOT necessarily mean speeding up!

Example 1: An object moves along the x-axis and its position is given by the function $s(t) = t^3 - 9t^2 + 24t + 8$. Its initial position is: $s(0) = 8$, eight units to the right of the origin.

a. Find its velocity at $t = 2$ seconds. Which way is the object moving?

b. When does the object change directions?

c. Find its acceleration at time $t = 2$ seconds.

d. When is the object speeding up? Slowing down?

Free Fall

One-dimensional motion: Up and Down.

Three cases:

- Drop an object from a certain height, but no initial velocity.
- Throw an object upward with initial velocity, but no initial height.
- Throw an object upward with initial height and initial velocity.

Formulas

The height of an object in free fall is given by $h(t) = -16t^2 + v_0t + h_0$ (distance in feet) or $h(t) = -4.9t^2 + v_0t + h_0$ (distance in meters), where v_0 is the initial velocity, h_0 is the initial height and t is time. The derivative of either one of these functions is velocity.

Example 2: An object is dropped from a height of 400 feet. How long does it take for the object to hit the ground and what is its velocity on impact?

Example 3: Supplies are dropped from a stationary helicopter and seconds later it hits the ground at 98 m/sec. How high was the helicopter?

Example 4: A stone is thrown straight up from a height of 6 feet with initial velocity of 112 feet per second.

- When will it hit the ground?
- Determine the greatest height the stone will reach.

The Derivative as a Rate of Change

In this section, we'll work through some problems where we will be interested in answer questions like,

“What is the rate of change of _____ with respect to _____.”

or

“How fast is _____ changing/decreasing/increasing with respect to _____.”

Recall: $\frac{dy}{dx}$ is read, “The derivative of y with respect to x .” We will translate words into math symbols.

You will need to know your geometric formulas for circles, squares, rectangles, triangles, spheres, cones, cylinders, cubes, etc., even Pythagorean Theorem.

Steps to solving related rate problems:

- Draw a “picture”.
- What do you know?
- What do you need to find?
- Write an equation involving the variables whose rates of change either are given or are to be determined. (This is an equation that relates the parts of the problem.)
- Implicitly differentiate both sides of the equation with respect to time. This FREEZES the problem.
- Solve for what you need.

Example 5: Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 feet??

Given:

Question:

Relation:

Example 6: Suppose a spherical balloon is inflated at the rate of 10 cubic centimeters per minute. How fast is the radius increasing when the radius is 5 centimeters?

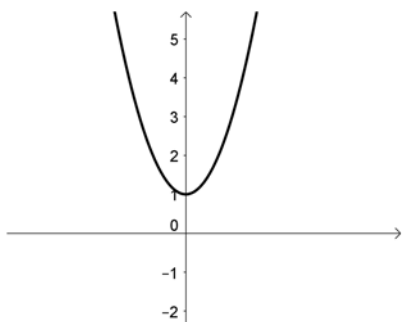
Given:

Question:

Relation:

Example 7: A point moves along the curve $y = 4x^2 + 1$ in such a way that the y value is decreasing at the rate of 2 units per second. At what rate is x changing when $x = 5$?

Picture:



Given:

Question:

Relation:

Example 8: A 13-foot ladder is leaning against a vertical wall. If the bottom of the ladder is being pulled away from the wall at the rate of 2 feet per second, how fast will the top of the ladder be moving along the wall when the bottom of the ladder is 12 feet from the wall?

Picture:

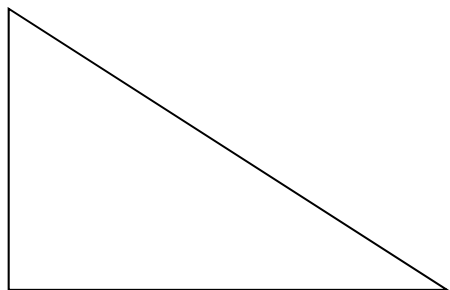
Given:

Question:

Relation:

Example 9: A 6-foot man is walking towards a 25 foot lamp post at the rate of 10 ft/sec. How fast is the length of his shadow changing when he is 20 feet from the lamp post? *Hint: This example will use similar triangles from Geometry.*

Picture:



Given:

Question:

Relation:

Example 10: On a morning when the sun will pass directly overhead, the shadow of a 60-ft tower on level ground is 45 feet long. At the moment in question, the angle the sun makes with the ground is increasing at the rate of $\pi/600$ radians/minute. Find the rate of change of the shadow's length.

Picture:

Given:

Question:

Relation:

Extra problems:

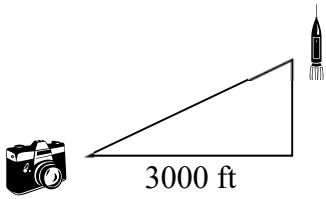
Suppose a person standing at the top of a building 112 feet high throws a ball vertically upward with an initial velocity of 96 ft/sec. How far does the ball travel during its flight?

A falling stone is at a certain instant 120 feet above the ground. Two seconds later it is only 40 feet above the ground. If it was thrown upward with an initial speed of 3 feet per second, from what height was it thrown?

A stone is thrown upward from ground level. With what minimum speed should the stone be thrown so as to reach a height of 36 feet?

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If a rocket is rising vertically at the rate of 880 ft/sec when it is 4000 feet up, how fast is the camera-to-rocket distance changing at the instant?



Two cars are moving towards the same point. The first car started from a point that is 100 miles away and it travels south at a rate of 40 mi/h. The second one started from a point 105 miles away and it travels east at a rate of 60 mi/h. At what rate is the distance between the cars changing one hour later?