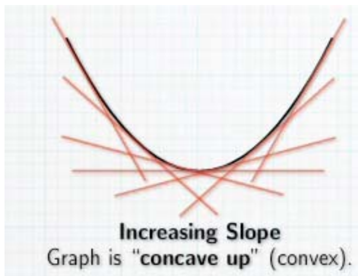
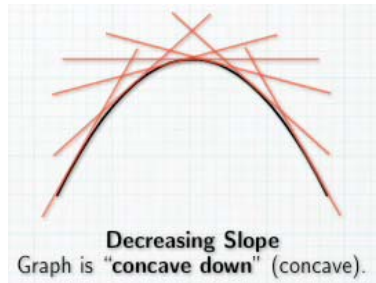
Section 3.5
Concavity and Points of Inflection

Let f be a function that is differentiable on an open interval I.

The graph of f is **concave up** if f' is increasing on I.

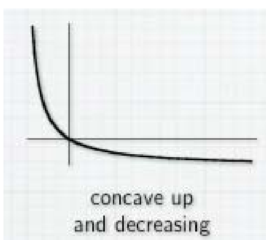
The graph of f is **concave down** if f' is decreasing on I.

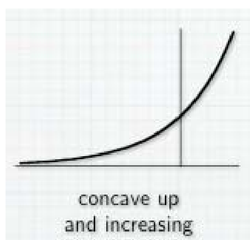


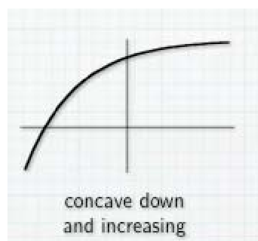


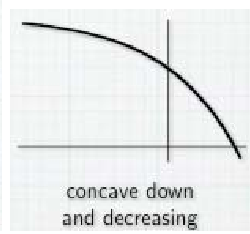
Even though both pictures indicate a local extreme value, note that that need not be the case.

Here are some graphs where the functions are concave up or down without any local extreme values.

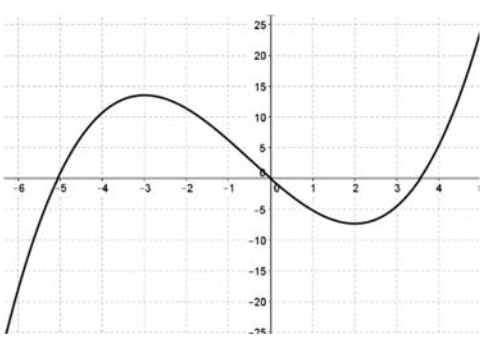








Example 1: The graph of $f'(x)$ (first derivative!) of a polynomial function f is given.



a. When is $f(x)$ concave up?

b. When is $f(x)$ concave down?

Theorem: Let f be a function that is twice differentiable on an open interval I.

- If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
- If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I

Determining the Intervals of Concavity for a Function

1. Find any value of x for which $f''(x) = 0$ or $f''(x)$ is undefined. Identify the intervals determined by these points.

2. Choose a test point c in each interval found in Step 1 and determine the sign of f'' in that interval.

    a. Wherever $f''(c) > 0$, then the function f is concave up on that interval.

    b. Wherever $f''(c) < 0$, then the function f is concave down on that interval.

Example 2: Determine the concavity of $f(x) = x^3 + 2x$. The domain of $f(x)$ is $(-\infty, \infty)$.

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

Concave Up:

Concave Down:

Example 3: Determine the concavity of $f(x) = cos^2 x - sin^2 x$, $x \in (0, \pi)$. The domain of $f(x)$ is $(-\infty, \infty)$.

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

Concave Up:

Concave Down:

A point in the domain of a differentiable function f at which the concavity changes is called a **point of inflection**.

Finding Inflection Points

1. Find any value of x in the domain of the function for which $f''(x)=0$ or $f''(x)$ is undefined.

2. Determine the sign of $f''(x)$ to the left and to the right of each point $x = a$ found in Step 1. If there is a sign change across the point $x = a$, then $(a, f(a))$ is a point of inflection of f.

Example 4: Given $f(x)$, determine any points of inflection $f(x)=\dfrac{x}{x^2-1}$ The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ and $f''(x)=\dfrac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3}$.

Find when $f''(x)=0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$:

POI:

Example 5: Find any points of inflection of $f\left(x\right)=2+x^{1/3}$. The domain of *f* is $(-\infty,\infty)$.

Find when $f''(x)=0$:

Find when $f''(x)$ is undefined:

$f(x)$:

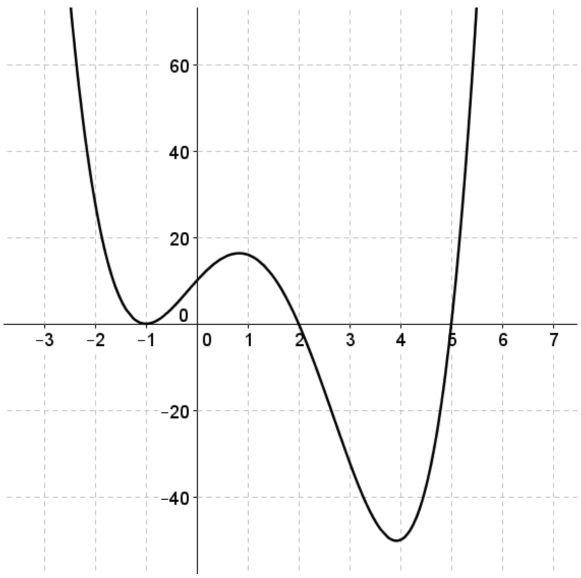
$f''(x)$:

POI:

The graph of f'':

When the graph of the second derivative is given, we can gather information about whether f is concave up or down, and any points of inflection for f.

Example 6: The graph of f'' (second derivative!) of a polynomial function *f* is given. Determine whether each of the following statements is/are true or false.



a. The function $f(x)$ concave down over one interval.

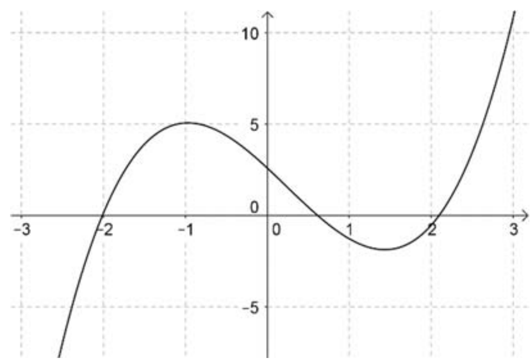
b. The x-values of the points of inflection are: x = -1, x = 2, x = 5.

Sometimes it is difficult to study the sign of the derivative function. For some cases, it may be easier to use the following test:

The Second-Derivative Test

Let c be a critical point for f where $f'(c)=0$ and $f''(c)$ exists.

* If $f''(c)>0$, then $f(c)$ is a local minimum value.
*If $f''(c)<0$, then $f(c)$ is a local maximum value.
*If $f''(c)=0$, then this test is inconclusive.



Example 7: Given $f\ ''(x)=6x-12$, $f\ '(1)=0$ and $f\ '(4)=0$. Classify these critical numbers as local min/max.

Try this one: Find any critical points and classify them as local min/max on $\left(0, \frac{2\pi}{3}\right)$ using the second derivative test.

$f(x) = 2\sin x + \cos(2x)$,

Local Max: Local Min: