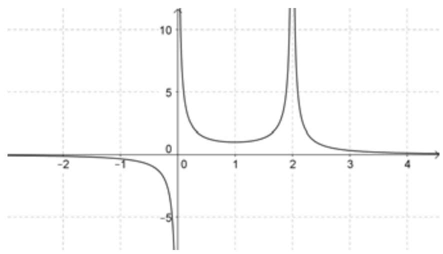
Section 3.6
Curve Sketching

Vertical Asymptotes

If $f(x) \to \pm\infty$ as $x \to c^+$ or $x \to c^-$, then the line $x = c$ is a **vertical asymptote** for $f(x)$.

The graph of $f(x) = \dfrac{1}{x|x-2|}$ is given below.



We can see the vertical asymptotes very easily from its graph. But also recall how to find them algebraically. *Recall: Simplify the function. Any variable factor left in the denominator, set equal to 0 and solve for x.*

- A function may have no vertical asymptotes, such as: $f(x) = \dfrac{x}{\sqrt{x^2+4}}$

- A function may have only one vertical asymptote, such as: $f(x) = \dfrac{\sqrt{x}}{4\sqrt{x} - x}$

- A function may have many vertical asymptotes, such as: $f(x) = \dfrac{x^2}{1 - 2\sin x}$

Horizontal Asymptotes

As we saw in Section 1.3, the behavior of a function as $x \to \pm\infty$ determines the **horizontal asymptotes**.

- If $\lim_{x\to\infty} f(x) = L$, then the line $y = L$ is a (rightward) horizontal asymptote.
- If $\lim_{x\to -\infty} f(x) = L$, then the line $y = L$ is a (leftward) horizontal asymptote.

Recall the shortcut for rational functions: Compare the degrees.

$f(x) = \dfrac{x+1}{x^2-4}$ H. A.:

$f(x) = \dfrac{x^2}{5x^2+1}$; H. A.:

$f(x) = \dfrac{x^5}{x^3-2x}$; H. A.:

These rules work because for $p > 0$ and provided $\dfrac{1}{x^p}$ is defined, $\lim_{x\to\infty} \dfrac{1}{x^p} = 0$ and $\lim_{x\to -\infty} \dfrac{1}{x^p} = 0$.

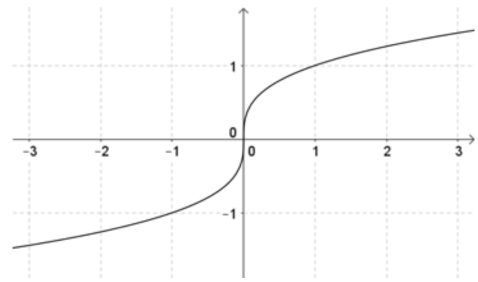
For example, $f(x) = \dfrac{x^2}{5x^2+1}$ has H.A. $y = \dfrac{1}{5}$ because:

Example 1: Find the horizontal asymptotes for each of the following functions.

a. $f(x) = \dfrac{x}{\sqrt{x^2 + 4}}$

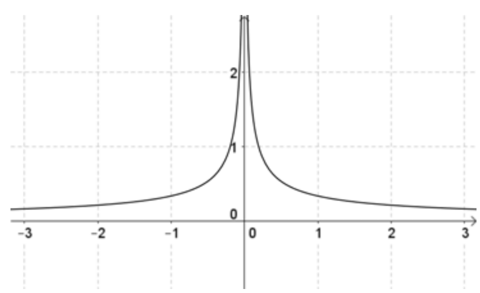
b. $f(x) = \dfrac{\sqrt{x}}{4\sqrt{x} - x}$

Vertical Tangents $f(x) = x^{1/3}$



Suppose that $f(x)$ is continuous at $x = c$. If $f'(x) \to \infty$ or $f'(x) \to -\infty$ as $x \to c$, then we say that the function has a **vertical tangent** at the point $(c, f(c))$.

$f'(x)$



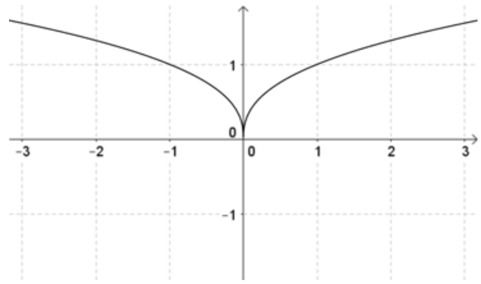
Vertical tangents will only happen with ***some*** radical functions. They may be found by observing that:

- $f(c)$ is defined.
- $f'(c)$ is undefined.
- The sign chart for f' across $x = c$ has ***no sign change.***

  Be careful when creating a sign chart for some radicals, don't forget find any critical points for the function.

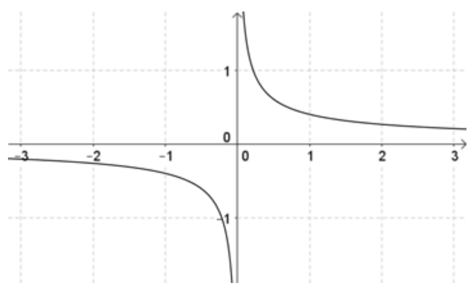
Vertical Cusps

$f(x) = x^{2/5}$



Suppose that $f(x)$ is continuous at $x = c$. If $f'(x) \to \infty$ as $x \to c$ from one side and $f'(x) \to -\infty$ as $x \to c$ from the other side, then we say that the function has a **vertical cusp** at the point $(c, f(c))$.

$f'(x)$



Cusps will only happen with ***some*** radical functions. They may be found by observing that:

- $f(c)$ is defined.
- $f'(c)$ is undefined.
- The sign chart for f' across $x = c$ has ***a sign change.***

  Be careful when creating a sign chart for some radicals, don't forget find any critical points for the function.

Example 2: For the following functions, determined whether the function has a vertical tangent, cusp or neither at the given value.

a. $f(x) = 5(x-8)^{4/5}$ at $c = 8$

Check list:

- Is $f(c)$ is defined?

- Is $f'(c)$ is undefined?

- Create a sign chart. Does the sign chart for f' across $x = c$ have ***a sign change or not?***

b. $f(x) = 9x^{3/5} - 2x^{6/5}$ at $c = 0$

Check list:

- Is $f(c)$ is defined?

- Is $f'(c)$ is undefined?

- Create a sign chart. Does the sign chart for f' across $x = c$ have ***a sign change or not?***

Curve Sketching

Using Calculus to Graph a Function.

1. Determine the **domain** of the function f.

   For radicals:
   - *The domain of any odd root will be* $(-\infty,\infty)$.
   - *The domain of any even root, set the radicand (inside) ≥ 0 and solve.*

2. Find any **asymptotes**—for functions with fractions.
3. Determine any **intercepts** of the function. To find the $x-$intercepts, we need to solve the equation $f(x)=0$ and to find the $y-$intercepts, evaluate the function at 0 (if 0 is in the domain of f).
4. Find the **first derivative,** f'. Determine any critical points, intervals of increase/decrease, local extreme points, vertical tangents and cusps.
5. Find the **second derivative,** f''. Study the sign of f'' to understand concavity of the function and determine any points of inflection.
6. Plot the **points of interest** (intercepts, local or absolute extreme points, points of inflection).
7. Sketch the graph of f using the information gathered in the previous steps. Make sure that the function has the right shape (concaves up/down, rises/falls) on the corresponding intervals.

Example 3: Use the information given to sketch the graph of function *f*.

$f(x)=\dfrac{4x-4}{x^2}$

Domain: $(-\infty,0)\cup(0,\infty)$

Intercept: x-intercept: 1

Asymptotes: x-axis and y-axis

Increasing: $(0,2)$

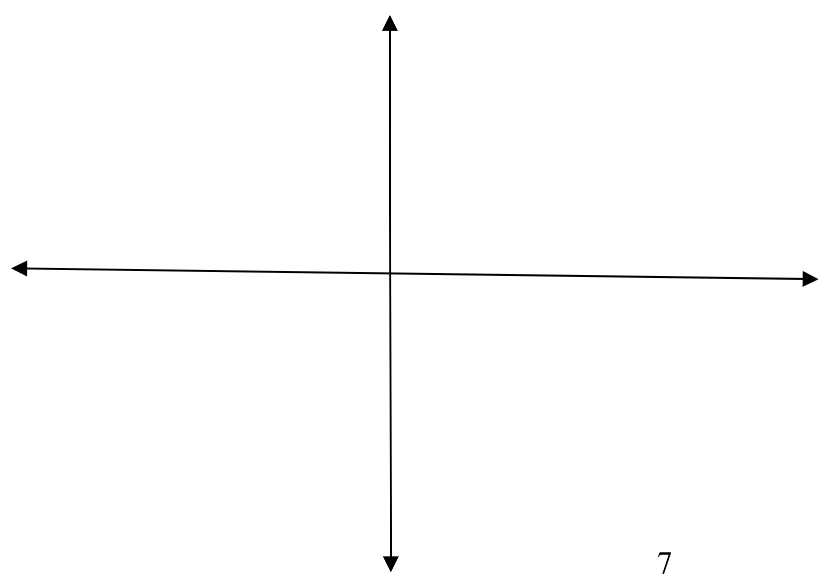
Decreasing: $(-\infty,0)$ and $(2,\infty)$

Relative Extrema: Relative Max at (2, 1)

Concave Down: $(-\infty,0)$ and $(0,3)$

Concave Up: $(3,\infty)$

Points of Inflection: $\left(3,\frac{8}{9}\right)$



Example 4: Use the guide to curve sketching to sketch $f(x) = x^4 - 4x^3$.

Domain of $f(x)$:

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:

$f'(x) = 4x^3 - 12x^2$ Find when $f'(x) = 0$:

Find when $f'(x)$ is undefined:

$f(x)$:

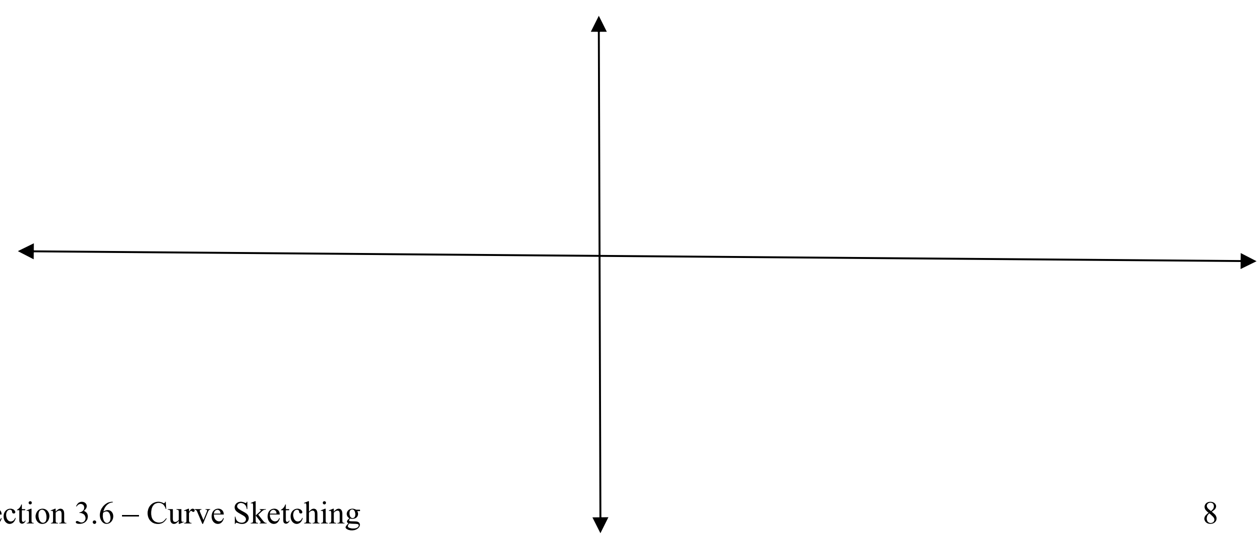
$f'(x)$:

$f''(x) = 12x^2 - 24x$ Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f'(x)$:



Example 5: Sketch the graph of $f\left(x\right)=\dfrac{2x^2}{x^2-1}$.

Domain of $f(x)$:

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:

$f'(x)=\dfrac{-4x}{(x^2-1)^2}$

Find when $f'(x)=0$:

Find when $f'(x)$ is undefined:

$f(x)$:

$f'(x)$:

$f''(x)=\dfrac{12x^2+4}{(x^2-1)^3}$

Find when $f''(x)=0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f'(x)$:

Try this one: Sketch the graph of $f\left(x\right)=x\left(x-1\right)^{\frac{1}{3}}$.

Domain of $f(x)$:

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:

$f'(x)=\dfrac{4x-3}{3(x-1)^{2/3}}$

Find when $f'(x)=0$:

Find when $f'(x)$ is undefined:

$f(x)$:

$f'(x)$:

$f''(x)=\dfrac{4x-6}{9(x-1)^{5/3}}$

Find when $f''(x)=0$:

Find when $f''(x)$ is undefined:

$f(x)$:

$f''(x)$: