Section 4.1: Inverse Functions

A function is said to be one-to-one (1-1) if there are no two distinct numbers in the domain of \( f \) that produce the same value. In other words, two different \( x \) values cannot have the same \( y \) value. If a function has an inverse, then we say it’s invertible.

If a function is 1-1, then it has an inverse function, denoted as \( f^{-1} \), which reverses what the first function did. The domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \).

**Example:** Celsius to Fahrenheit: \( \frac{9}{5}C + 32 = F \)

and

Fahrenheit to Celsius: \( \frac{5}{9}(F - 32) = C \)

Geometrically
**Property of Inverse Functions**
Let $f$ and $g$ be two functions such that $(f \circ g)(x) = x$ for every $x$ in the domain of $g$ and $(g \circ f)(x) = x$ for every $x$ in the domain of $f$ then $f$ and $g$ are inverses of each other.

Given a functions whose graph is known or the given the graph of a function, we can use the Horizontal Line Test to determine if the function is 1-1.

**Example 1:** Is the following graph of $f(x) = 2^{\frac{x}{3}}$ 1-1?

![Graph of $f(x) = 2^{\frac{x}{3}}$](image)

**Example 2:** Is the function, $f(x) = (x + x^3)^\frac{1}{7}$ 1-1?

**Example 3:** Is $f(x) = 3\sin x$ invertible on $\left[ \frac{\pi}{2}, \frac{\pi}{2} \right]$?
A function is **monotonic** if it is always increasing or always decreasing on its domain.

*Recall:*

- If $f'(x) > 0$ on its domain, then $f$ is increasing and; hence, monotonic.
- If $f'(x) < 0$ on its domain, then $f$ is decreasing and; hence, monotonic.

**Theorem:** If $f$ is monotonic, then $f$ is an invertible function.

**Example 4:** Is the following function 1-1? If so, give the equation of the inverse function.

\[
f(x) = \frac{x - 1}{x + 1}
\]
Sometimes it’s too long or too difficult to find the equation of the inverse, yet we may want to know if a function has an inverse or not.

**Example 5:** Is \( f(x) = x^3 + 3x \) invertible?

**Example 6:** Let \( f(x) = x^3 - kx^2 + 2x \). For what values of \( k \) is \( f(x) \) one-to-one?
Math 1431 Section 4.1

**Finding the Derivative of the Inverse Function**

**Theorem:** If $f(x)$ is continuous and invertible then $f^{-1}(x)$ is continuous.

**Theorem:** If $f(x)$ is differentiable (so must be continuous) and invertible, and $f'(x) \neq 0$, then $f^{-1}(x)$ is differentiable.

If $f(a) = b$ and $f'(a) \neq 0$, then 

\[
(f^{-1})'(b) = \frac{1}{f'(a)}.
\]

**Example 7:** For $f(x) = x^3$, we know that $f(2) = 8$. Find $(f^{-1})'(8)$.

**Example 8:** If $f$ is invertible, and $f(1) = 2$, $f(3) = 1$, $f'(1) = 4$, $f'(3) = 5$, $f'(2) = 6$, find $(f^{-1})'(1)$.
Example 9: Given $f(x) = x^5 + 1$, find $(f^{-1})'(33)$ if possible.

Example 10: If $f(x) = \sin x + 5\cos x$, $x \in \left[0, \frac{\pi}{2}\right]$, find $(f^{-1})'(3\sqrt{2})$. 
Example 11: Let \( f(x) = x^5 + 2x^3 + 2x \). The point \((-5, -1)\) is on the graph of \( f^{-1}(x) \). Find \( (f^{-1})'(-5) \), then give an equation for the tangent line to the graph of \( f^{-1}(x) \) at the point \((-5, -1)\).
Try this one: Is the following function 1-1? If so, give the equation of the inverse function.

\[ g(x) = x + \frac{4}{x} \]

Let \( f(x) = \frac{1}{3}x^3 - x^2 + kx \). For what values of \( k \) is \( f(x) \) invertible?