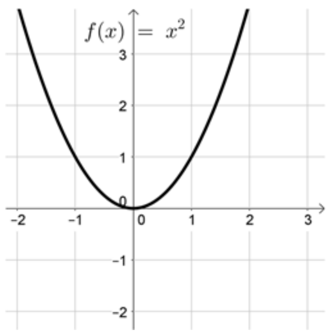
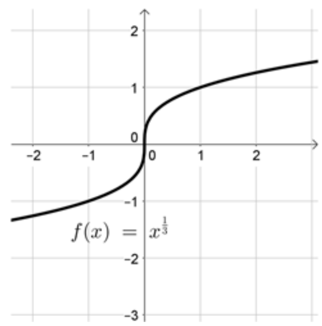
Section 4.1: Inverse Functions

A function is said to be **one-to-one** (1-1) if there are no two distinct numbers in the domain of *f* that produce the same value. In other words, two different *x* values cannot have the same *y* value. If a function has an inverse, then we say it's **invertible**.





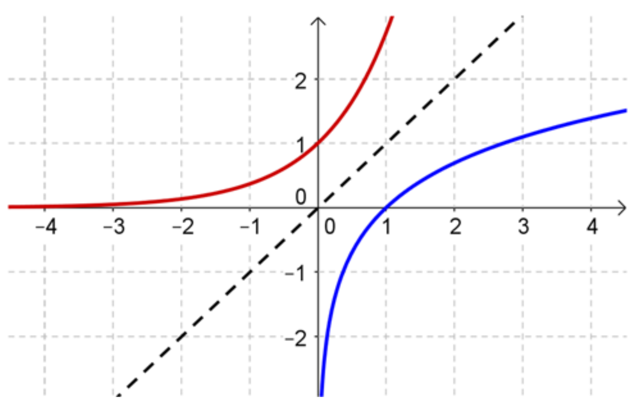
If a function is 1-1, then it has an inverse function, denoted as f^{-1}, which reverses what the first function did. The domain of *f* is the range of f^{-1} and the range of *f* is the domain of f^{-1}.

Example: *Celsius to Fahrenheit:* $\frac{9}{5}C + 32 = F$

and

Fahrenheit to Celsius: $\frac{5}{9}(F - 32) = C$

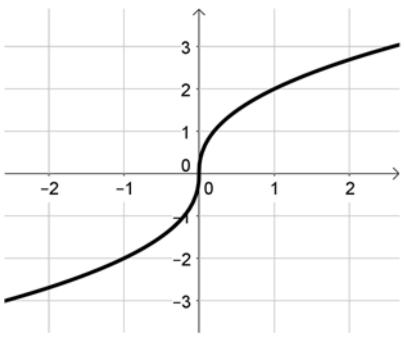
Geometrically



Property of Inverse Functions
Let f and g be two functions such that $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f then ***f* and *g* are inverses of each other.**

Given a functions whose graph is known or the given the graph of a function, we can use the Horizontal Line Test to determine if the function is 1-1.

Example 1: Is the following graph of $f(x) = 2x^{3/7}$ 1-1?



Example 2: Is the funtion, $f(x) = (x + x^2)^7$ 1-1?

Example 3: Is $f(x) = 3\sin x$ invertible on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

A function is **monotonic** if it is always increasing or always decreasing on its domain.

Recall:

- If $f'(x) > 0$ on its domain, then *f* is increasing and; hence, monotonic.
- If $f'(x) < 0$ on its domain, then *f* is decreasing and; hence, monotonic.

Theorem: If *f* is monotonic, then *f* is an invertible function.

Example 4: Is the following function 1-1? If so, give the equation of the inverse function.

$$f(x) = \frac{x-1}{x+1}$$

Sometimes it's too long or too difficult to find the equation of the inverse, yet we may want to know if a function has an inverse or not.

Example 5: Is $f(x) = x^3 + 3x$ invertible?

Example 6: Let $f(x) = x^3 - kx^2 + 2x$. For what values of k is $f(x)$ one-to-one?

Finding the Derivative of the Inverse Function

Theorem: If $f(x)$ is continuous and invertible then $f^{-1}(x)$ is continuous.

Theorem: If $f(x)$ is differentiable (so must be continuous) and invertible, and $f'(x) \neq 0$, then $f^{-1}(x)$ is differentiable.

If $f(a) = b$ and $f'(a) \neq 0$, then $\left(f^{-1}\right)'(b) = \dfrac{1}{f'(a)}$.

Example 7: For $f(x) = x^3$, we know that $f(2) = 8$. Find $\left(f^{-1}\right)'(8)$.

Example 8: If f is invertible, and $f(1) = 2,\quad f(3) = 1,\quad f'(1) = 4,\quad f'(3) = 5,\quad f'(2) = 6$,

find $\left(f^{-1}\right)'(1)$.

Example 9: Given $f(x) = x^5 + 1$, find $\left(f^{-1}\right)'(33)$ if possible.

Example 10: If $f(x) = \sin x + 5\cos x$, $x \in \left[0, \frac{\pi}{2}\right]$, find $\left(f^{-1}\right)'(3\sqrt{2})$.

Example 11: Let $f(x) = x^5 + 2x^3 + 2x$. The point (–5, –1) is on the graph of $f^{-1}(x)$. Find $\left(f^{-1}\right)'(-5)$, then give an equation for the tangent line to the graph of $f^{-1}(x)$ at the point (–5, –1).

Try this one: Is the following function 1-1? If so, give the equation of the inverse function.

$$g(x) = x + \frac{4}{x}$$

Let $f(x) = \frac{1}{3}x^3 - x^2 + kx$. For what values of k is $f(x)$ invertible?