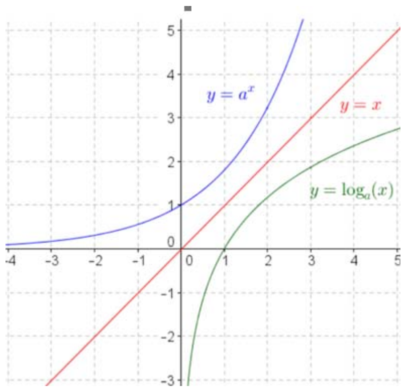
Section 4.3: Logarithmic Function

The exponential function is 1-1; therefore, it has an inverse function. The inverse function of the exponential function with base a is called the **logarithmic function with base *a***. The function $f(x) = \log_a x$ is the **logarithmic function with base *a*** with $x > 0$, $a > 0$ and $a \neq 1$.

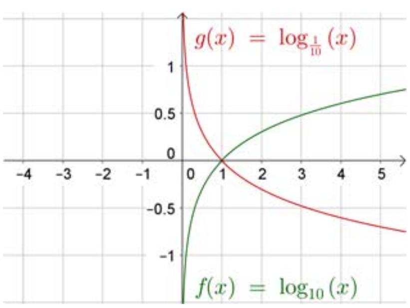
For example, for a > 1:



- If $a = 10$ then $f(x) = \log x$. This is known as the **common base**.
- If $a = e$ then $f(x) = \ln x$. This is known as the **natural logarithm**.
- $\ln(0)$ is undefined.
- $\ln(1) = 0$

Graphs

The orientation of the logarithmic function depends on its base, $0 < a < 1$ or $a > 1$.



Both graphs have a vertical asymptote of $x = 0$ (the y-axis).

Their domain is $(0, \infty)$ and range is $(-\infty, \infty)$.

Example 1: Find the domain of $y = \ln(\ln(x))$.

The logarithmic function with base *a* satisfies:

- $\log_a a^x = x$ for all $x \in (-\infty, \infty)$

  and

- $a^{\log_a x} = x$, for all $x \in (0, \infty)$

Examples:

$\log_2 2^3 =$

$\ln e =$

$e^{\ln 5} =$

$3^{\log_3 x} =$

Converting Expressions

Fact: $y = \log_a x$ is equivalent to $a^y = x$

For example:

$2^5 = 32$ is equivalent to $\log_2(32) = 5$

$\log_4(16) = 2$ is equivalent to $16 = 4^2$

Laws of Logarithms

Product Rule for Logs: $\log_a(x \cdot y) = \log_a x + \log_a y$

Quotient Rule for Logs: $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

Power Rule for Logs: $\log_a\left(x^p\right) = p \cdot \log_a x$

Example 2: Expand using the Laws of Logarithms:

$$\ln\left(\frac{x^3 y}{z\sqrt{x+1}}\right)$$

Example 3: Simplify using the Laws of Logarithms: $\dfrac{1}{3}\log x - \log\left(x^2+1\right) - 4\log z$

Differentiating Natural Logarithmic Function:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

If u is a function of x, then use the chain rule.

Example 4: Differentiate the following functions.

a. $y = \ln\left(4x^3 - 5x\right)$

b. $y = \ln(\sin x)$

c. $f(x) = \ln(\ln(2x))$

Sometimes it's easier to use the rules of logarithms to differentiate a log function.

Example 5: Differentiate $g(x) = \ln\left(\dfrac{x^4}{3x^2 - 1}\right)$. Then find the equation of the tangent line at the point $\left(1, \ln\left(\dfrac{1}{2}\right)\right)$.

Example 6: Determine where $f(x) = x \ln x$ is increasing/decreasing, any relative extrema, concavity, and any points of inflection.

Different Bases

Change of base formula: $\log_a x = \dfrac{\ln x}{\ln a}$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x} \cdot \frac{1}{\ln a}$$

If u is a function of x, then use the chain rule.

Example 7: Find the derivative of each function.

a. $y = \log_2 x$

b. $y = \log_3\left(4x^3 + 2x\right)$

c. $f(x)=\dfrac{\log_9 x}{x^2}$

We know how to find the derivative of functions such as:

- x^5 ---use the power rule
- 5^x ---use exponential rule

But how about x^x? We will use a method called Logarithmic Differentiation.

Example 8: Find the derivative of $y = x^{\tan x}$.

Step 1: Take natural logs of both sides of the equation.

Step 2: Use any rules of logs to simplify the equation.

Step 3: Take the derivative of both sides of the equation.

Step 4: Solve for y'. *Remember what y was equal to originally.*

Example 9: Find the derivative of $y = x^{2x+1}$.

Step 1: Take natural logs of both sides of the equation.

Step 2: Use any rules of logs to simplify the equation.

Step 3: Take the derivative of both sides of the equation.

Step 4: Solve for y'. *Remember what y was equal to originally.*

Try these:

Find the derivative of: $y = \log\left(\cos\left(4x\right)\right)$.

Let $f\left(x\right) = x\ln\left(\cos\left(2x\right)\right)$, find $f'\left(\pi\right)$.

Determine where $f(x) = 2x^2 \ln\left(\frac{x}{4}\right)$ is increasing/decreasing.

Find the points of inflection for the function $f(x) = 4x^2 \ln\left(\frac{x}{4}\right)$.

Find the derivative of $f\left(x\right) = e^{2x}\ln(2x)$.

Find the derivative of $f\left(x\right) = \ln\left(5^{-x^2+x}\right)$.

Find the slope of the tangent line to the curve $y = \left(2 + \cos x\right)^{4+\sin x}$ at $x = 2\pi$.