Section 4.5: Hyperbolic Functions

We will now look at six special functions which are defined using the exponential functions e^x and e^{-x}. These functions have similar names, identities, and differentiation properties as the trigonometric functions. While the trigonometric functions are closely related to circles, the hyperbolic functions earn their names due to their relationship with hyperbolas. The first two functions we will define are the hyperbolic sine and hyperbolic cosine functions.

Hyperbolic Cosine: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic Sine: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

The four remaining hyperbolic functions are defined as you would expect given their names. That is:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

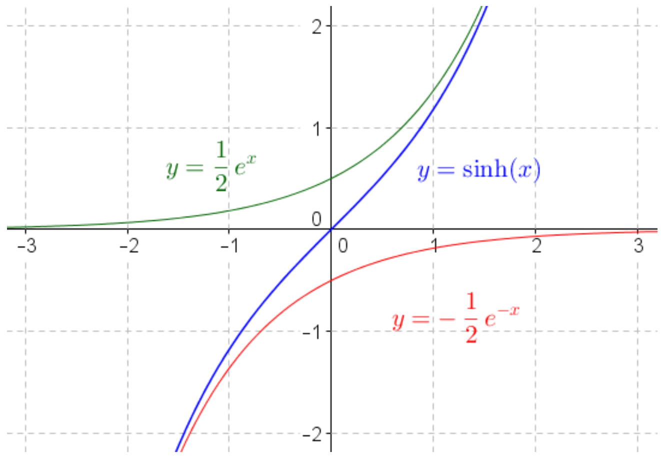
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

- These functions are not about trigonometry nor angles, they're about e^x.
- These functions act like sine, cosine, etc, so naming them this way makes it easier to remember their properties.

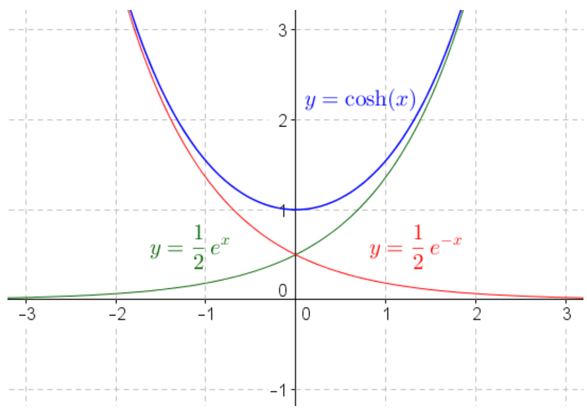
Graphs of Hyperbolic Sine and Hyperbolic Cosine

Below we have the graph of the **hyperbolic sine function**, as well as the two exponential functions used to define it.



- The function is an ODD function, i.e. *sinh* (–x) = – *sinh* x .
- Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
- A clear point on the graph is (0,0) or f(0) = 0.

Next we will look at the graph of the **hyperbolic cosine function** below, as well as the two exponential functions used to define it.



- The function is an EVEN function, i.e. *cosh* (–x) = *cosh* x.
- Domain: $(-\infty, \infty)$; Range: $[1, \infty)$
- A clear point on the graph is (0,1) or f(0) = 1.

Example 1: Given $f(x) = \cosh(2x)$, find $f(\ln(3))$.

Recall: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Some Important Identities

- $\cosh x + \sinh x = e^x$
- $\cosh x - \sinh x = e^{-x}$
- $\cosh^2 x - \sinh^2 x = 1$

These can easily be proven by using the definition of each function cosh(x) and sinh(x).

We should notice this identity is similar to the Pythagorean trigonometric identity satisfied by the sine and cosine functions, namely $\cos^2 t + \sin^2 t = 1$. Also, we can see how these functions earn the hyperbolic distinction by setting $x = \cosh(t)$ and $y = \sinh(t)$ and substituting into $x^2 - y^2 = 1$, which is the equation of a hyperbola.

Derivatives of the Hyperbolic Functions

$\dfrac{d}{dx} \sinh x = \cosh x$ $\qquad \dfrac{d}{dx} \cosh x = \sinh x$

$\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$ $\qquad \dfrac{d}{dx} \operatorname{sech} x = -\tanh x \operatorname{sech} x$

$\dfrac{d}{dx} \coth x = -\operatorname{csch}^2 x$ $\qquad \dfrac{d}{dx} \operatorname{csch} x = -\coth x \operatorname{csch} x$

If the input is not just x alone, you'll need to use the chain rule. Other times, you may need to use the quotient or product rule or a combination of multiple rules.

Example 2: Differentiate: $f(x) = \cosh\left(\ln(4x^3)\right)$

Example 3: Differentiate: $y = \arctan(\sinh(x))$

Example 4: Differentiate: $\frac{d}{dx}\left[\frac{cosh(x)}{1+sinh(x)}\right]$

Example 5: Differentiate: $y = e^{-x}\cosh x$

Example 6: Differentiate: $y = (\sinh x)^x$

The hyperbolic functions arise in many problems of mathematics and mathematical physics in which integrals involving $\sqrt{1+x^2}$ arise (whereas the circular functions involve $\sqrt{1-x^2}$). For instance, the hyperbolic sine arises in the gravitational potential of a cylinder and the calculation of the Roche limit. The hyperbolic cosine function is the shape of a hanging cable (the so-called catenary). The hyperbolic tangent arises in the calculation of and rapidity of special relativity. All three appear in the Schwarzschild metric using external isotropic Kruskal coordinates in general relativity. The hyperbolic secant arises in the profile of a laminar jet. The hyperbolic cotangent arises in the Langevin function for magnetic polarization.

http://mathworld.wolfram.com/HyperbolicFunctions.html

Try these in lab:

1. Differentiate: $y = \cosh^3(e^x)$

2. Find the extreme values of $f(x) = -\frac{17}{2}\cosh x + \frac{15}{2}\sinh x$