Section 5.3: L'Hospital's Rule

Recall that in Section 1.3 we first saw the indeterminate form of 0/0. We were able to evaluate them further by either factoring, rationalizing, graphing the functions, etc, but that sometimes is not enough to calculate the limit. Where these elementary methods are difficult to apply, we may be able to use L'Hospital's rule. This rule makes the calculation much easier.

L'Hospital's Rule (for the indeterminate form 0/0)

Let f, g be two functions differentiable on an open interval I that contains c and suppose that $g'(x) \neq 0$ on I (except possibly at c). If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$, provided that the limit on the right hand side exists.

On the other hand, if $\frac{f'(x)}{g'(x)} \to \pm\infty$ as $x \to c$, then $\frac{f(x)}{g(x)} \to \pm\infty$ as well.

Note: You are NOT using the quotient rule here!

In some cases, applying L'Hospital's rule once still produces an indeterminate form. We can apply the rule several times as long as the conditions of the rule are met. As soon as we get a limit that is not indeterminate, we should stop.

Example 1: Evaluate each limit, if it exists.

a. $\lim_{x\to 4} \frac{x-4}{x^4 - 256}$

b. $\lim_{x\to 0} \frac{10\arctan x}{x}$

c. $\lim_{x\to \pi} \frac{1+\cos(x)}{x-\pi}$

d. $\lim_{x\to 0} \frac{5e^{3x}-5}{x^2}$

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L'Hospital's Rule (for the form ∞ / ∞)

Let f, g be two functions differentiable on an open interval I that contains c and suppose that $g'(x) \neq 0$ on I (except possibly at c). If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$, provided that the limit on the right hand side exists.

On the other hand, if $\frac{f'(x)}{g'(x)} \to \pm\infty$ as $x \to c$, then $\frac{f(x)}{g(x)} \to \pm\infty$ as well.

We saw some examples of this type in Section 1.3, such as $\lim\limits_{x \to \infty} \frac{x^2 + 3x}{x^4 - 2}$. For more complicated functions, we will use L'Hospital's rule to evaluate such limits. But first recall that $\lim\limits_{x \to \pm\infty} \frac{1}{x} = 0$. So we say, $\frac{1}{\infty} \to 0$ and $\frac{1}{0} \to \infty$.

Example 2: Evaluate each limit, if it exists.

a. $\lim\limits_{x \to \infty} \frac{5\sqrt{1 + x^2}}{6x^2}$

b. $\lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\sec x}{1 + \tan x}$

c. $\lim\limits_{x \to \infty} \left(\dfrac{\ln^2 x}{x} \right)$

Other indeterminate forms exist. Let's see what we do…

Indeterminate Form: $0 \cdot \infty \Rightarrow$ **Rewrite the function so that you get 0/0 or** ∞ / ∞ **.**

Generally in the case of $0 \cdot \infty$, you'll need to rewrite the expression A$\cdot$ B as $\dfrac{A}{1/B}$ or $\dfrac{B}{1/A}$ so

that it either becomes 0/0 or ∞ / ∞ and then you'll be able to use L'Hospital's rule. Deciding on which form to rewrite it into, remember you'll need to take their derivatives, reciprocate the one that's easier to differentiate.

The $0 \cdot \infty$ **Case**

Example 3: Evaluate the limit, if it exists. $\displaystyle\lim_{x \to \infty} \left(x^2 \sin \frac{1}{x} \right)$

Indeterminate Forms: $\infty - \infty \Rightarrow$ **Rewrite the function so that you get 0/0 or** ∞ / ∞**.**

Example 4: Evaluate the limit, if it exists.

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

Indeterminate Powers 0^0, 1^∞ **and** ∞^0 **-- Use a logarithmic "trick".** These indeterminate powers arise if the function is of the form $[f(x)]^{g(x)}$, where $f(x) > 0$. We'll apply the natural logarithm function.

First recall a couple of logarithmic rules: $e^{\ln x} = x$ and $\ln x^p = p \ln x$.

The 1^∞ **Case**

Example 5: Evaluate the limit, if it exists. $\lim_{x \to 1} x^{\frac{3}{x-1}}$

The 0^0 Case

Example 6: Evaluate the limit, if it exists.

$$\lim_{x\to 0^+} (\sin x)^x$$

The ∞^0 Case

Example 7: Evaluate the limit, if it exists.

$$\lim_{x\to\infty}\left(x^4+1\right)^{\frac{1}{\ln x}}$$

Keep in mind: When evaluating limits, you should ALWAYS begin by substituting, as you may not even need to use L'Hospital's rule!

In some cases you may even be able to use: $\lim\limits_{x\to 0}\frac{\sin(ax)}{bx}=\frac{a}{b}$

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Indeterminate forms: $0/0$ or ∞ / ∞

$\Rightarrow$ **Use L'Hospital's rule.**

Indeterminate Forms: $0 \cdot \infty$ or $\infty - \infty$

$\Rightarrow$ **Rewrite the function so that you get 0/0 or ∞ / ∞ then you can use L'Hospital's rule.**

Indeterminate Powers: 0^0, 1^∞ or ∞^0

$\Rightarrow$ **Use a logarithmic "trick" to see which indeterminate form you have and only use L'Hospital's rule when you get** $0/0$ or ∞ / ∞ **.**

The following cases are "determinate".

$\infty + \infty \to \infty$

$-\infty - \infty \to -\infty$

$0^\infty \to 0$

$0^{-\infty} \to \infty$

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Try these:

$$\lim_{x\to 0} \frac{4+4x-4e^x}{5x(e^x-1)}$$

$$\lim_{x\to 0^+} \frac{2^x-1}{\sqrt{x}}$$

$$\lim_{x\to 0} \frac{x-\tan x}{x-\sin x}$$

$$\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x$$

$$\lim_{x\to \infty} \cos\left(\frac{7}{x}\right)^x$$