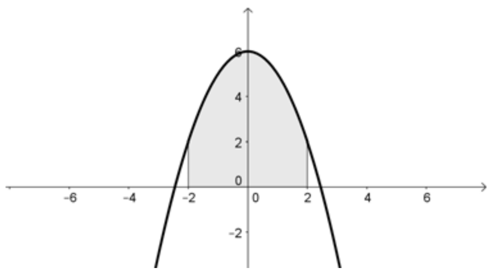
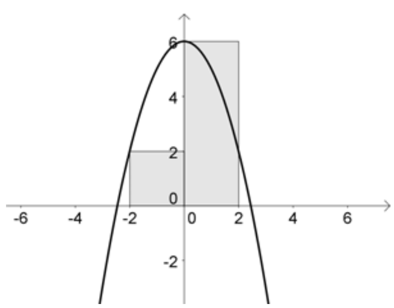
Section 6.1: Definite Integral

Suppose we want to find the area of a region that is not so nicely shaped. For example, consider the function shown below. The area below the curve and above the *x* axis cannot be determined by a known formula, so we'll need a method for approximating the area.



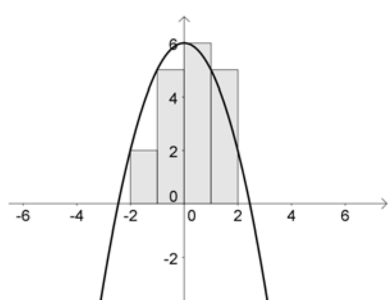
Suppose we want to find the area under the parabola and above the *x* axis, between the lines $x = 2$ and $x = -2$.

We can approximate the area under the curve by subdividing the interval [-2, 2] into smaller intervals and then draw rectangles extending from the *x* axis up to the curve. Suppose we divide the region into two parts and draw two rectangles. We can find the area of each rectangle and add them together. That will give us an approximation of the area under the curve.

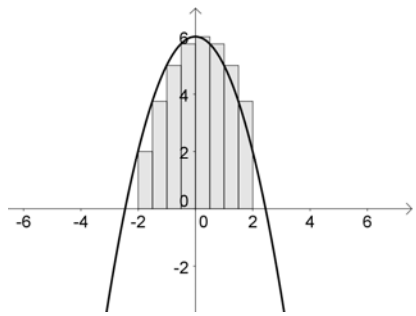


This would not give a very good approximation, as a large region in Quadrant 2 will be left out in the approximation of the area, and a large region in Quadrant 1 will be included and should not be.

Now suppose we increase the number of rectangles that we draw to four. We'll find the area of each of the four rectangles and add them up. Here's the graph for this situation.



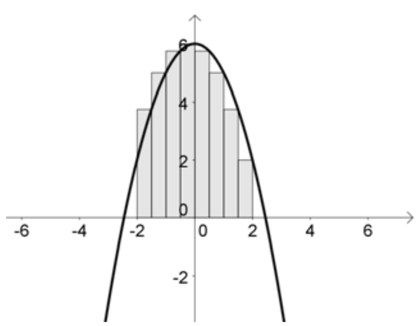
The approximation will be more accurate, but it still isn't perfect. Let's increase the number of rectangles to 8:



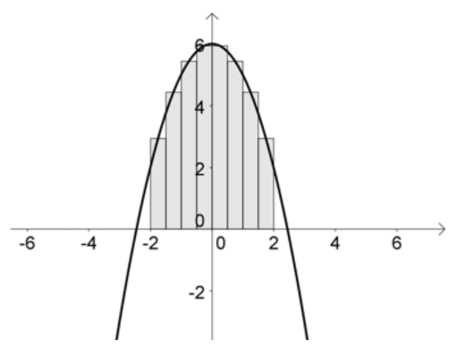
As we add more and more rectangles, the accuracy improves. We're still not to an exact area, but the area we'd find using more rectangles is clearly more accurate than the area we'd find if we just used 2 rectangles.

Suppose we let the number of rectangles increase without bound. If we do this, the width of each rectangle becomes smaller and smaller, as the number of rectangles approaches infinity, there will be no area that is included that shouldn't be and none left out that should be included

Using left endpoints is not the only option we have in working these problems. We can also use right endpoints or midpoints. The first graph below shows the region with eight rectangles, using right endpoints. The second graph below shows the region with eight rectangles, using midpoints.



Right Endpoints



Midpoints

To get an exact area, we would need to let the number of rectangles increase without bound:

$$A = \lim_{n\to\infty} \left[f\left(x_1\right) + f\left(x_2\right) + \ldots + f\left(x_n\right) \right] \cdot \Delta x$$

This last computation is quite difficult, we will not work problem of this type. Instead, we will use a limited number of rectangles in the problems that we work.

The process we are using to approximate the area under the curve is called "finding a Riemann sum." These sums are named after the German mathematician who developed them.

Approximating the area under a curve given the type of Riemann sums

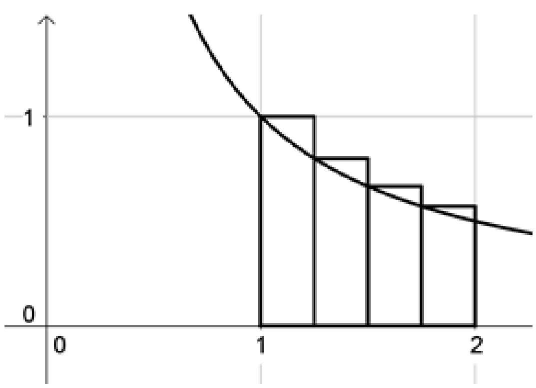
1. **Start by finding the width of each rectangle**. A **partition** of a closed interval $[a,b]$ is a finite subset of $[a,b]$ that contains the points a and b. The lengths of these subintervals may or may not be equal. If the lengths are equal, it is called a "**regular partition**" and $\Delta x = \dfrac{b-a}{n}$.

2. **Now find the height of the rectangles**. Use the appropriate point in each subinterval to compute the value of the function at each of these points (gives the heights of the rectangles).

3. **Find the area of each rectangle and add them up.**
$S^*(P) = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n$

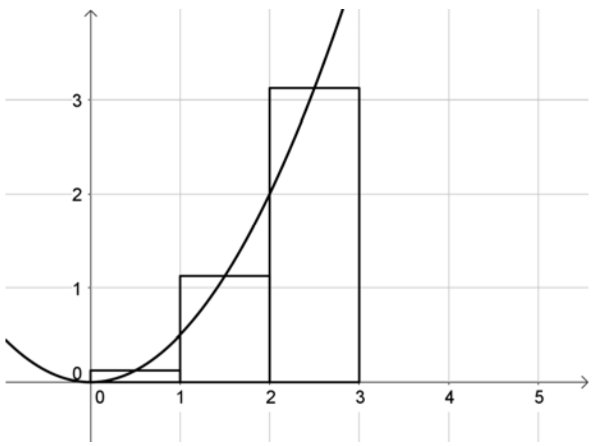
Example 1: For each problem, approximate the area under the curve over the given interval, with the given number of partitions and type of Riemann sums.

a. Given $f(x) = \dfrac{1}{x}$, use left endpoints from $[1,2]$ with n = 4.



Now try it again, but use the right endpoints of each subdivision.

b. Given $f(x) = 0.5x^2$, use midpoints from $[0,3]$ with n = 3.



We can also approximate this area by using Upper Sums or Lower Sums.

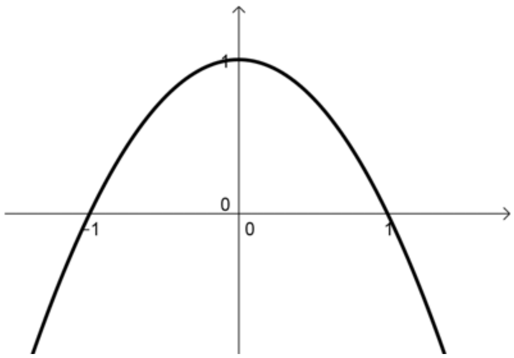
Upper Sums and Lower Sums

Let f be a continuous function on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$.

The **upper sum of *f*** is $U_f(P) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + \ldots + M_n\Delta x_n$. The value M_i is the maximum value of the function for a partition.

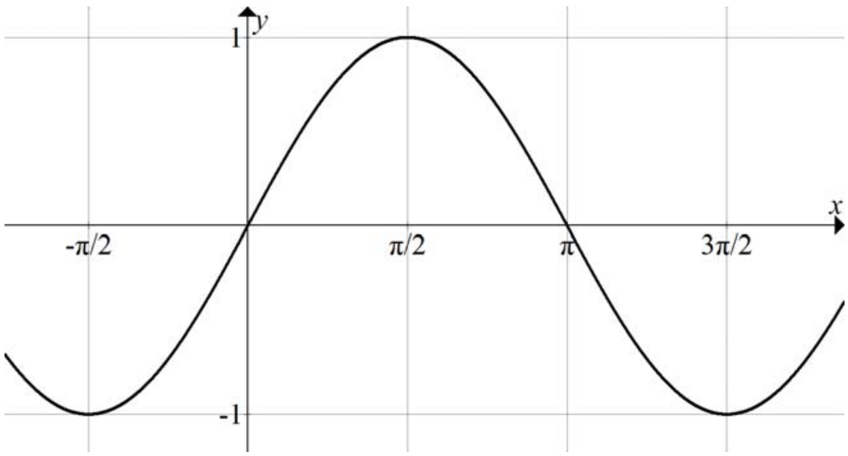
The **lower sum of *f*** is $L_f(P) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + \ldots + m_n\Delta x_n$. The value m_i is the minimum value of the function for a partition.

Example 2: Find the upper sum for $f(x) = 1 - x^2$, $x \in [-1, 1]$ if the partition is $P = \left[-1, \frac{-3}{4}, \frac{1}{2}, 1\right]$.



Keep in mind that the max or min does not have to happen at an endpoint of a subdivision. You'll need to graph the original function to figure this out.

Example 3: Find $L_f(P)$ given $f(x) = \sin x$ over $[0, \pi]$ and $P = \left[0, \frac{\pi}{4}, \frac{2\pi}{3}, \pi\right]$.



As the number of partitions are added, the upper sum tends to get smaller. As the number of partitions are added, the lower sum tends to get bigger. The number they meet at is called the **definite integral**.

For a function f which is continuous on $[a,b]$, there is one and only one number that satisfies the inequality

$$L_f(P) \le I \le U_f(P), \text{ for all partitions } P \text{ of } [a,b].$$

This unique number *I* is called the **definite integral** (or just the integral) of *f* from *a* to *b* and is denoted by $\int_a^b f(x)dx$.

We read $\int_a^b f(x)dx$ as: "the integral from a to b of f with respect to x".

The component parts have these names:

- $\int$: the integral sign
- a: lower limit of integration
- b: upper limit of integration
- $f(x)$: integrand
- dx indicates the independent variable in discussion and denotes the widths are getting smaller.

The procedure of calculating the integral is called **integration**.

In general, the integral can be negative, positive or zero.

Important Properties of a Definite Integral

Assume that f and g are continuous functions.

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

When we defined the definite integral $\int_a^b f(x)dx$*, we assumed that* $a < b$*. However, the integral makes sense even if* $a > b$*. We can integrate from right to left.*

3. $\int_a^b \left[f(x) \pm g(x) \right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

4. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant number.

Example 4: Given $\int_1^5 f(x)\,dx = 10$, $\int_1^5 g(x)\,dx = 4$, $\int_5^{10} f(x)\,dx = 6$, $\int_6^{10} f(x)\,dx = 8$. Evaluate the following integrals.

a. $\int_{10}^{10} f(x)\,dx$

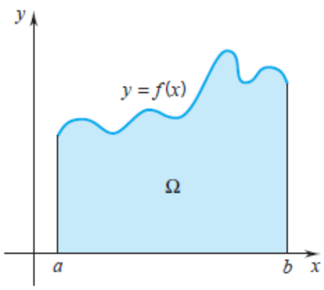
b. $\int_1^5 \left[f(x) - 2g(x) \right] dx$

c. $\int_5^6 f(x)\,dx$

d. $\int_5^1 g(x)\,dx$

Area Under the Graph of a Nonnegative Function

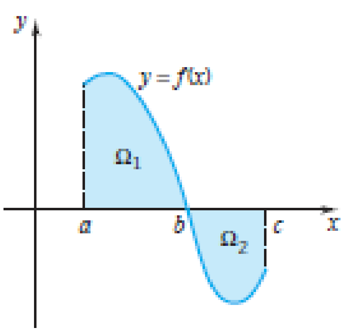
If $y = f(x)$ is nonnegative and integrable over the interval $[a,b]$, then the **area under the curve** $y = f(x)$ over $[a,b]$ is given by $\int_a^b f(x)\,dx \geq 0$.



$$\text{area of } \Omega = \int_a^b f(x)\,dx.$$

If the curve is sometimes negative, then one can split the region into pieces using the roots of the function as the limits of the integral.

Consider the function whose graph is given below:



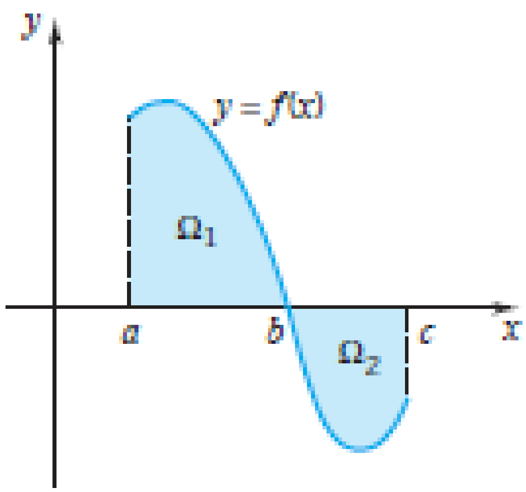
Theorem: If f is continuous on $[a,b]$ and if $a < b < c$, then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

For the function shown above, Area of $\Omega_1 = \int_a^b f(x)dx$

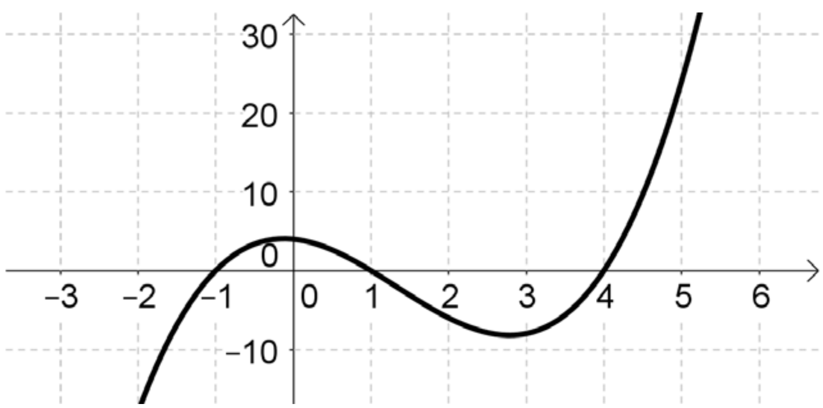
and

Area of $\Omega_2 = \left|\int_b^c f(x)dx\right| = -\int_b^c f(x)dx$.

Example 5: Given the graph of f, if the area of Ω_1 is 12 and area of Ω_2 is 8, find $\int_a^c f(x)\,dx$.



Example 6: Given $\int_{-1}^{1} f(x)\ dx = 5$, $\int_{1}^{4} f(x)\ dx = -12$, $\int_{4}^{5} f(x)\ dx = 4$. Find the **area** between the curve and the x-axis from x = -1 to x = 5.



Other Properties of Definite Integral

Assume that f and g are continuous functions.

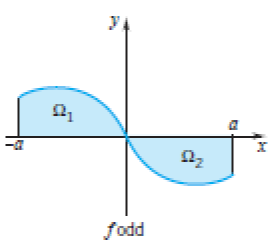
1. $\int_a^b k\ dx = k(b-a)$, where k is a constant number.

2. If $f(x) \geq g(x)$ over $[a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

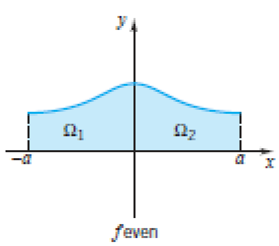
3. If $m \leq f(x) \leq M$ over $[a,b]$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

4. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

5. If f is an odd function, then $\int_{-a}^a f(x)dx = 0$.

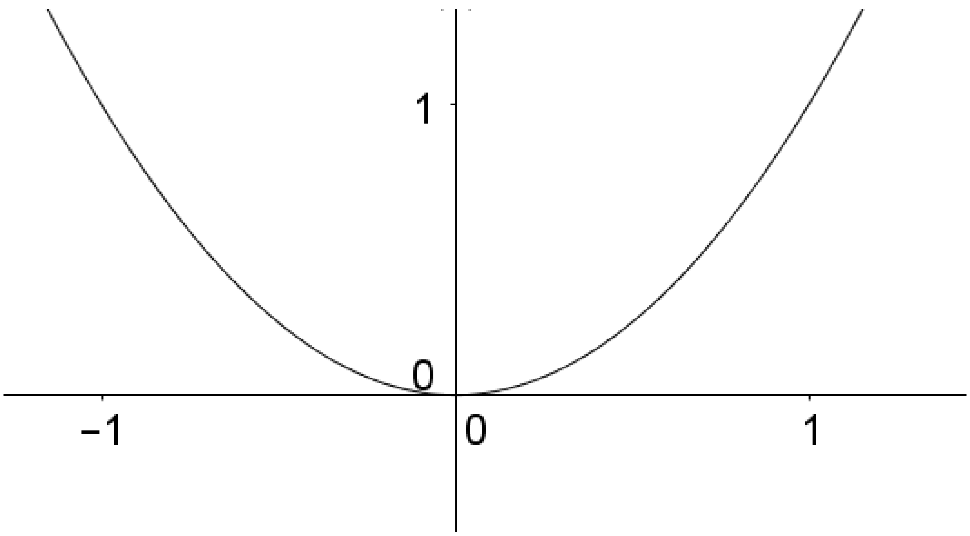


If f is an even function, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.



Try this one: Find the lower sum for $f(x) = x^2$, $x \in [-1, 1]$ if the partition is

$$P = \left[-1, \frac{-3}{4}, \frac{1}{4}, \frac{1}{2}, 1\right]$$



Try this one: Estimate $\int_0^6 3x^2 dx$ by using left endpoint estimates, where n = 6.

Try this one: Given $\int_0^1 f(x)\,dx = 3, \int_0^3 f(x)\,dx = 5, \int_3^6 f(x)\,dx = 9,$ find $\int_6^1 f(x)\,dx.$