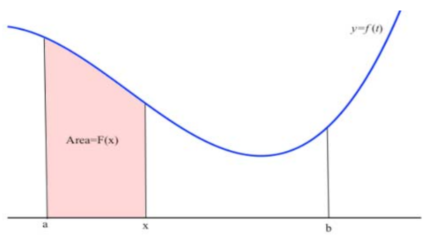
Section 6.2: The Fundamental Theorem of Calculus

Recall from the previous section that:

For a function f which is continuous on $[a,b]$, there is one and only one number that satisfies the inequality $L_f(P) \le I \le U_f(P)$, for all partitions P of $[a,b]$. This unique number *I* is called the **definite integral** (or just the integral) of *f* from *a* to *b* and is denoted by $\int_a^b f(x)dx$. This number can be positive, negative or zero.

Let f be a continuous function over the interval $[a,b]$. Define a new function by $F(x) = \int_a^x f(t)\, dt$. Here, the upper limit x varies between a and b.

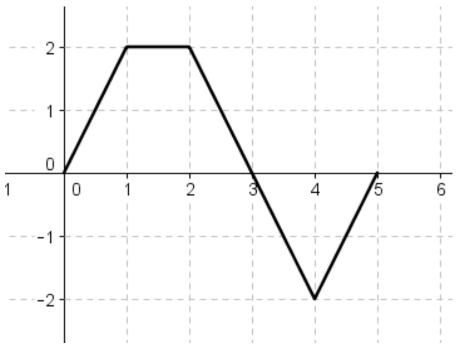
If f happens to be a nonnegative function, then $F(x)$ can be seen as the "area under the graph of f from a to x". We can think of $F(x)$ as the **"accumulated area"** function.



So for example,

- if $F(1) = \int_0^1 f(t)\, dt$ then this is "area under the function *f* from x = 0 to x = 1".
- if $F(5) = \int_0^5 f(t)\, dt$ then this is "area under the function *f* from x = 0 to x = 5".

Example 1: If f is the function whose graph is given below, and $F(x)=\int_0^x f(t)\,dt$, find $F(5)$.



Fundamental Theorem of Calculus Part 1

If f is a continuous function over the interval $[a,b]$, then the function $F(x)=\int_a^x f(t)\,dt$

is continuous on $[a,b]$ and differentiable on (a,b). Moreover, $F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, for all x in

(a,b). *Note: If a = 0 then* $F(0)=\int_0^0 f(t)\,dt = 0$.

Example 2: Let f be a continuous function satisfying $x^3+x^2-x=\int_1^x f(t)\,dt$.

a. State F(x).

b. Find $F'(x)=f(x)$.

c. Find $F''(x)=f'(x)$.

Hence,

- where f is positive, F is increasing
- where f is negative, F is decreasing
- where f is zero, F has possible max, min or inflection point
- where f is increasing, F is concave up
- where f is decreasing, F is concave down

Example 3: Find $F'(x)$ given $F(x)=\int_0^x (t^2+2t)\ dt$.

Example 4: Find $\frac{d}{dx}\int_2^x 5\cos(2s)\ ds$. Then find $F'(4\pi)$.

Example 5: Find $\frac{d}{dx}\int_x^0 \sqrt{3t+1}\ dt$.

Recall the following definite integral property from Section 6.1: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Other times if the upper limit is not just x, we'll need to use the chain rule. That is,

$$\frac{d}{dx}\left(\int_a^{v(x)} f(t)dt\right) = f(v(x))\cdot v'(x)$$

Example 6: Find $F'(x)$ given $F(x) = \int_{\pi}^{5x^2} \frac{1}{1+t^2}\, dt$.

And yet, other times, if both limits of integration are a functions of x we'll apply the following rule:

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)dt\right) = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)$$

Example 7: Given $F(x) = \int_{x^2}^{2x^3} t^2 dt$, find $F'(x)$.

Let f be a continuous function over the interval $[a,b]$. A function F is called an **antiderivative** for f over the interval $[a,b]$ if F is continuous on $[a,b]$ and $F'(x) = f(x)$ for all x in (a,b).

Example 8: Give a few antiderivatives for $f(x) = 2x$.

Theorem: Fundamental Theorem of Calculus Part 2

Let f be a continuous function over the interval $[a,b]$. If G is any antiderivavite for f over the interval $[a,b]$, then $\int_a^b f(x)\ dx = G(b) - G(a)$.

Example 9: Calculate the definite integral $\int_1^5 2x\ dx$ using FTOC.

Try this one: Let the function F be defined by $F(x)=\int_a^x (t^2-4t^3)\ dt$.

a. Find any critical numbers for F.

Recall: $F'(x)=\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$

b. Discuss the concavity of F.