Math1431   Section 3.1

Example 4: A stone is thrown straight up from a height of 6 feet with initial velocity of 112 feet per second.

\[ h(t) = -16t^2 + 112t + 6 \]

a. When will it hit the ground?  

\[ h(t) = 0 \]

\[ -16t^2 + 112t + 6 = 0 \]

\[ -2 \left( 8t^2 - 56t - 3 \right) = 0 \]

\[ t = \frac{56 \pm \sqrt{(56)^2 - 4(-2)(-3)}}{16} \]

\[ t \geq 0 \]

b. Determine the greatest height the stone will reach.  

Velocity is zero

\[ h'(t) = -32t + 112 \]

\[ -32t + 112 = 0 \]

\[ t = \frac{7}{2} \]

\[ h\left(\frac{7}{2}\right) = -16\left(\frac{7}{2}\right)^2 + 112\left(\frac{7}{2}\right) + 6 \]
Math1431 Section 3.1

The Derivative as a Rate of Change

In this section, we’ll work through some problems where we will be interested in answer questions like,

“What is the rate of change of __________ with respect to __________.”

or

“How fast is __________ changing/decreasing/increasing with respect to __________.”

Recall: \( \frac{dy}{dx} \) is read, “The derivative of y with respect to x.” We will translate words into math symbols.

You will need to know your geometric formulas for circles, squares, rectangles, triangles, spheres, cones, cylinders, cubes, etc., even Pythagorean Theorem.

Steps to solving related rate problems:

- Draw a “picture”.
- What do you know?
- What do you need to find?
- Write an equation involving the variables whose rates of change either are given or are to be determined. (This is an equation that relates the parts of the problem.)
- Implicitly differentiate both sides of the equation with respect to time. This FREEZES the problem.
- Solve for what you need.

Question 23: Two airplanes left the same airport traveling in opposite directions. If one airplane average 400 miles per hour and the other airplane averages 250 miles per hour, in how many hours will the distance between the two planes be 1625 miles?

a. 2.5
b. 4
c. 5
d. 10.8
e. None of the above
Example 5: Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 feet?

Given:
\[
\frac{dr}{dt} = \frac{2\text{ ft}}{\text{sec}}
\]

Question:
\[
\frac{dA}{dt} = ??
\]

\[
A = \pi r^2
\]

Relation:
Find derivative with respect to time

\[
\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}
\]

\[
= \pi \cdot 2 \cdot (60) \cdot \frac{2 \text{ ft}}{\text{sec}}
\]

\[
= 240\pi \frac{\text{ft}^2}{\text{sec}}
\]

Question 33: What is the formula for the volume of a sphere?

a. \( V = \pi r^2 \)  
Area of a circle

b. \( V = 2\pi r \)  
Circumference of a circle

c. \( V = \frac{1}{3} \pi r^3 h \)  
Volume of a cone

d. \( V = \frac{4}{3} \pi r^3 \)  
Volume of a sphere

e. \( V = \pi r^2 h \)  
Volume of a cylinder
Math 1431  Section 3.1

Example 6: Suppose a spherical balloon is inflated at the rate of 10 cubic centimeters per minute. How fast is the radius increasing when the radius is 5 centimeters?

Given: 

\[ \frac{dv}{dt} = 10 \text{ cm}^3 / \text{min} \]

Question: 

\[ \frac{dr}{dt} = ? \]

Relation:

\[ V = \frac{4}{3} \pi r^3 \]

\[ \frac{dv}{dt} = \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt} \]

\[ 10 \text{ cm}^3 / \text{min} = \frac{4}{3} \pi \cdot 3(5 \text{ cm})^2 \cdot \frac{dr}{dt} \]

\[ 10 = 100 \pi \frac{dr}{dt} \]

\[ \frac{dr}{dt} = \frac{10}{100\pi} = \frac{1}{10\pi} \text{ cm/min} \]

\[ \frac{dv}{dt} \left[ (r^3) \right] = 3(r^2) \cdot \frac{dr}{dt} \]
Example 7: A point moves along the curve \( y = 4x^2 + 1 \) in such a way that the \( y \) value is decreasing at the rate of 2 units per second. At what rate is \( x \) changing when \( x = 5 \)?

Given: \( \frac{dy}{dt} = -2 \)

Question: \( \frac{dx}{dt} = ? \)

Relation: \( y = 4x^2 + 1 \)

\( \frac{dy}{dt} (y) = \frac{dy}{dt} (4x^2 + 1) \)

\( \frac{dy}{dt} = 8x \cdot \frac{dx}{dt} \)

\( -2 = 8(5) \cdot \frac{dx}{dt} \)

\( \frac{-1}{20} = \frac{dx}{dt} \)
Example 8: A 13-foot ladder is leaning against a vertical wall. If the bottom of the ladder is being pulled away from the wall at the rate of \(2\) feet per second, how fast will the top of the ladder be moving along the wall when the bottom of the ladder is 12 feet from the wall?

\[
\begin{align*}
x &= 12 \\
x &= \frac{2 \text{ ft}}{\text{sec}}
\end{align*}
\]

Relation:
\[
x^2 + y^2 = c^2
\]
\[
x^2 + y^2 = 13^2
\]

Given:
\[
\frac{dx}{dt} = \frac{2 \text{ ft}}{\text{sec}}
\]

Question:
\[
\frac{dy}{dt} = ??
\]

\[
2x \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0
\]

\[
2(12)(2) + 2(5) \cdot \frac{dy}{dt} = 0
\]
\[
10 \cdot \frac{dy}{dt} = -48
\]
\[
\frac{dy}{dt} = \frac{-48}{10} = \frac{-24}{5}
\]

\[
\frac{dy}{dt} = \frac{-48}{10} = \frac{-24}{5}
\]
Example 9: A 6-foot man is walking towards a 25 foot lamp post at the rate of 10 ft/sec. How fast is the length of his shadow changing when he is 20 feet from the lamp post? 

Hint: This example will use similar triangles from Geometry.

Picture:

Given: 
\frac{dx}{dt} = 10 \text{ ft/sec}

Question: 
\frac{dy}{dt} = ???

\begin{align*}
h_1 &= 25 \\
h_2 &= 6 \\
x &= 20
\end{align*}

Relation:
\begin{align*}
h_1 \cdot y &= h_2 (x+y) \\
25y &= 6 (x+y) \\
25y &= 6x + 6y \\
19y &= 6x \\
19 \frac{dy}{dt} &= 6 \frac{dx}{dt}
\end{align*}

\begin{align*}
19 \frac{dy}{dt} &= 6 \cdot 10 \\
\frac{dy}{dt} &= \frac{60}{19} \text{ ft/sec}
\end{align*}
Example 10: On a morning when the sun will pass directly overhead, the shadow of a 60-ft tower on level ground is 45 feet long. At the moment in question, the angle the sun makes with the ground is increasing at the rate of \( \frac{\pi}{600} \) radians/minute. Find the rate of change of the shadow's length.

Given:
\[
\frac{d\theta}{dt} = \frac{\pi}{600}
\]

Question:
\[
\frac{dx}{dt} = ?
\]

Relation:
\[
\tan \theta = \frac{60}{x} \quad \text{(Moving Parts)}
\]
\[
\tan \theta = \frac{60}{x} \quad \Rightarrow \quad x = \frac{60}{\tan \theta}
\]

\[
\sec^2(\theta) \frac{d\theta}{dt} = -\frac{60}{x^2} \cdot \frac{dx}{dt}
\]

\[
\left( \frac{5}{3} \right)^2 \cdot \frac{\pi}{600} = -\frac{60}{(45)^2} \cdot \frac{dx}{dt}
\]

\[
\frac{25}{9} \cdot \frac{\pi}{600} \cdot \frac{(45)^2}{-60} = \frac{dx}{dt}
\]

\[
3 : 4 : 5
\]

\[
\sec \theta = \frac{75}{45} = \frac{5}{3}
\]

\[
\frac{dx}{dt} = -3 \text{ ft/minute}
\]
An explosion causes debris to rise vertically from the ground with an initial velocity of 72 feet per second.

25. Which of the following equations models the situation given?
   A. \( h(t) = -16t^2 \)
   B. \( h(t) = -16t^2 + 72t \)
   C. \( h(t) = -16t^2 + 72 \)
   D. \( h(t) = -16t^2 - 72t \)

26. What is the velocity after 2 seconds?
   A. 112 ft/sec      B. -64 ft/sec      C. 80 ft/sec      D. 8 ft/sec
Suppose a person standing at the top of a building 112 feet high throws a ball vertically upward with an initial velocity of 96 ft/sec. How far does the ball travel during its flight?

A falling stone is at a certain instant 120 feet above the ground. Two seconds later it is only 40 feet above the ground. If it was thrown upward with an initial speed of 3 feet per second, from what height was it thrown?

A stone is thrown upward from ground level. With what minimum speed should the stone be thrown so as to reach a height of 36 feet?
If a rocket is rising vertically at the rate of 880 ft/sec when it is 4000 feet up, how fast is the camera-to-rocket distance changing at the instant?

Two cars are moving towards the same point. The first car started from a point that is 100 miles away and it travels south at a rate of 40 mi/h. The second one started from a point 105 miles away and it travels east at a rate of 60 mi/h. At what rate is the distance between the cars changing one hour later?
Section 3.2: Rolle’s Theorem and the Mean Value Theorem

Rolle’s Theorem

Suppose that \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). If \( f(a) = f(b) \), then there is at least one number \( c \) in \((a, b)\) for which \( f'(c) = 0 \).

The essence of Rolle’s theorem may be seen on these pictures:

Rolle’s Theorem sometimes states:

Suppose that \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). If \( f'(a) = f'(b) = 0 \), then there is at least one number \( c \) in \((a, b)\) for which \( f'(c) = 0 \). That is, Rolle’s theorem tells us that between two roots of \( f \), there must be a root to \( f' \).

Example 1: Verify that the Rolle’s Theorem applies to the function \( f(x) = x^2 - x - 20 \) over \([-3, 4]\). Then find all points in this interval that satisfy Rolle’s Theorem.

- If \( f \) continuous on \([-3, 4]\)?
- If \( f \) differentiable on \((-3, 4)\)?
- \( f(-3) = \) \( f(4) = \)
Example 2: Verify that the Rolle’s Theorem applies to the function \( f(x) = \cos(2x) \) over \([0, \pi]\). Then find all points in this interval that satisfy Rolle’s Theorem.

- If \( f \) continuous on \([0, \pi]\)?
- If \( f \) differentiable on \((0, \pi)\)?
- \( f(0) = \)
- \( f(\pi) = \)

The mean-value theorem is a generalization of the Rolle’s Theorem.

**The Mean-Value Theorem**

Suppose that \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). There is at least one number \( c \) in \((a, b)\) for which \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

The conclusion of the Mean Value Theorem states that there exists a point \( c \) in the interval \((a, b)\) such that the tangent line is parallel to the line passing through \((a, f(a))\) and \((b, f(b))\).
Example 3: At how many points between 0 and 10 does the function satisfy the Mean Value theorem?

Example 4: Verify that the Mean Value Theorem applies to the function \( f(x) = x^3 + x - 4 \) over \([-1, 2]\). Then find all points in this interval that satisfy Mean Value Theorem.

- If \( f \) is continuous on \([-1, 2]\)?
- If \( f \) is differentiable on \((-1, 2)\)?
Math1431  Section 3.2

Example 5: Does the Mean Value Theorem apply to the function \( f(x) = \frac{x+2}{x} \) over \([-1, 2]\).

- If \( f \) continuous on \([-1,2]\)?
- If \( f \) differentiable on \((-1,2)\)?

Example 6: Verify that the Mean Value Theorem applies to the function \( f(x) = \sqrt{16-x^2} \) over \([0, 4]\). Then find all points in this interval that satisfy Mean Value Theorem.

- If \( f \) continuous on \([0,4]\)?
- If \( f \) differentiable on \((0,4)\)?
Example 7: Does the Mean Value Theorem apply to the function $f(x) = x^{1/3}$ over [-1, 1]?

*Be careful with any form of $f(x) = |x|$ as well, as this function has a sharp corner!*