Example 5: Does the Mean Value Theorem apply to the function $f(x) = \frac{x+2}{x}$ over [-1, 2].

- If $f$ continuous on [-1,2]? \textbf{No} \quad x \neq 0
- If $f$ differentiable on (-1,2)? MUT does not apply

Example 6: Verify that the Mean Value Theorem applies to the function $f(x) = \sqrt{16-x^2}$ over [0, 4]. Then find all points in this interval that satisfy Mean Value Theorem.

- If $f$ continuous on [0,4]? \textbf{Yes}
- If $f$ differentiable on (0,4)? \textbf{Yes}

\[ f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{0 - 4}{4} = -1 \]

\[ f'(x) = \frac{-x}{\sqrt{16-x^2}} \]

\[ \frac{-c}{\sqrt{16-c^2}} = -1 \]

\[ c = \sqrt{16-c^2} \]

\[ c^2 = 16 - c^2 \]

\[ c = -\sqrt{2}, \sqrt{2} \quad 0 < c < 4 \]

\[ \left( 2\sqrt{2}, \frac{f(2\sqrt{2})}{\sqrt{2}} \right) \]

\[ \left( 2\sqrt{2}, 2\sqrt{2} \right) \]
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**Question 27:** Given \( f(x) = 8 - \frac{7}{x} \), find all \( c \) in the interval \((1, 7)\) such that \( f'(c) = \frac{f(7) - f(1)}{7 - 1} \)

a. \( \sqrt{7} \)
b. \( 4 \)
c. \( \pm \sqrt{7} \)
d. \( \frac{7}{8} \)
e. None of the above

**Example 7:** Does the Mean Value Theorem apply to the function \( f(x) = x^{\frac{2}{3}} \) over \([-1, 1]\)?

![Graph of f(x) = x^{2/3}]

Be careful with any form of \( f(x) = |x| \) as well, as this function has a sharp corner!

**Question 37:** Find all value(s) of \( c \) (if any) that satisfy the conclusion of the Mean Value Theorem for the function \( f(x) = \frac{1}{1+x} \) on the interval \([-2, 1]\).

a. 0 
b. 1 
c. \( -1 \pm \sqrt{2} \) 
d. \( \sqrt{2} - 1 \) 
e. No value
Section 3.3: Increasing and Decreasing Functions

Let $f'$ be a function whose domain includes an interval $I$.

We say that $f$ is **increasing** on $I$ if for every two numbers $x_1, x_2$ in $I$,

$x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

We say that $f$ is **decreasing** on $I$ if for every two numbers $x_1, x_2$ in $I$,

$x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

**Example 1:** Given the graph of a polynomial function below, give the interval(s) of increasing and decreasing.

Increasing: $(-\infty, -2) \cup (3, \infty)$

Decreasing: $(-2, 3)$
One way we can find intervals of increase and decrease is to graph the function.

**Example 2:** Given \( f(x) = 5|x - 2| + 1 \), when is this function increasing? When is it decreasing?

Increasing: \((2, \infty)\) \hspace{1cm} Decreasing: \((-\infty, 2)\)

**Example 3:** Given \( f(x) = \begin{cases} 
    x^2 + 1, & x < 0 \\
    2x, & 0 \leq x \leq 5 \\
    -x, & 5 < x
\end{cases} \), when is this function increasing? When is it decreasing?

Increasing: \((0, 5)\) \hspace{1cm} Decreasing: \((-\infty, 0) \cup (5, \infty)\)
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Let’s use the graph below to observe the slopes of the tangent lines as the graph increases and decreases.

Over the intervals where the function is increasing, the tangent lines have positive slope. On the other hand, over the intervals of decrease, the tangent lines have negative slope.

**Theorem:** Suppose that $f$ is differentiable on the interior of an interval $I$ and continuous on all of $I$.

- If $f'(x) > 0$ for all $x$ in $I$, then $f$ increases on $I$.
- If $f'(x) < 0$ for all $x$ in $I$, then $f$ decreases on $I$.

**Finding intervals where a Function is Increasing or Decreasing**

1. Find all values of $x$ for which $f'(x) = 0$ or $f'(x)$ is undefined. Identify the intervals determined by these points.

2. Choose a test point $c$ in each interval found in Step 1 and determine the sign of $f'$ in that interval.
   - Wherever $f'(c) > 0$, then the function $f$ is increasing on that interval.
   - Wherever $f'(c) < 0$, then the function $f$ is decreasing on that interval.

**Example 4:** Given $f(x) = 6x^5 - 40x^3 + 10$, when is this function increasing? When is it decreasing?

$$f'(x) = 30x^4 - 120x^2$$

$$f'(x) = 0 \quad ?$$

$$30x^4 - 120x^2 = 0$$

$$30x^2(x^2 - 4) = 0$$

$$30x^2(x + 2)(x - 2) = 0$$

$$x = 0, -2, 2$$
Use the following problem to answer popper Questions 5 and 6.

A heap of rubbish in the shape of a cube is compacted into a smaller cube in such a way that the volume decreases at the rate of 2 cubic meters per min. Find the rate of change of an edge of the cube when the volume is 27 cubic meters. Let $V = \text{volume of the cube}$ and $x = \text{edge}$.

5. The GIVEN information may be denoted as:
   A. $\frac{dx}{dt} = 2$
   B. $\frac{dV}{dt} = -2$
   C. $\frac{dx}{dt} = -2$
   D. $\frac{dV}{dt} = 2$

6. The derivative that needs to be found (QUESTION) is:
   A. $\frac{dS}{dt}$
   B. $\frac{dV}{dt}$
   C. $\frac{dt}{dt}$
   D. $\frac{dx}{dt}$
Example 5: Given $f(x) = (x - 5)^{2/3}$, determine the intervals over which the function is increasing/decreasing.

$$f'(x) = \frac{2}{3(x - 5)^{1/3}}$$

Roots? $f'(x) = 0$
- Numerator $= 0$
- $x = 0 \times

$$\begin{array}{c|c|c|c|c|c|c|}
\hline
\text{Interval} & -\infty & 0 & 5 & 5.1 & 5.2 & \infty \\
\hline
f'(x) & - & - & + & + & + & + \\
\hline
\text{Increasing:} & (5, \infty) & \text{Decreasing:} & (-\infty, 5) \\
\hline
\end{array}$$

Example 6: Given $f(x) = \frac{\sin x}{1 + \cos x}$, when is this function increasing on $(0, \pi)$? When is it decreasing on $(0, \pi)$?

$$f'(x) = \frac{1}{1 + \cos x}$$

Roots? $f'(x) = 0$
- No where

$$\begin{array}{c|c|c|c|c|c|c|}
\hline
\text{Interval} & 0 & \frac{\pi}{2} & \pi \\
\hline
f'(x) & + & + & + \\
\hline
\text{Increasing:} & \left(0, \frac{\pi}{2}\right) \quad \text{Decreasing:} & \left(\frac{\pi}{2}, \pi\right) \\
\hline
\end{array}$$
Section 3.4: Extreme Values

Local Extreme Values

Suppose that \( f \) is a function defined on open interval \( I \) and \( c \) is an interior point of \( I \). The function \( f \) has a local minimum at \( x = c \) if \( f(c) \leq f(x) \) for all \( x \) in \( I \) (that is, for all \( x \) sufficiently close to \( c \)). The function \( f \) has a local maximum at \( x = c \) if \( f(c) \geq f(x) \) for all \( x \) in \( I \) (that is, for all \( x \) sufficiently close to \( c \)).

In general, if \( f \) has a local minimum or maximum at \( x = c \), we say that \( f(c) \) is a local extreme value of \( f \).

Identify the local extreme values for the following function.

This graph suggests that local maxima or minima occur at the points where the tangent line is horizontal or where the function is not differentiable, and this is true.

If \( c \) is in the domain of a function \( f \) for which \( f'(c) = 0 \) or \( f'(c) \) does not exist, then \( c \) is called a critical point for \( f \).

First Derivative Test

Let \( c \) be a critical number of a function \( f \) that is continuous on an open interval \( I \) containing \( c \).

1. If \( f'(c) \) changes from negative to positive at \( c \), then \( f(c) \) is a local minimum of \( f \).
2. If \( f'(c) \) changes from positive to negative at \( c \), then \( f(c) \) is a local maximum of \( f \).

So you want to make a sign chart when using the first derivative test.
Example 1: Find any critical points (value) and local extreme points: \( f(x) = x^3 - 3x^2 - 24x + 32 \)

Domain of \( f(x) \):

\[
\left( -\infty, \infty \right)
\]

\[
\frac{df}{dx} = 3x^2 - 6x - 24
\]

Find when \( \frac{df}{dx} = 0 \):

\[
3x^2 - 6x - 24 = 0
\]

\[
3(x - 4)(x + 2) = 0
\]

\[
x = 4, -2
\]

Critical #’s

Apply First Derivative Test:

Local Extreme Points

Local Max:

\(( -2, f(-2) )\)

Local Min:

\(( 4, f(4) )\)
Example 2: Find any critical points (value) and local extreme points \( f(x) = 64x^2 + \frac{54}{x} - 2 \).

Domain of \( f(x) \):
\[
\mathbb{R} \setminus \{0\} \quad (-\infty, 0) \cup (0, \infty)
\]

Find when \( f'(x) = 0 \):
\[
128x - \frac{54}{x^2} = 0
\]
\[
128x = \frac{54}{x^2}
\]
\[
x^3 = \frac{54}{128} = \frac{27}{64}
\]
\[
x = \frac{3}{4}
\]
\( x = \frac{3}{4} \) is only critical number

Apply First Derivative Test:

\[
f'(x): \quad - - - - \quad \text{Not in Domain} \quad + + + +
\]

Local Extreme Points
Local Max: \( \text{None} \)  
Local Min:
\[
\left( \frac{3}{4}, f\left( \frac{3}{4} \right) \right)
\]
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Use the following sign chart to answer questions 34 – 36 about the polynomial function $f$.

35. Over the interval $(0, 2)$, the function $f$ is:
   A. zero  B. increasing  C. decreasing

36. Over the interval $(2, \infty)$, the function $f$ is:
   A. zero  B. increasing  C. decreasing

38. $x = 0$ a. Max  b. Min  c. Neither
39. $x = 2$ classify
40. A
Example 3: Find any critical points (value) and local extreme points: \( f(x) = \frac{x^2}{(x - 4)} \)

Domain of \( f(x) \):

\[
f'(x) = \frac{x(x-8)}{(x-4)^2}
\]

Find when \( f'(x) = 0 \):

Apply First Derivative Test:

\[
f'(x) :
\]

Find when \( f'(x) \) = undefined:

Local Extreme Points
Local Max: 
Local Min:
Example 4: Find any critical points (value) and local extreme points: \( f(x) = (x - 3)^{2/5} \)

Domain of \( f(x) \):

\[
f'(x) = \frac{2}{5(x - 3)^{3/5}}
\]

Find when \( f'(x) = 0 \):

Find when \( f'(x) = \text{undefined} \):

Apply First Derivative Test:

Local Extreme Points
Local Max: Local Min:
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Sometimes it is difficult to study the sign of the derivative function. For some cases, it may be easier to use the following test:

**The Second Derivative Test**

Let $f$ be a function such that $f'(c) = 0$ and the second derivative of $f$ exists on an open interval containing $c$.

- If $f''(c) > 0$, then $f(c)$ is a local minimum.
- If $f''(c) < 0$, then $f(c)$ is a local maximum.
- If $f''(c) = 0$, then the test fails. In such cases, you can use the First Derivative Test.

**Example 5:** Find any critical points (value) and local extremum on $\left(0, \frac{3\pi}{4}\right)$ using the second derivative test.

$$f(x) = 2 \sin x + \cos(2x)$$

Find when $f'(x) = 0$:

Find when $f'(x) = \text{undefined}$:

$$f''(x) =$$

**Apply Second Derivative Test:**

**Local Extreme Points**

Local Max:

Local Min:
What can we say about $f$ given the graph of $f'$? Given the graph of $f'$ we can find the critical numbers of $f$ and where it’s increasing or decreasing.

Example 6: Below is the graph of the derivative of a polynomial function $f$. Which of the following statements is/are true or false?

![Graph of the derivative of a polynomial function](image)

a. The critical numbers for $f$ are 0, 1 and 2.

b. The function $f$ has two minimums.

c. The function $f$ has one maximum.

d. The function $f$ is decreasing over one interval.