Section 3.5
Concavity and Points of Inflection

Let \( f \) be a function that is differentiable on an open interval \( I \).

The graph of \( f \) is **concave up** if \( f' \) is increasing on \( I \).

The graph of \( f \) is **concave down** if \( f' \) is decreasing on \( I \).

Even though both pictures indicate a local extreme value, note that that need not be the case.

Here are some graphs where the functions are concave up or down without any local extreme values.

**Example 1:** The graph of \( f'(x) \) (first derivative!) of a polynomial function \( f \) is given.

a. When is \( f(x) \) concave up? \((-\infty, -3) \cup (2, \infty)\)

b. When is \( f(x) \) concave down? \((-3, 2)\)
**Theorem:** Let $f$ be a function that is twice differentiable on an open interval $I$.

- If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave up on $I$.
- If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave down on $I$.

**Determining the Intervals of Concavity for a Function**

1. Find any value of $x$ for which $f''(x) = 0$ or $f''(x)$ is undefined. Identify the intervals determined by these points.

2. Choose a test point $c$ in each interval found in Step 1 and determine the sign of $f''$ in that interval.
   - Wherever $f''(c) > 0$, then the function $f$ is concave up on that interval.
   - Wherever $f''(c) < 0$, then the function $f$ is concave down on that interval.

**Example 2:** Determine the concavity of $f(x) = x^3 + 2x$. The domain of $f(x)$ is $(-\infty, \infty)$.

$f'(x) = 3x^2 + 2$

$f''(x) = 6x$

Find when $f''(x) = 0$:

$6x = 0$

$x = 0$

$f''(x) = 6x$

Find when $f''(x)$ is undefined:

$x = \text{undefined}$

$f(x)$:

$f''(x)$:

- - - - - - + + + + + +

$-\infty \ -1 \ 0 \ 201 \ \infty$

Concave Up: $(0, \infty)$

Concave Down: $(-\infty, 0)$
Example 3: Determine the concavity of \( f(x) = \cos^2 x - \sin^2 x \), \( x \in (0, \pi) \). The domain of \( f(x) \) is \( (-\infty, \infty) \).

\[
\begin{align*}
f'(x) &= -2 \cos x \sin x - 2 \sin x \cos x \\
&= -2 \sin(2x)
\end{align*}
\]

\[
\begin{align*}
f''(x) &= -4 \cos^2 x + 4 \sin^2 x \\
&= -4 \cos(2x)
\end{align*}
\]

\[-4 \cos^2 x + 4 \sin^2 x = 0 \\
-4 (\cos^2 x - \sin^2 x) = 0 \\
\overbrace{\cos(2x)}^{\text{cos } x = 0}
\]

\[
\begin{align*}
\theta &= \frac{\pi}{2}, \frac{3\pi}{2} \\
\theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{align*}
\]

Find when \( f''(x) = 0 \):

\( 2x = \theta \)

\( x = \frac{\theta}{2} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \)

\( f(x) \):

\[
\begin{array}{cccccccc}
\theta & 0 & \frac{\pi}{4} & \frac{3\pi}{4} & \frac{5\pi}{4} & \frac{7\pi}{4} & \frac{9\pi}{4} \\
- & - & + & + & + & - & -
\end{array}
\]

Concave Up:

\( \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \)

Concave Down:

\( (0, \frac{\pi}{4}) \cup \left( \frac{3\pi}{4}, \frac{5\pi}{4} \right) \)

Section 3.5 – Concavity and Points of Inflection
A point in the domain of a differentiable function $f$ at which the concavity changes is called a point of inflection.

**Finding Inflection Points**

1. Find any value of $x$ in the domain of the function for which $f''(x) = 0$ or $f''(x)$ is undefined.

2. Determine the sign of $f''(x)$ to the left and to the right of each point $x = a$ found in Step 1. If there is a sign change across the point $x = a$, then $(a, f(a))$ is a point of inflection of $f$.

**Example 4**: Given $f(x)$, determine any points of inflection $f(x) = \frac{x}{x^2 - 1}$ The domain of $f$ is $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$ and $f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$.

Find when $f''(x) = 0$:

\[
\text{Numerator} = 0 \\
2x(x^2 + 3) = 0 \\
x = 0, \pm \sqrt{-3} \\
\text{No Real}
\]

\[
x = 0
\]

Find when $f''(x)$ is undefined:

\[
\text{Denominator} = 2x \\
x = -1, 1 \\
\text{Not in Domain}
\]

\[
f(x):
\]

\[
f''(x): -- 0 + + + + 0 -- 0 +
\]

POI: $(0, f(0)) \rightarrow (0,0)$
**Question 1:** The following graph is the graph of the \textit{first derivative} of a polynomial function $f$. Over which interval(s) is the function $f$ decreasing?

A. $(1, 3)$  B. $(-\infty, 0)$  C. $(-\infty, 1) \cup (3, \infty)$  D. $(-\infty, 0) \cup (1, 3)$  E. None of these

**Question 2:** Over which interval(s) is the function $f$ increasing?

A. $(0, 1)$  B. $(-\infty, 1)$  C. $(-\infty, 1) \cup (3, \infty)$  D. $(0, \infty)$  E. None of these
Example 5: Find any points of inflection of \( f(x) = 2 + x^{1/3} \). The domain of \( f \) is \((-\infty, \infty)\).

\[ f'(x) = \frac{1}{3x^{2/3}} \]

\[ f''(x) = -\frac{2}{9x^{5/3}} \]

Find when \( f''(x) = 0 \):

\( x = 0 \)

Find when \( f''(x) \) is undefined:

\[ f''(x) = \frac{2}{9x^{5/3}} \]

POI:

\( x = 0 \) or \( (0, 2) \)

Section 3.5 – Concavity and Points of Inflection
The graph of \( f'' \):

When the graph of the second derivative is given, we can gather information about whether \( f' \) is concave up or down, and any points of inflection for \( f' \).

Example 6: The graph of \( f'' \) (second derivative!) of a polynomial function \( f \) is given. Determine whether each of the following statements is/are true or false.

\[
\text{D: } (-\infty, \infty) \\
f''(x) = 0 \\
x = -1, 2, 5
\]

a. The function \( f(x) \) concave down over one interval.

\[
\text{True} \\
\text{No sign change @ } x = -1
\]

b. The \( x \)-values of the points of inflection are: \( x = -1, x = 2, x = 5 \).

\[
\text{False} \\
\text{No sign change @ } x = -1
\]
Sometimes it is difficult to study the sign of the derivative function. For some cases, it may be easier to use the following test:

**The Second-Derivative Test**

Let \( c \) be a critical point for \( f \) where \( f'(c) = 0 \) and \( f''(c) \) exists.

- If \( f''(c) > 0 \), then \( f(c) \) is a local minimum value.
- If \( f''(c) < 0 \), then \( f(c) \) is a local maximum value.
- If \( f''(c) = 0 \), then this test is inconclusive.

**Example 7:** Given \( f''(x) = 6x - 12 \), \( f'(1) = 0 \) and \( f'(4) = 0 \). Classify these critical numbers as local min/max.

\[ x = 1, 4 \quad \text{Critical Numbers} \]

\[ f'(1) = 6(1) - 12 < 0 \quad \text{at } x = 1 \quad \text{Local Max} \]

\[ f''(4) = 6(4) - 12 > 0 \quad \text{at } x = 4 \quad \text{Local Min} \]
Try this one: Find any critical points and classify them as local min/max on \( \left( 0, \frac{2\pi}{3} \right) \) using the second derivative test.

\[
f(x) = 2\sin x + \cos(2x),
\]

Local Max: 

Local Min:
Question 17  Find all intervals on which \( f(x) = x + \frac{4}{x^2} \) is decreasing.

\[ f'(x) = 1 - \frac{8}{x^3} \]
\[ f''(x) = 0 \]
\[ 1 - \frac{8}{x^3} = 0 \]
\[ 1 = \frac{8}{x^3} \]
\[ x^3 = 8 \]
\[ x = 2 \]

a)  \((2, \infty)\)

b)  \((0, 2)\)

c)  \((-\infty, -2), (-2, 1)\) and \((1, \infty)\)

d)  \((-\infty, 0)\) and \((2, \infty)\)

e)  None of these

Question 10  If \( y \) is a function of \( x \) such that \( y' < 0 \) for all \( x \), and \( y'' < 0 \) for all \( x \), which of the following could be a part of the graph of \( y \)?

A.          B.     C .     D .
Vertical Asymptotes

If \( f(x) \to \pm \infty \) as \( x \to c^+ \) or \( x \to c^- \), then the line \( x=c \) is a **vertical asymptote** for \( f(x) \).

The graph of \( f(x) = \frac{1}{x|x-2|} \) is given below.

We can see the vertical asymptotes very easily from its graph. But also recall how to find them algebraically. **Recall:** Simplify the function. Any variable factor left in the denominator, set equal to 0 and solve for \( x \).

- A function may have no vertical asymptotes, such as: \( f(x) = \frac{x}{\sqrt{x^2 + 4}} \)

- A function may have only one vertical asymptote, such as: \( f(x) = \frac{x\sqrt{x}}{4\sqrt{x} - x} \)

- A function may have many vertical asymptotes, such as: \( f(x) = \frac{x^2}{1 - 2\sin x} \)
Horizontal Asymptotes

As we saw in Section 1.3, the behavior of a function as \( x \to \pm \infty \) determines the horizontal asymptotes.

- If \( \lim_{x \to \infty} f(x) = L \), then the line \( y = L \) is a (rightward) horizontal asymptote.
- If \( \lim_{x \to -\infty} f(x) = L \), then the line \( y = L \) is a (leftward) horizontal asymptote.

Recall the shortcut for rational functions: Compare the degrees.

\[
f(x) = \frac{x+1}{x^2-4} \quad \text{H. A.:}
\]

\[
f(x) = \frac{x^2}{5x^2+1} \quad \text{H. A.:}
\]

\[
f(x) = \frac{x^5}{x^3-2x} \quad \text{H. A.:}
\]

These rules work because for \( p > 0 \) and provided \( \frac{1}{x^p} \) is defined, \( \lim_{x \to \infty} \frac{1}{x^p} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x^p} = 0 \).

For example, \( f(x) = \frac{x^2}{5x^2+1} \) has H.A. \( y = \frac{1}{5} \) because:
Example 1: Find the horizontal asymptotes for each of the following functions.

a. \[ f(x) = \frac{x}{\sqrt{x^2 + 4}} \]

b. \[ f(x) = \frac{\sqrt{x}}{4\sqrt{x} - x} \]
Use the following graph below which is the graph of the *first derivative* of a polynomial function $f$ to answer questions 2 – 6.

**Question 40:** How many critical numbers does the function $f$ have?

**Question 43:** Over how many intervals is $f$ increasing?

**Question 46:** Over how many intervals is $f$ decreasing?

**Question 49:** How many relative maximums does $f$ have?

**Question 50:** How many relative minimums does $f$ have?

A. None    B. One    C. Two    D. Three
Vertical Tangents

$f(x) = x^{\frac{1}{3}}$

Suppose that $f(x)$ is continuous at $x = c$. If $f'(x) \to \infty$ or $f'(x) \to -\infty$ as $x \to c$, then we say that the function has a vertical tangent at the point $(c, f(c))$. 

Vertical tangents will only happen with some radical functions. They may be found by observing that:

- $f(c)$ is defined.
- $f'(c)$ is undefined.
- The sign chart for $f'$ across $x = c$ has no sign change.

*Be careful when creating a sign chart for some radicals, don’t forget find any critical points for the function.*
Vertical Cusps

\[ f(x) = x^{\frac{2}{3}} \]

Suppose that \( f(x) \) is continuous at \( x = c \). If \( f'(x) \to \infty \) as \( x \to c \) from one side and \( f'(x) \to -\infty \) as \( x \to c \) from the other side, then we say that the function has a **vertical cusp** at the point \((c, f(c))\).

Cusps will only happen with *some* radical functions. They may be found by observing that:

- \( f(c) \) is defined.
- \( f'(c) \) is undefined.
- The sign chart for \( f' \) across \( x = c \) has a **sign change**.

*Be careful when creating a sign chart for some radicals, don’t forget find any critical points for the function.*