Sometimes it is difficult to study the sign of the derivative function. For some cases, it may be easier to use the following test:

**The Second-Derivative Test**

Let \( c \) be a critical point for \( f \) where \( f'(c) = 0 \) and \( f''(c) \) exists.

* If \( f''(c) > 0 \), then \( f(c) \) is a **local minimum value**.

* If \( f''(c) < 0 \), then \( f(c) \) is a **local maximum value**.

* If \( f''(c) = 0 \), then this test is inconclusive.

**Example 7:** Given \( f''(x) = 6x - 12 \), \( f'(1) = 0 \) and \( f'(4) = 0 \). Classify these critical numbers as local min/max.

Critical \# ≤ \( x = 1, 4 \)

Apply 2nd Der. Test:

\[
\begin{align*}
    f''(1) &= -6 \quad < 0 \quad @ \quad x = 1 \quad L. \quad Max \\
    f''(4) &= 12 \quad > 0 \quad @ \quad x = 4 \quad L. \quad Min
\end{align*}
\]
Try this one: Find any critical points and classify them as local min/max on \( \left( 0, \frac{2\pi}{3} \right) \) using the second derivative test.

\[
f(x) = 2\sin x + \cos(2x),
\]

\[
f'(x) = 2\cos(x) - 2\sin(2x)
\]

\[
= 2\cos(x) - 2\cdot 2\sin(x)\cos(x)
\]

\[
= 2\cos(x) - 4\sin(x)\cos(x) = 2\cos(x)(1 - 2\sin(x))
\]

\[
2\cos(x) = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \quad \text{Not in Domain}
\]

\[
\sin(x) = \frac{1}{2}
\]

\[
\Rightarrow \quad x = \frac{\pi}{6}, \quad \frac{5\pi}{6}
\]

\[
f''(x) = -2\sin(x) - 4\cos(2x)
\]

\[
f''\left(\frac{\pi}{6}\right) = -2\sin\left(\frac{\pi}{6}\right) - 4\cos\left(2\cdot\frac{\pi}{6}\right)
\]

\[
= -2\cdot\frac{1}{2} - 4\cdot\frac{1}{2} = -3 < 0 \quad \text{L. Max}
\]

\[
f''\left(\frac{\pi}{2}\right) = -2\sin\left(\frac{\pi}{2}\right) - 4\cos\left(2\cdot\frac{\pi}{2}\right)
\]

\[
= -2\cdot1 - 4\cdot(-1) = 2 > 0 \quad \text{L. Min}
\]

Local Max:

\[
\left( \frac{\pi}{6}, \frac{2}{3} \right)
\]

Local Min:

\[
\left( \frac{\pi}{2}, 1 \right)
\]
Vertical Asymptotes

If \( f(x) \to \pm \infty \) as \( x \to c^+ \) or \( x \to c^- \), then the line \( x = c \) is a vertical asymptote for \( f(x) \).

The graph of \( f(x) = \frac{1}{x|x-2|} \) is given below.

We can see the vertical asymptotes very easily from its graph. But also recall how to find them algebraically. Recall: Simplify the function. Any variable factor left in the denominator, set equal to 0 and solve for \( x \).

- A function may have no vertical asymptotes, such as: \( f(x) = \frac{x}{\sqrt{x^2 + 4}} \)

- A function may have only one vertical asymptote, such as: \( f(x) = \frac{\sqrt{x}}{4\sqrt{x} - x} \)

- A function may have many vertical asymptotes, such as: \( f(x) = \frac{x^2}{1 - 2\sin x} \)
Horizontal Asymptotes

As we saw in Section 1.3, the behavior of a function as \( x \to \pm \infty \) determines the **horizontal asymptotes**.

- If \( \lim_{x \to \infty} f(x) = L \), then the line \( y = L \) is a (rightward) **horizontal asymptote**.
- If \( \lim_{x \to -\infty} f(x) = L \), then the line \( y = L \) is a (leftward) **horizontal asymptote**.

Recall the shortcut for rational functions: Compare the degrees.

\[
f(x) = \frac{x^2 + 1}{x^2 - 4}, \quad \text{H. A.: } y = 0
\]

\[
f(x) = \frac{1 \cdot x^2}{5x^2 + 1}, \quad \text{Same Power} \quad \text{H. A.: } y = \frac{1}{5}
\]

\[
f(x) = \frac{x^5}{x^3 - 2x}, \quad \text{Top Heavy} \quad \text{H. A.: } \text{None}
\]

These rules work because for \( p > 0 \) and provided \( \frac{1}{x^p} \) is defined, \( \lim_{x \to \infty} \frac{1}{x^p} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x^p} = 0 \).

For example, \( f(x) = \frac{x^2}{5x^2 + 1} \) has H.A. \( y = \frac{1}{5} \) because:

\[
\lim_{x \to \infty} \frac{x^2}{5x^2 + 1} = \frac{\lim_{x \to \infty} \frac{x^2}{x^2}}{\lim_{x \to \infty} \frac{5x^2}{x^2} + \frac{1}{x^2}} = \frac{1}{5}
\]
Example 1: Find the horizontal asymptotes for each of the following functions.

a. \( f(x) = \frac{x}{\sqrt{x^2 + 4}} \)

\[
\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 4}} = \lim_{x \to -\infty} \frac{1}{\sqrt{1 + \frac{4}{x^2}}} = \frac{1}{\sqrt{1}} = 1
\]
Right H. A.

b. \( f(x) = \frac{\sqrt{x}}{4\sqrt{x} - x} \)

\[
\lim_{x \to \infty} \frac{\sqrt{x}}{4\sqrt{x} - x} = \lim_{x \to \infty} \frac{1}{4\sqrt{x} - 1} = \frac{1}{1} = 1
\]
HA. > 0
Vertical Tangents

Suppose that \( f(x) \) is continuous at \( x = c \). If \( f'(x) \to \infty \) or \( f'(x) \to -\infty \) as \( x \to c \), then we say that the function has a **vertical tangent** at the point \( (c, f(c)) \).

Vertical tangents will only happen with **some** radical functions. They may be found by observing that:

- \( f(c) \) is defined.
- \( f'(c) \) is undefined.
- The sign chart for \( f' \) across \( x = c \) has **no sign change**.

*Be careful when creating a sign chart for some radicals, don’t forget find any critical points for the function.*
Vertical Cusps

Suppose that \( f(x) \) is continuous at \( x = c \). If \( f'(x) \to \infty \) as \( x \to c \) from one side and \( f'(x) \to -\infty \) as \( x \to c \) from the other side, then we say that the function has a **vertical cusp** at the point \( (c, f(c)) \).

Cusps will only happen with *some* radical functions. They may be found by observing that:

- \( f(c) \) is defined.
- \( f'(c) \) is undefined.
- The sign chart for \( f' \) across \( x = c \) has a sign change.

*Be careful when creating a sign chart for some radicals, don’t forget find any critical points for the function.*
Example 2: For the following functions, determine whether the function has a vertical tangent, cusp or neither at the given value.

a. \( f(x) = 5(x - 8)^{4/5} \) at \( c = 8 \)

\[
\frac{d}{dx} (x) = \frac{4}{(x - 8)^{1/5}}
\]

Check list:

- Is \( f(c) \) is defined?
  \[
f(8) = 5(8 - 8)^{4/5} = 0
\]

- Is \( f'(c) \) is undefined?
  \[
f'(8) = \frac{4}{(8 - 8)^{1/5}} = \frac{4}{0} = \text{undefined}
\]

- Create a sign chart. Does the sign chart for \( f' \) across \( x = c \) have a sign change or not?

\[
\begin{array}{c|c|c|c|c}
& - & - & + & + & + \\
\hline
f'(x) & & & & & \\
\hline
x & -\infty & 8 & 8 & \infty
\end{array}
\]

b. \( f(x) = 9x^{3/5} - 2x^{6/5} \) at \( c = 0 \)

Check list:

- Is \( f(c) \) is defined? Does \( f(0) \) exist?
  \[
f(0) = 0\]

- Is \( f'(c) \) is undefined?
  \[
f'(x) = \frac{27}{5x^{2/5}} - 12x^{1/5}
\]

\[
f'(0) = \frac{27}{0} - 12 (0) = \text{undefined} \quad \text{yes}
\]

- Create a sign chart. Does the sign chart for \( f' \) across \( x = c \) have a sign change or not?

\[
\begin{array}{c|c|c|c|c|c}
& + & + & + & + & + \\
\hline
f'(x) & & & & & \\
\hline
x & -0.1 & 0 & 0.1
\end{array}
\]
Curve Sketching

Using Calculus to Graph a Function.

1. Determine the domain of the function \( f \).
   
   For radicals:
   
   - The domain of any odd root will be \((-\infty, \infty)\).
   - The domain of any even root, set the radicand (inside) \( \geq 0 \) and solve.

2. Find any asymptotes— for functions with fractions.

3. Determine any intercepts of the function. To find the \( x \) -intercepts, we need to solve the equation \( f(x) = 0 \) and to find the \( y \) -intercepts, evaluate the function at 0 (if 0 is in the domain of \( f \)).

4. Find the first derivative, \( f' \). Determine any critical points, intervals of increase/decrease, local extreme points, vertical tangents and cusps.

5. Find the second derivative, \( f'' \). Study the sign of \( f'' \) to understand concavity of the function and determine any points of inflection.

6. Plot the points of interest (intercepts, local or absolute extreme points, points of inflection).

7. Sketch the graph of \( f \) using the information gathered in the previous steps. Make sure that the function has the right shape (concaves up/down, rises/falls) on the corresponding intervals.

Example 3: Use the information given to sketch the graph of function \( f \).

\[
 f(x) = \frac{4x - 4}{x^2}
\]

Domain: \(( -\infty, 0 ) \cup (0, \infty)\)

Intercept: \( x \)-intercept: 1

Asymptotes: \( x \)-axis and \( y \)-axis

Increasing: \((0,2)\)

Decreasing: \((-\infty,0)\) and \((2,\infty)\)

Relative Extrema: Relative Max at \((2, 1)\)

Concave Down: \(( -\infty, 0 ) \) and \((0, 3) \)

Concave Up: \((3, \infty)\)

Points of Inflection: \((3, \frac{8}{9})\)
**Example 4:** Use the guide to curve sketching to sketch \( f(x) = x^4 - 4x^3 \).

Domain of \( f(x) \):

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:

\[ f'(x) = 4x^3 - 12x^2 \]

Find when \( f'(x) = 0 \):

Find when \( f'(x) \) is undefined:

\[ f''(x) = 12x^2 - 24x \]

Find when \( f''(x) = 0 \):

Find when \( f''(x) \) is undefined:

\[ f'''(x) \]

Find when \( f'''(x) \) is undefined:

\[ f(x) : \]

\[ f'(x) : \]

\[ f''(x) : \]

\[ f'''(x) : \]

Section 3.6 – Curve Sketching
Example 5: Sketch the graph of \( f(x) = \frac{2x^2}{x^2 - 1} \).

Domain of \( f(x) \): 

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:
\[
\begin{align*}
\frac{d}{dx} f(x) &= \frac{-4x}{(x^2-1)^2} \\
\text{Find when } f'(x) &= 0 : & \text{Find when } f'(x) \text{ is undefined:}
\end{align*}
\]

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) &= \frac{12x^2 + 4}{(x^2-1)^3} \\
\text{Find when } f''(x) &= 0 : & \text{Find when } f''(x) \text{ is undefined:}
\end{align*}
\]
Try this one: Sketch the graph of \( f(x) = \frac{1}{x(x-1)^3} \).

Domain of \( f(x) \):

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:

\[
f'(x) = \frac{4x-3}{3(x-1)^{2/3}}
\]

Find when \( f'(x) = 0 \):

Find when \( f'(x) \) is undefined:

\[
f'(x):
\]

\[
f''(x):
\]

\[
f''(x) = \frac{4x-6}{9(x-1)^{5/3}}
\]

Find when \( f''(x) = 0 \):

Find when \( f''(x) \) is undefined:

\[
f(x):
\]

\[
f''(x):
\]