<table>
<thead>
<tr>
<th>Score Range</th>
<th>Number of Students</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Median</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 10</td>
<td>2</td>
<td>38.46</td>
<td>17.115</td>
<td>44</td>
<td>56</td>
</tr>
<tr>
<td>10 to 20</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 to 30</td>
<td>22</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 to 40</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 to 50</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 to 60</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60 to 70</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70 to 80</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 to 90</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90 to 100</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 to 110</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Score Range</th>
<th>Number of Students</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Median</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 5</td>
<td>2</td>
<td>19.57</td>
<td>11.133</td>
<td>19</td>
<td>45</td>
</tr>
<tr>
<td>5 to 10</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 to 15</td>
<td>27</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15 to 20</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 to 25</td>
<td>21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25 to 30</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 to 35</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35 to 40</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 to 45</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45 to 50</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 to 55</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Section 3.6
Curve Sketching

Vertical Asymptotes

If \( f(x) \to \pm \infty \) as \( x \to c^+ \) or \( x \to c^- \), then the line \( x = c \) is a vertical asymptote for \( f(x) \).

The graph of \( f(x) = \frac{1}{x|x-2|} \) is given below.

We can see the vertical asymptotes very easily from its graph. But also recall how to find them algebraically. Recall: Simplify the function. Any variable factor left in the denominator, set equal to 0 and solve for \( x \).

- A function may have no vertical asymptotes, such as: \( f(x) = \frac{x}{\sqrt{x^2 + 4}} \)
  \[ x^2 + 4 \geq 0 \]
  \[ x^2 \geq -4 \]
  \[ D: (-\infty, \infty) \]

- A function may have only one vertical asymptote, such as: \( f(x) = \frac{\sqrt{x}}{4\sqrt{x} - x} \)
  \[ 4\sqrt{x} - x \neq 0 \]
  \[ 4\sqrt{x} = x \]
  \[ 16x = x^2 \]
  \[ 0 = x^2 - 16x \]
  \[ 0 = x(x - 16) \]
  \[ VA: x = 16 \]

- A function may have many vertical asymptotes, such as: \( f(x) = \frac{x}{1 - 2\sin x} \)
  \[ 1 - 2\sin x \neq 0 \]
  \[ \frac{1}{2} = \sin x \]
  \[ x = \frac{\pi}{6} + 2\pi k \]
  \[ \frac{5\pi}{6} + 2\pi k \]
Horizontal Asymptotes

As we saw in Section 1.3, the behavior of a function as $x \to \pm \infty$ determines the **horizontal asymptotes**.

- If $\lim_{x \to \infty} f(x) = L$, then the line $y = L$ is a (rightward) horizontal asymptote.
- If $\lim_{x \to -\infty} f(x) = L$, then the line $y = L$ is a (leftward) horizontal asymptote.

**Recall the shortcut for rational functions: Compare the degrees.**

$$f(x) = \frac{x+1}{x^2-4} \quad \text{H. A.:} \quad y = 0$$

$$f(x) = \frac{1+x^2}{5x^3+1} \quad \text{H. A.:} \quad y = -\frac{1}{5}$$

$$f(x) = \frac{x^5}{x^3-2x} \quad \text{H. A.:} \quad \text{None}$$

These rules work because for $p > 0$ and provided $\frac{1}{x^p}$ is defined, $\lim_{x \to \infty} \frac{1}{x^p} = 0$ and $\lim_{x \to -\infty} \frac{1}{x^p} = 0$.

For example, $f(x) = \frac{x^2}{5x^2+1}$ has H.A. $y = \frac{1}{5}$ because: **Highest Degree**: $x \to x^2$

$$= \lim_{x \to \infty} \frac{x^2}{5x^2+1} \cdot \frac{1}{x^2}$$

$$= \lim_{x \to \infty} \frac{x^2}{5x^2} + \frac{1}{x^2}$$

$$= \lim_{x \to \infty} \frac{1}{5x^2 + 0} = \frac{1}{5}$$
Example 1: Find the horizontal asymptotes for each of the following functions.

a. \( f(x) = \frac{x}{\sqrt{x^2 + 4}} \)

\[
\lim_{x \to \infty} \frac{\sqrt{x^2}}{\sqrt{x^2 + 4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{x \cdot \frac{1}{x}}{\sqrt{x^2 + 4} \cdot \frac{1}{x}} = \lim_{x \to \infty} \frac{x}{\sqrt{\frac{x^2}{x^2} + \frac{4}{x^2}}} = \lim_{x \to \infty} \frac{x}{\sqrt{1 + \frac{4}{x^2}}} = \frac{1}{\sqrt{1 + 0}} = 1
\]

HA: \( y = 1 \)

\[
\lim_{x \to -\infty} \frac{\sqrt{x^2}}{\sqrt{x^2 + 4}} \cdot \frac{x}{\sqrt{x^2}} = \lim_{x \to \infty} \frac{\sqrt{x^2}}{\sqrt{x^2 + 4} \cdot \frac{x^2}{x^2}} = \lim_{x \to \infty} -\frac{1}{\sqrt{1 + \frac{4}{x^2}}} = y = -1
\]

Left side HA: \( y = -1 \)

b. \( f(x) = \frac{\sqrt{x}}{4\sqrt{x} - x} \)

Use shortcut

Degree in numerator = \( \frac{1}{2} \)
Degree in denominator = 1

HA: \( y = 0 \)
Vertical Tangents

Suppose that \( f(x) \) is continuous at \( x = c \). If \( f'(x) \to \infty \) or \( f'(x) \to -\infty \) as \( x \to c \), then we say that the function has a **vertical tangent** at the point \( (c, f(c)) \).

Vertical tangents will only happen with **some radical functions**. They may be found by observing that:

- \( f(c) \) is defined.
- \( f'(c) \) is undefined.
- The sign chart for \( f' \) across \( x = c \) has no sign change.

Be careful when creating a sign chart for some radicals, don’t forget find any critical points for the function.
**Vertical Cusps**

Suppose that \( f(x) \) is continuous at \( x = c \). If \( f'(x) \to \infty \) as \( x \to c \) from one side and \( f'(x) \to -\infty \) as \( x \to c \) from the other side, then we say that the function has a **vertical cusp** at the point \( (c, f(c)) \).

Cusps will only happen with *some* radical functions. They may be found by observing that:

- \( f(c) \) is defined.
- \( f'(c) \) is undefined.
- The sign chart for \( f' \) across \( x = c \) has a **sign change**.

*Be careful when creating a sign chart for some radicals, don’t forget find any critical points for the function.*
Example 2: For the following functions, determine whether the function has a vertical tangent, cusp or neither at the given value.

a. \( f(x) = 5(x - 8)^{4/5} \) at \( c = 8 \)

\[ f'(x) = \frac{4}{(x-8)^{1/5}} \]

Check list:
- Is \( f(c) \) is defined?
  \[ f(8) = 5(4)^{4/5} = 5(2) = 10 \]
  \( \text{No} \)

- Is \( f'(c) \) is undefined?
  \( f'(8) \) is undefined, \( c = 8 \) is a critical number.

- Create a sign chart. Does the sign chart for \( f' \) across \( x = c \) have a sign change or not?
  \[ f'(x) \]
  \[ - - - + + + + + \]
  \[ -10 \quad 9 \quad 243 \]
  \( \text{No other CN} \)
  \( \text{Sign change} \)
  \( \text{Vertical Cusp at} \ x = 9 \)

b. \( f(x) = 9x^{3/5} - 2x^{9/5} \) at \( c = 0 \)

Check list:
- Is \( f(c) \) is defined?
  \[ f(0) = 0 \]
  \( \text{Yes} \)

- Is \( f'(c) \) is undefined?
  \[ f'(x) = \frac{27}{5}x^{-2/5} - \frac{12}{5}x^{4/5} \]
  \[ f'(0) \] is undefined

- Create a sign chart. Does the sign chart for \( f' \) across \( x = c \) have a sign change or not?
  \[ f'(x) \]
  \[ \frac{27}{5}x^{-2/5} - \frac{12}{5}x^{4/5} = 0 \]
  \[ 27x^{-1/5} - 12x^{1/5} = 0 \]
  \[ \frac{27}{x} = 12x^{1/5} \]
  \( x = \left(\frac{9}{4}\right)^{5/3} \)
  \[ \text{Save Sign} \]
  \[ \text{Vertical Target} \]
Question ___: The graph of $f'(x)$ is shown below. Give the number of critical numbers for $f(x)$.

a. 2
b. 3
c. 4
d. 5
e. 6

Question ___. The graph of $f'(x)$ is shown below. Give the number of local minimums for $f(x)$.

a. 0
b. 1
c. 2
d. 3
e. 4
Curve Sketching
Using Calculus to Graph a Function.

1. Determine the **domain** of the function \( f \).
   
   *For radicals:*
   
   - The domain of any odd root will be \((-\infty, \infty)\).
   - The domain of any even root, set the radicand (inside) \( \geq 0 \) and solve.

2. Find any **asymptotes**—for functions with fractions.

3. Determine any **intercepts** of the function. To find the \( x \) – intercepts, we need to solve the equation \( f(x) = 0 \) and to find the \( y \) – intercepts, evaluate the function at 0 (if 0 is in the domain of \( f \)).

4. Find the **first derivative**, \( f' \). Determine any critical points, intervals of increase/decrease, local extreme points, vertical tangents and cusps.

5. Find the **second derivative**, \( f'' \). Study the sign of \( f'' \) to understand concavity of the function and determine any points of inflection.

6. Plot the **points of interest** (intercepts, local or absolute extreme points, points of inflection).

7. Sketch the graph of \( f \) using the information gathered in the previous steps. Make sure that the function has the right shape (concaves up/down, rises/falls) on the corresponding intervals.

**Example 3:** Use the information given to sketch the graph of function \( f \).

\[
f(x) = \frac{4x - 4}{x^2}
\]

Domain: \((-\infty, 0) \cup (0, \infty)\)

Intercept: \( x \)-intercept: 1

Asymptotes: \( x \)-axis and \( y \)-axis

Increasing: \((0, 2)\)

Decreasing: \((-\infty, 0) \) and \((2, \infty)\)

Relative Extrema: Relative Max at \((2, 1)\)

Concave Down: \((-\infty, 0) \) and \((0, 3)\)

Concave Up: \((3, \infty)\)

Points of Inflection: \((3, \frac{8}{9})\)
Example 4: Use the guide to curve sketching to sketch \( f(x) = x^4 - 4x^3 \).

Domain of \( f(x) \): \((-\infty, \infty)\)

Asymptotes: None

x-intercept(s):
\[ x^4 - 4x^3 = 0 \]
\[ x^3(x - 4) = 0 \]
\[ x = 0, 4 \]

y-intercept(s): \( x = 0 \)

\[ f(0) = 0 \]

Critical Points:
\[ f'(x) = 4x^3 - 12x^2 \]

Find when \( f'(x) = 0 \):
\[ 4x^3 - 12x^2 = 0 \]
\[ 4x^2(x - 3) = 0 \]
\[ x = 0, 3 \]

Dec.: \((-\infty, 0) \cup (0, 3)\)

Incr.: \((3, \infty)\)

R. Min.: \((3, -27)\)

Find when \( f'(x) \) is undefined:

\[ f''(x) = 12x^2 - 24x \]

Find when \( f''(x) = 0 \):
\[ 12x^2 - 24x = 0 \]
\[ 12x(x - 2) = 0 \]
\[ x = 0, 2 \]

Find when \( f''(x) \) is undefined:

R. Max.: \((0, 10)\)

Points of Inflection:
\((0, 0)\)
\((3, -27)\)

Section 3.6 – Curve Sketching

BYE
Example 5: Sketch the graph of \( f(x) = \frac{2x^2}{x^2 - 1} \).

Domain of \( f(x) \):
\[ x \neq -1, 1 \quad (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \]

Asymptotes:
- \( VA: x = -1, 1 \)
- \( HA: y = 2 \)

\( y \)-intercept(s):
\( f(0) = \frac{0}{-1} = 0 \)

\( y \)-intercept:
\( (0, 0) \)

Critical Points:
\[ f'(x) = \frac{-4x}{(x^2 - 1)^2} \]
Find when \( f'(x) = 0 \):
\[ \text{Numerical Derivative} = \frac{2\cdot \text{zero}}{} \]
\[ -4x = 0 \]
\[ x = 0 \]

\( x = 0 \)

\( f(x) \):

\[ f'(x) : \]

Find when \( f'(x) \) is undefined:
\[ x^2 - 1 = 0 \]
\[ x = -1, 1 \]

\( x = -1, 1 \)

\[ f''(x) = \frac{12x^2 + 4}{(x^2 - 1)^3} \]
Find when \( f''(x) = 0 \):
\[ 12x^2 + 4 = 0 \]
\[ 12x^2 = -4 \]
\[ No \, \text{possible} \]

\( f(x) \):

\[ f'(x) : \]

Find when \( f''(x) \) is undefined:
\[ x^2 - 1 = 0 \]
\[ x = \pm 1 \]

\[ f''(x) : \]

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)

\( f'(x) : \)

\( f''(x) : \)

\( f(x) : \)
Try this one: Sketch the graph of \( f(x) = x(x - 1)^{\frac{1}{3}} \).

Domain of \( f(x) \):

Asymptotes:

x-intercept(s):

y-intercept(s):

Critical Points:

\[
f'(x) = \frac{4x - 3}{3(x - 1)^{2/3}}
\]

Find when \( f'(x) = 0 \):

Find when \( f'(x) \) is undefined:

\[
f'(x):
\]

\[
f''(x):
\]

\[
f''(x) = \frac{4x - 6}{9(x - 1)^{5/3}}
\]

Find when \( f''(x) = 0 \):

Find when \( f''(x) \) is undefined:

\[
f'(x):
\]

\[
f''(x):
\]
**Question 47.** The graph of $f'(x)$ is shown below. Classify the critical number for $f(x)$ between the given points.

$f'(x)$

- a. local minimum
- b. local maximum
- c. neither
- d. there is no critical number between the points

**Questions**

Match the function with its **first** derivative.

Functions:

<table>
<thead>
<tr>
<th>34.</th>
<th>35.</th>
<th>36.</th>
<th>37.</th>
<th>38.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
<td><img src="image5.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

Derivatives:

<table>
<thead>
<tr>
<th>A.</th>
<th>B.</th>
<th>C.</th>
<th>D.</th>
<th>E.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image6.png" alt="Graph" /></td>
<td><img src="image7.png" alt="Graph" /></td>
<td><img src="image8.png" alt="Graph" /></td>
<td><img src="image9.png" alt="Graph" /></td>
<td><img src="image10.png" alt="Graph" /></td>
</tr>
</tbody>
</table>