Section 4.1: Inverse Functions

A function is said to be **one-to-one** (1-1) if there are no two distinct numbers in the domain of \( f \) that produce the same value. In other words, two different \( x \) values cannot have the same \( y \) value. If a function has an inverse, then we say it’s **invertible**.

If a function is 1-1, then it has an **inverse function**, denoted as \( f^{-1} \), which reverses what the first function did. The domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \).

**Example:**

Celsius to Fahrenheit: \( \frac{9}{5} C + 32 = F \)

and

Fahrenheit to Celsius: \( \frac{5}{9} (F - 32) = C \)

Geometrically
Property of Inverse Functions
Let $f$ and $g$ be two functions such that $(f \circ g)(x) = x$ for every $x$ in the domain of $g$ and $(g \circ f)(x) = x$ for every $x$ in the domain of $f$ then $f$ and $g$ are inverses of each other.

Given a functions whose graph is known or the given the graph of a function, we can use the Horizontal Line Test to determine if the function is 1-1.

Example 1: Is the following graph of $f(x) = 2x^{\frac{3}{2}}$ 1-1?

Example 2: Is the function, $f(x) = (x + x^2)^7$ 1-1?

Example 3: Is $f(x) = 3\sin x$ invertible on $[-\frac{\pi}{2}, \frac{\pi}{2}]$?
A function is **monotonic** if it is always increasing or always decreasing on its domain.

**Recall:**
- If \( f'(x) > 0 \) on its domain, then \( f \) is increasing and; hence, monotonic.
- If \( f'(x) < 0 \) on its domain, then \( f \) is decreasing and; hence, monotonic.

**Theorem:** If \( f \) is monotonic, then \( f \) is an invertible function.

**Example 4:** Is the following function 1-1? If so, give the equation of the inverse function.

\[
f(x) = \frac{x-1}{x+1}
\]

\[
f'(x) = \frac{(x+1)(1) - (1)(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0 \quad \text{for all } x \text{ in Domain of } f
\]

\[
\text{Inverse } f^{-1} \\
x = \frac{y-1}{y+1} \\
x = \frac{y(1-x)}{y+1} \\
y = \frac{-x-1}{x-1} \quad \text{or} \quad \frac{x+1}{1-x} \\

\text{P.F.: 13}
\]

**Question 21:** Is \( y = (1 + x^2)^7 \) one-to-one?
A. Yes  
B. No  

**Question 22:** Is \( y = x^{\frac{1}{3}} \) one-to-one?
A. Yes  
B. No
Sometimes it’s too long or too difficult to find the equation of the inverse, yet we may want to know if a function has an inverse or not.

Example 5: Is \( f(x) = x^3 + 3x \) invertible?

\[
\begin{align*}
  f'(x) &= 3x^2 + 3 > 0 \quad \text{ALWAYS ... something} \\
  f'(x) &= 0 \quad \text{or} \quad f'(x) = \text{undefined} \\
  3x^2 + 3 &= 0 \\
  3x^2 &= -3 \\
  x^2 &= -1 \\
\end{align*}
\]

No Solution

Example 6: Let \( f(x) = x^3 - kx^2 + 2x \). For what values of \( k \) is \( f(x) \) one-to-one?

\[
\begin{align*}
  f'(x) &= 3x^2 - 2kx + 2 \quad \text{Parabola} \\
  \text{Discriminant} &= b^2 - 4ac \quad (\text{Inside of } f') \\
  D &= b^2 - 4ac \\
  D > 0 &\rightarrow 2 \text{ Real Solutions} \\
  D = 0 &\rightarrow 1 \text{ Real Solution} \\
  D < 0 &\rightarrow 0 \text{ Real Solutions} \\
  D &\leq 0 \\
  b^2 - 4ac &\leq 0 \\
  (-2k)^2 - 4(3)(2) &\leq 0 \\
  4k^2 - 24 &\leq 0 \\
  k^2 - 6 &\leq 0 \\
  (k + \sqrt{6})(k - \sqrt{6}) &\leq 0 \\
  k &= -\sqrt{6}, +\sqrt{6} \quad (\text{Zeros}) \\
  -\sqrt{6} &\leq k \leq \sqrt{6}
\end{align*}
\]
Question 24: Is \( y = x^2 \) monotonic?
A. Yes
B. No

Question 36: Is \( y = -x \) monotonic?
A. Yes
B. No

Finding the Derivative of the Inverse Function

Theorem: If \( f(x) \) is continuous and invertible then \( f^{-1}(x) \) is continuous.

Theorem: If \( f(x) \) is differentiable (so must be continuous) and invertible, and \( f'(x) \neq 0 \), then \( f^{-1}(x) \) is differentiable.

If \( f(a) = b \) and \( f'(a) \neq 0 \), then
\[
(f^{-1})'(b) = \frac{1}{f'(a)}.
\]

Example 7: For \( f(x) = x^3 \), we know that \( f(2) = 8 \).
Find \( (f^{-1})'(8) \).

\[
(f^{-1})'(8) = \frac{1}{f'(2)} = \frac{1}{12}
\]

\[
(f^{-1})(x) = \sqrt[3]{x}
\]

\[
(f^{-1})'(x) = \frac{1}{3x^{\frac{2}{3}}}
\]

\[
(f^{-1})'(8) = \frac{1}{3(8)^{\frac{2}{3}}} = \frac{1}{3 \cdot 4} = \frac{1}{12}
\]
Example 8: If \( f \) is invertible, and \( f(1) = 2, \ f(3) = 1, \ f'(1) = 4, \ f'(3) = 5, \ f'(2) = 6, \)
find \( (f^{-1})'(1). \)

\[
\begin{align*}
\text{If } b \text{ is } y\text{-value in } f(x), \quad & f(3) = 1 \checkmark \\
\frac{f''(3)}{f'(3)} &= ? \\
\frac{f''(5)}{f'(5)} &= 5 \\
(f^{-1})'(1) &= \frac{1}{f'(3)} = \frac{1}{5}
\end{align*}
\]

Example 9: Given \( f(x) = x^3 + 1 \), find \( (f^{-1})'(33) \) if possible.

\[
\begin{align*}
a^5 + 1 &= 33 \\
a^5 &= 32 \\
a &= 2 \\
(f^{-1})'(33) &= \frac{1}{f'(2)} \\
\frac{f'(x)}{f'(2)} &= 5x^4 \\
\frac{f'(2)}{f'(2)} &= 5(2)^4 \\
&= 80
\end{align*}
\]
Example 10: If \( f(x) = \sin x + 5 \cos x, \quad x \in \left[ 0, \frac{\pi}{2} \right] \), find \( (f^{-1})'(3\sqrt{2}) \).

\[
\sin a + 5 \cos a = 3\sqrt{2} \\
a = \frac{\pi}{4} \\
\frac{\sqrt{2}}{2} + 5 \left( \frac{\sqrt{2}}{2} \right) = \frac{6\sqrt{2}}{2} \\
\frac{f'(x)}{f'(\frac{\pi}{4})} = -\frac{1}{2\sqrt{2}} \\
\frac{f'(\frac{\pi}{4})}{f'(\frac{\pi}{4})} = -\frac{\sqrt{2}}{4}
\]

Example 11: Let \( f(x) = x^5 + 2x^3 + 2x \). The point \((-5, -1)\) is on the graph of \( f^{-1}(x) \). Find \( (f^{-1})'(-5) \), then give an equation for the tangent line to the graph of \( f^{-1}(x) \) at the point \((-5, -1)\).

\[
f^{-1}(-5) = -1 \quad \Rightarrow \quad f'(-1) = -5 \\
(f^{-1})'(-5) = \frac{1}{f'(-1)} \\
\]

\[
f'(x) = 5x^4 + 6x^2 + 2 \\
f'(-1) = 13
\]

\[
\text{Tangent line} \\
y - y_1 = m (x - x_1) \\
y - (-1) = \frac{1}{13} (x - (-5)) \\
y + 1 = \frac{1}{13} (x + 5)
\]
Try this one: Is the following function 1-1? If so, give the equation of the inverse function.

\[ g(x) = x + \frac{4}{x} \]

Let \( f(x) = \frac{1}{3}x^3 - x^2 + kx \). For what values of \( k \) is \( f(x) \) invertible?
Section 4.2: The Exponential Function

The exponential function $f$ with base $a$ is defined by $f(x) = a^x (a > 0 \text{ and } a \neq 1)$ and $x$ is any real number.

If $a = e$ (the natural base, $e \approx 2.7183$), then we have $f(x) = e^x$.

- $e^0 = 1$
- $e^1 = e$
- $e^{-1} = \frac{1}{e}$

Graphs

- If $a > 1$, the graph of $f(x) = a^x$ looks like (larger $a$ results in a steeper graph):
- If $0 < a < 1$, the graph of $f(x) = a^x$ looks like (smaller $a$ results in a steeper graph):

Their domain is $(-\infty, \infty)$ and range is $(0, \infty)$.

Both graphs have a horizontal asymptote of $y = 0$ (the x-axis).
Derivatives of Exponential Functions

The exponential function with base “e” has the unique property that it is its own derivative.

\[
\frac{d}{dx}(e^x) = e^x
\]

If the exponent is more than just \(x\), you’ll need to use the chain rule to differentiate.

**Example 1:** Differentiate the following functions.

a. \( y = 10e^x \)

b. \( g(x) = e^{\sqrt{x}} + e^{10/x} \)

c. \( f(x) = 4x^3e^{x^2} \) then find equations of the tangent and normal line at \((2,32e^4)\).