Example 6: Determine where \( f(x) = x \ln x \) is increasing/decreasing, any relative extrema, concavity, and any points of inflection.

\[
\frac{d}{dx} \left[ \ln \left( u(x) \right) \right] = \frac{1}{u} \cdot u'
\]

\( D: (0, \infty) \)

\( \text{Exp} \) = \( \frac{\ln x}{x} \)

\( \text{Base} \) = \( x \)

\( \text{Log} \) \( \text{(Value)} \) = \( \ln x \)

\( \text{True} \)

\[
f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1
\]

\[
f'(x) = 0 \quad \text{or} \quad f'(x) = \text{undefined}
\]

\[
\ln x + 1 = 0
\]

\[
\ln x = -1
\]

\[
e -1
\]

\[
x = e^{-1} (N)
\]

\[
f''(x) = \frac{1}{x}
\]

\[
f''(x) = 0 \quad \text{or} \quad f''(x) = \text{undefined}
\]

\[
x = 0 \quad (\text{Not in Domain})
\]

\[
\frac{1}{x}
\]

\[
\text{Concave Up: } (0, \infty)
\]

Question 42: Find \( \frac{d}{dx} \left( \ln \left( 5 - x \right)^6 \right) \)

a. \( \frac{1}{(5-x)^6} \)

b. \( \frac{6}{x-5} \)

c. \( -6(5-x)^5 \)

d. \( 6(5-x)^5 \)
Different Bases

Change of base formula: \( \log_a b = \frac{\ln b}{\ln a} \)

\[
\frac{d}{dx} (\log_a x) = \frac{1}{x} \cdot \frac{1}{\ln a}
\]

\[
\frac{d}{dx} \left( \frac{1}{\ln a} \cdot \ln x \right) = \frac{1}{\ln a} \cdot \frac{1}{x}
\]

If \( u \) is a function of \( x \), then use the chain rule.

Example 7: Find the derivative of each function.

a. \( y = \log_2 x \) \[ y' = \frac{1}{\ln 2} \cdot \frac{1}{x} \]

Rewrite

b. \( y = \log_3(4x^3 + 2x) \) \[ y' = \frac{1}{\ln 3} \cdot \frac{1}{4x^3 + 2x} \cdot (12x^2 + 2) \]

Rewrite

\[
\frac{\ln (4x^3 + 2x)}{\ln 3} = \frac{1}{\ln 3} \cdot \ln (4x^3 + 2x)
\]

Quotient rule

\[
f(x) = \frac{\log_9 x}{x^2} = \frac{\ln x}{\ln 9} \cdot \frac{1}{\ln 9} \cdot \frac{\ln x}{x^2}
\]

\[
f'(x) = \frac{1}{\ln 9} \cdot \left[ \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{(x^2)^2} \right]
\]

\[
f'(x) = \frac{1}{\ln 9} \cdot \frac{X - 2x \ln x}{x^4}
\]
We know how to find the derivative of functions such as:

- $x^5$ --- use the power rule
- $5^x$ --- use exponential rule

But how about $x^x$? We will use a method called Logarithmic Differentiation.

**Example 8:** Find the derivative of $y = x^{\tan x}$.

Step 1: Take **natural logs of both sides** of the equation.

$$\ln y = \ln x^\tan x$$

Step 2: Use any rules of logs to simplify the equation.

$$\ln y = \tan x \cdot \ln x$$

Step 3: Take the derivative of both sides of the equation.

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sec^2 x \cdot \ln x + \tan x \cdot \frac{1}{x}$$

Step 4: Solve for $y'$. *Remember what $y$ was equal to originally.*

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \left[ \sec^2 x \cdot \ln x + \tan x \cdot \frac{1}{x} \right]$$

$$= x^{\tan x} \cdot \left[ \sec^2 x \cdot \ln x + \frac{\tan x}{x} \right]$$
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**Example 9:** Find the derivative of \( y = x^{2x+1} \).

**Step 1:** Take natural logs of both sides of the equation.

\[
\ln y = \ln x^{2x+1}
\]

**Step 2:** Use any rules of logs to simplify the equation.

\[
\ln y = (2x+1) \cdot \ln x
\]

**Step 3:** Take the derivative of both sides of the equation.

\[
\frac{1}{y} \cdot y' = (2x+1) \cdot \ln x + (2x+1) \cdot \frac{1}{x}
\]

**Step 4:** Solve for \( y' \). *Remember what \( y \) was equal to originally.*

\[
y' = y \cdot \left[ 2 \ln x + \frac{2x+1}{x} \right]
\]

\[
= x^{2x+1} \cdot \left( 2 \ln x + 2 + \frac{1}{x} \right)
\]
Try these:

Find the derivative of: \( y = \log(\cos(4x)) \).

Let \( f(x) = x \ln(\cos(2x)) \), find \( f'(\pi) \).

Determine where \( f(x) = 2x^2 \ln\left(\frac{x}{4}\right) \) is increasing/decreasing.

Find the points of inflection for the function \( f(x) = 4x^2 \ln\left(\frac{x}{4}\right) \).

Find the derivative of \( f(x) = e^{2x} \ln(2x) \).

Find the derivative of \( f(x) = \ln\left(5^{-x^2+x}\right) \).

Find the slope of the tangent line to the curve \( y = (2 + \cos x)^{4+\sin x} \) at \( x = 2\pi \).
Question 1: Let \( f(x) = 2x + \ln x \). Find \( (f^{-1})'(2) \).

\[
\frac{d}{dx} f(2) = 2 + \frac{1}{x}
\]

a. 1  b. 1/2  c. 1/3  d. 1/(ln2)

\[ f'(1) = \frac{3}{2} \]

Question 17. Given the graph of the derivative of \( f \), classify the point at \( x=a \).

A. local maximum
B. local minimum
C. point of inflection
D. none of these

Question 37. Given the graph of the derivative of \( f \), classify the point at \( x=b \).

A. local maximum
B. local minimum
C. point of inflection
D. none of these
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**Question** . Given the graph of the derivative of $f$, classify the point at $x=c$.

A. local maximum  
B. local minimum  
C. point of inflection  
D. none of these

**Question** . Given the graph of the derivative of $f$, classify the point at $x=d$.

A. local maximum  
B. local minimum  
C. point of inflection  
D. none of these

**Question** . Find the slope of the tangent line to $y = e^{\cos(2x)}$ at $x=0$.

A. 1  
B. $e$  
C. 0  
D. $e^2$  
E. none of these

\[
y' = e^{\cos(2x)} \cdot (-\sin(2x) \cdot 2)
\]
\[
e^{\cos(0)} \cdot (-\sin(0) \cdot 2)
\]
\[
e^1 \cdot (-0 \cdot 2) = 0
\]
Section 4.4: The Inverse Trigonometric Functions

In section 4.1, we learned that in order to have an inverse, a function must be one-to-one. Since trigonometric functions are periodic, they are NOT one-to-one. To define the inverse trig functions, we must restrict the usual domains.

The function \( \sin(x) \) is graphed below. Notice that this graph does not pass the horizontal line test; therefore, it is not invertible.

However, if we restrict it from \( x = -\frac{\pi}{2} \) to \( x = \frac{\pi}{2} \), then we have created the “Restricted” sine function and it’s one-to-one. Since the restricted sine function is one-to-one, it has an inverse \( f(x) = \sin^{-1}(x) = \arcsin(x) \).
The function \( \tan(x) \) is graphed below. Notice that this graph does not pass the horizontal line test; therefore, it is not invertible.

However, if we restrict it from \( x = -\frac{\pi}{2} \) to \( x = \frac{\pi}{2} \) then we have created the "Restricted" tangent function and it’s one-to-one. Since the restricted tangent function is one-to-one, it has an inverse \( f(x) = \tan^{-1}(x) = \arctan(x) \).

\[
\tan(x) \\
D: \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \\
R: (-\infty, \infty)
\]

\[
\tan^{-1}(x) \\
D: (-\infty, \infty) \\
R: \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)
\]
The function $\cos(x)$ is graphed below. Notice that this graph does not pass the horizontal line test; therefore, it does not have an inverse.

However, if we restrict it from $x = 0$ to $x = \pi$ then we have created the “Restricted” cosine function and it’s one-to-one. Since the restricted cosine function is one-to-one, it has an inverse $f(x) = \cos^{-1}(x)$.

**Domain:** $[0, \pi]

**Range:** $[-1, 1]$
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The restrictions when working with arcsine are: \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) which are angles from QUADRANTS 1 AND 4.

Example 1: Compute \( \sin^{-1}\left( -\frac{1}{\sqrt{2}} \right) \).

\[ \sin^{-1}\left( -\frac{\sqrt{2}}{2} \right) = -\frac{\pi}{4} \]

The restrictions when working with arccosine are:

\[ [0, \pi] \] which are angles from QUADRANTS 1 AND 2.

Example 2: Compute \( \cos^{-1}\left( \frac{1}{2} \right) \).

The restrictions for arcsec are: \( \left[ 0, \frac{\pi}{2} \right] \cup \left( \frac{\pi}{2}, \pi \right] \). The restrictions for arccsc are: \( \left[ -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right] \).
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The restrictions when working with arctangent are: \( \left( \frac{-\pi}{2}, \frac{\pi}{2} \right) \), which are angles from QUADRANTS 1 AND 4.

\[ \tan(\theta) = \frac{\sqrt{3}}{3} \]

**Example 3:** Compute \( \arctan \left( \frac{1}{\sqrt{3}} \right) \).

\[ \arctan \left( \frac{\sqrt{3}}{3} \right) = \frac{\pi}{6} \]

**Example 4:** Find \( \sin^{-1} \left( \cos \left( \frac{2\pi}{3} \right) \right) \).

\[ \sin^{-1} \left( -\frac{1}{2} \right) = -\frac{\pi}{6} \]
Example 5: Find $\cot \left( \tan^{-1} \left( -\sqrt{3} \right) \right)$.

\[
= \cot \left( -\frac{\pi}{3} \right)
\]

\[
= -\frac{\sqrt{3}}{3} \quad \text{or} \quad -\frac{1}{\sqrt{3}}
\]

For some problem we’ll need to recall the following identities:

\[
\sin(2\alpha) = 2\sin \alpha \cos \alpha
\]
\[
\cos(2\alpha) = 1 - 2\sin^2 \alpha
\]
\[
\cos(2\alpha) = 2\cos^2 \alpha - 1
\]
\[
\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha
\]

Example 6: Find $\sin \left( 2 \arcsin \left( \frac{5}{13} \right) \right)$.

\[
= \sin \left( 2 \cdot \alpha \right)
\]

\[
= 2 \sin \alpha \cdot \cos \alpha
\]

\[
= 2 \cdot \frac{5}{13} \cdot \frac{12}{13}
\]

\[
= \frac{120}{169}
\]
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**Question 2:** \( \sin \left( \sin^{-1} \left( \frac{\pi}{6} \right) \right) \)

a. \( \frac{1}{2} \)
b. \( \frac{\sqrt{3}}{2} \)
c. \( \frac{\pi}{6} \)
d. \( \frac{\pi}{3} \)
e. None of the Above

**Question 3:** \( \frac{d}{dx} \left( e^{5 \ln x^2} \right) \)

a. \( e^{5 \ln x^2} \)
b. \( 10 e^{5 \ln x^2} \)
c. \( \frac{e^{5 \ln x^2}}{x} \)
d. \( 10x^9 \)
e. None of the Above
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Derivative Formulas \((u \text{ is a function of } x)\):

\[
\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}
\]

\[
\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \quad \frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}
\]

\[
\frac{d}{dx}[\text{arcsec } x] = \frac{1}{|x|\sqrt{x^2-1}} \quad \frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}}
\]

Example 7: Differentiate: \(y = \cos(\arcsin (2x))\).

\[
y' = -\sin(\arcsin (2x)) \cdot \frac{1}{\sqrt{1-(2x)^2}} \\
\]

\[
= -\frac{2 \sin(\arcsin (2x))}{\sqrt{1-4x^2}} \\
\]

Example 8: Differentiate: \(y = \sec^{-1}(\sqrt{7})\).

\[
y' = -\frac{14x}{|7x^2| \cdot \sqrt{(7x^2)^2-1}} = \frac{u'}{|u|\sqrt{u^2-1}}
\]
Example 9: Differentiate: \( f(x) = e^{\arctan(x)} + \arcsin(\ln x) \)

Question 4: Evaluate \( \tan^{-1} \left( \tan \frac{7\pi}{4} \right) \)

a. 1  b. -1  c. \( \frac{\pi}{4} \)  d. \( -\frac{\pi}{4} \)  e. \( \frac{7\pi}{4} \)

Example 10: Differentiate: \( f(x) = 6e^{\arcsin x} \)
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**Question 5:** Evaluate \( \sin \left( \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) \right) \)

a. \( \frac{\pi}{6} \)  b. \( -\frac{\pi}{6} \)  c. \( \frac{11\pi}{6} \)  d. \( -\frac{\sqrt{3}}{2} \)  e. \( -\frac{\sqrt{3}}{2} \)

**Example 11:** Given \( g(x) = \arcsin \left( \frac{e^x}{2} \right) \), find the equation for the tangent line to the graph of this function at \( x = 0 \).

**Example 12:** Differentiate: \( f(x) = \sqrt{25 - x^2} + 5 \arcsin \left( \frac{x}{5} \right) \)
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Try these:

Find \( \tan^{-1}\left( \sin\left(\frac{5\pi}{6}\right) \right) \).

Find \( \cos\left(2\arcsin\left(\frac{3}{5}\right)\right) \).

Differentiate:

a. \( y = \arcsin(2x^2 + 5x) \)

b. \( f(x) = \ln(\arctan(3x^2 + 2x)) \)

c. \( g(x) = \frac{x}{\sqrt{36 - x^2}} - \arcsin\left(\frac{x}{6}\right) \)

d. \( h(x) = \arcsin\left(\frac{e^{6x}}{3}\right) \)