Section 1.5: The Intermediate Value Theorem

If \( f(x) \) is continuous on the closed interval \([a, b]\) and \( N \) is a real number such that \( f(a) \leq N \leq f(b) \), then there is at least one value \( c \) in \((a, b)\) so that \( f(c) = N \).

Use the intermediate value theorem to show that there is a solution to the given equation in the indicated interval.

**Example 1:** \( x^2 - 4x + 3 = 0 \), \([2, 4]\)

\( f(x) \) is continuous between \([2, 4]\)

\[
\begin{align*}
 f(2) &= (2)^2 - 4(2) + 3 = -1 \\
 f(4) &= (4)^2 - 4(4) + 3 = 3
\end{align*}
\]

\( f(2) < 0 < f(4) \) therefore there exist a \( c \)-value between \((2, 4)\) s.t. \( f(c) = 0 \)

**Example 2:** \( 2 \tan x - x = 1 \), \([0, \frac{\pi}{4}]\)

\( f(x) \) is continuous between \([0, \frac{\pi}{4}]\)

\[
\begin{align*}
 f(0) &= 0 \\
 f\left(\frac{\pi}{4}\right) &= 2 \cdot \tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} \\
 &= 2 - \frac{\pi}{4}
\end{align*}
\]

\( f(0) < 1 < f\left(\frac{\pi}{4}\right) \) so there is a \( c \)-value s.t. \( f(c) = 1 \) s.t. \( c \) between \((0, \frac{\pi}{4})\)
Example 3: Given \( f(x) = x^2 - 3x + 1 \), the IVT applies to the interval \([0, 6]\) for \( f(c) = 5 \). Find the value(s) that satisfy the conclusion of the theorem.

\[
\begin{align*}
 f(0) &= 0 - 3(0) + 1 = 1 \\
 f(6) &= 36 - 3(6) + 1 = 7
\end{align*}
\]

Since \( f(0) = 1 < 5 < f(6) = 7 \), there exists at least one \( c \) between \((0, 6)\) such that \( f(c) = 5 \).

\[
\begin{align*}
 f(c) &= 5 \\
 c^2 - 3c + 1 &= 5 \\
 c^2 - 3c - 4 &= 0 \\
 (c - 4)(c + 1) &= 0
\end{align*}
\]

So, \( c = 4 \) or \( c = -1 \). Since \( c = 4 \) is not between \((0, 6)\), we choose \( c = -1 \).

Example 4: Does the IVT guarantee at least one solution for \( f(x) = 2 \sin x - 8 \cos x - 3x^2 \) on the interval \([0, \frac{\pi}{4}]\)?

\[
\begin{align*}
 f(0) &= 2(0) - 8(1) - 3(0) = -8 \\
 f\left(\frac{\pi}{4}\right) &= 2\left(\frac{\pi}{4}\right) - 8\left(\frac{\sqrt{2}}{2}\right) - 3\left(\frac{\pi}{4}\right)^2 = -3\sqrt{2} - \frac{3\pi^2}{16}
\end{align*}
\]

Therefore, \( f(0) = -8 < 0 < f\left(\frac{\pi}{4}\right) \), and \( 0 \) is not between \( f(0) \) and \( f\left(\frac{\pi}{4}\right) \). Thus, the IVT does not guarantee a solution in \([0, \frac{\pi}{4}]\).

Example 5: Does the IVT guarantee at least one solution for \( f(x) = \frac{x - 1}{x - 4} \) on the interval \([0, \frac{\pi}{4}]\)?

\[
\begin{align*}
 f(0) &= \frac{0 - 1}{0 - 4} = \frac{1}{4} \\
 f\left(\frac{\pi}{4}\right) &= \frac{\frac{\pi}{4} - 1}{\frac{\pi}{4} - 4} = \frac{\frac{\pi - 4}{4}}{\frac{\pi - 16}{4}} = \frac{\pi - 4}{\pi - 16} > 0
\end{align*}
\]

Since \( f(0) = \frac{1}{4} > 0 \) and \( f\left(\frac{\pi}{4}\right) > 0 \), there is no guarantee of a solution in \([0, \frac{\pi}{4}]\).
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The Extreme-Value Theorem

If \( f \) is continuous on a bounded interval \([a,b]\), then \( f \) takes on both a \textbf{maximum value} and a \textbf{minimum value}.

If the function is not continuous, it may or may not take minimum or maximum value.

**Example 5:** State whether it is possible to have a function \( f \) defined on the indicated interval and meets the given conditions:

\( f \) is defined on \([3, 6]\); \( f \) is continuous on \([3, 6]\), takes on the values -3 and 3 but does not take on the value 0.

\[
\begin{array}{c}
\text{Not Possible}\\
\text{Would contradict IVT.}
\end{array}
\]

**Example 6:** State whether it is possible to have a function \( f \) defined on the indicated interval and meets the given conditions:

\( f \) is defined on \([4, 5]\); \( f \) is continuous on \([4, 5]\), \textbf{minimum value} \( f(5) = 4 \) and \textbf{no maximum value}.

\[
\begin{align*}
\frac{-1}{x-5} & \quad 4 \leq x < 5 \\
4 & \quad x > 5
\end{align*}
\]
Section 1.6: The Pinching Theorem; Trigonometric Limits

**Theorem:** Let \( p > 0 \) and \( c \) be a real number. Suppose \( f(x), g(x) \) and \( h(x) \) are defined in an open interval \((c - p, c + p)\) (except possibly at \( x = c \)). If \( f(x) \leq g(x) \leq h(x) \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L \) then \( \lim_{x \to c} g(x) = L \).

Before we move on, we should note that:

\[
\lim_{x \to 0} \sin(x) = 0 \quad \text{and} \quad \lim_{x \to 0} \cos(x) = 1
\]

These limits can be found by direct substitution or by simply recalling their graphs.

**Example 1:** Evaluate.

a. \[
\lim_{x \to 0} \frac{\sin(3x)}{4x - 1} = \frac{\sin(3 \cdot 0)}{4(0) - 1} = \frac{0}{1} = 0
\]

b. \[
\lim_{x \to 0} \frac{1 - 5 \cos(3x)}{12} = \frac{1 - 5 \cos(3 \cdot 0)}{12} = \frac{1 - 5(1)}{12} = \frac{-4}{12} = -\frac{1}{3}
\]
Take $\lim_{x \to 0} \frac{\sin(x)}{x}$. If we use direct substitution we get an indeterminate form $\frac{0}{0}$, but the Pinching Theorem allows us to prove that the limit is 1.

\[ f(x) = \cos(x) \]

\[ h(x) = 1 \]

\[ \lim_{x \to 0} h(x) = 1 \]

\[ \lim_{x \to 0} f(x) = 1 \]

Example 2: Evaluate.

a. \[ \lim_{x \to 0} \frac{\sin(7x)}{x} \cdot \frac{7}{7} = \lim_{x \to 0} \frac{\sin(7x)}{7x} \cdot 7 = 1 \cdot 7 = 7 \]

b. \[ \lim_{x \to 0} \frac{\sin(2x)}{5x} \cdot \frac{2}{2} = \lim_{x \to 0} \frac{\sin(2x)}{2x} \cdot \frac{2}{5} = \frac{2}{5} \]

In fact: \[ \lim_{x \to 0} \frac{\sin(ax)}{bx} = \frac{a}{b} \]

\[ \lim_{x \to 0} \frac{\sin(ax)}{bx} \cdot \frac{a}{a} = \lim_{x \to 0} \frac{\sin(ax)}{a \cdot x} = \frac{a}{b} \]

Example 3: Evaluate

a. \[ \lim_{x \to 0} \frac{\sin(7x)}{3x} = \frac{7}{3} \]
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b. \( \lim_{x \to 0} \frac{\sin(8x)}{2x} = \frac{8}{2} = 4 \)

c. \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \)

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d. \( \lim_{x \to 0} \frac{5x}{\sin(3x)} \cdot \frac{2}{3} = \lim_{x \to 0} \frac{3x}{\sin(3x)} \cdot \frac{5}{3} = 1 \)

e. \( \lim_{x \to 0} \frac{\sin(6x) + 3}{4x} = \lim_{x \to 0} \frac{\sin(6x)}{4x} + \frac{3}{4x} = \frac{6}{4} + \text{undefined} \rightarrow \text{DNE} \)

Take \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \). If we use direct substitution we get an indeterminate form \( \frac{0}{0} \), but the Pinching Theorem allows us to prove that the \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \)

In fact: \( \lim_{x \to 0} \frac{1 - \cos(ax)}{bx} = 0 \)

Example 4: Evaluate.

\( a. \lim_{x \to 0} \frac{1 - \cos(6x)}{8x} \)

\( b. \lim_{x \to 0} \frac{\cos(2x) - 1}{3x} \)
#10.
Find the values of $b$ and $c$ in the following figure. Then find a positive number $\delta$ such that $|\sqrt{x} - 2| < 0.05$ if $0 < |x - 4| < \delta$.

\[\varepsilon = 0.05\]
\[b = (1.95)^2 = 3.8025\]
\[c = (2.05)^2 = 4.2025\]
\[\delta = 0.1975\]

$1.95 = \sqrt{x}$

$2.05 = \sqrt{x}$

$\delta_1 = 0.1975$

$\delta_2 = 0.2025$
And in some cases you’ll need to use older information

\[
\begin{align*}
\sin^2(\theta) + \cos^2(\theta) &= 1, \\
\tan^2(\theta) + 1 &= \sec^2(\theta), \\
1 + \cot^2(\theta) &= \csc^2(\theta).
\end{align*}
\]

\[
\begin{align*}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta), \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta), \\
\cos(2\theta) &= 2\cos^2(\theta) - 1, \\
\cos(2\theta) &= 1 - 2\sin^2(\theta).
\end{align*}
\]

**Example 5:**

a. \[
\lim_{x \to 0} \frac{\cos x \tan x}{7x}
\]

b. \[
\lim_{x \to 0} \frac{1 - \cos^2(2x)}{5x}
\]

c. \[
\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}
\]

d. \[
\lim_{x \to 0} \frac{3x^2}{1 - \cos(3x)}
\]