Section 2.2: Derivatives of Polynomials and Trigonometric Functions

As you saw in several examples in Section 2.1, finding the derivative of a function using the definition is quite tedious. We now introduce some rules that will make our computations much easier.

A constant function is represented by a horizontal line. What is the slope of a horizontal line?

Then the derivative of any constant is zero!

**Rule 1: The Derivative of a Constant**

\[
\frac{d}{dx}[k] = 0, \text{ where } k \text{ is a constant.}
\]

**Example 1:** Find the derivative of \( f(x) = -17 \).

\[ f'(x) = 0 \]

**Rule 2: The Power Rule**

\[
\frac{d}{dx}[x^n] = nx^{n-1} \text{ for any real number } n
\]

**Example 2:** Find the derivative of each function

\[
\begin{align*}
  f(x) &= x^5 & f'(x) &= 5x^4 \\
  g(x) &= x^{-12} & g'(x) &= -12x^{-13} \\
  h(x) &= \sqrt{x} = x^{1/2} & h'(x) &= \frac{1}{2}x^{\frac{1}{2}} \\
  i(x) &= x^{3/5} & i'(x) &= \frac{3}{5}x^{1/5}
\end{align*}
\]

**Rule 3: Derivative of a Constant Multiple of a Function**

\[
\frac{d}{dx}[kf(x)] = k \frac{d}{dx}[f(x)] \text{ where } k \text{ is any real number}
\]

**Example 3:** Find the derivative of each function.

a. \( f(x) = 4x^5 \)

\[
\begin{align*}
  f'(x) &= 4 \cdot 5x^4 \\
  &= 20x^4
\end{align*}
\]
b. \( h(x) = \frac{2}{3x^3} = \frac{2}{3} x^{-4} \)

\[
h'(x) = \frac{2}{3} \cdot -4 x^{-5} = -\frac{8}{3} x^{-5}
\]

Evaluate the following:

21. \( \lim_{{x \to -3}} f(x) \)  
28. \( \lim_{{x \to 2}} f(x) \)

A. 1   B. -2   C. 0
D. -1   E. DNE

**Theorem:** Let \( k \) be any real number. If \( f \) and \( g \) are differentiable at \( x \), then so are \( f + g \) and \( f - g \). Moreover, 
\[
(f \pm g)'(x) = f'(x) \pm g'(x)
\]

**Example 4:** Find the derivative: 
\[ f(x) = -\frac{3}{4} x^6 + \frac{5}{x^3} + 6\sqrt{x^2} + 13 \]

\[
\frac{f(x)}{x^5} = -\frac{3}{4} x^5 + 5 x^{-4} + 6 x^{\frac{1}{2}} + 13
\]

\[
\frac{f'(x)}{x^5} = -\frac{3}{4} x^5 + 5 x^{-4} + \frac{12}{2} x^{-\frac{1}{2}} + 0
\]

\[
= -\frac{15}{4} x^5 - 15 x^{-4} + \frac{12}{3} x^{-\frac{1}{2}}
\]

\[
= -\frac{9}{2} x^5 - \frac{15}{4} x^{-4} + \frac{4}{\sqrt{x}}
\]
Example 5: Find the derivative: \( f(x) = \frac{3x^5 - 7x^2 + 3}{x^3} \) = \( \frac{3x^5}{x^3} - \frac{7x^2}{x^3} + \frac{3}{x^3} \)

\[
\frac{f(x)}{\frac{\text{d}x}{\text{d}x}} = 3x^2 - 7x^{-1} + 3x^{-3}
\]

\[
\frac{f''(x)}{\frac{\text{d}^2x}{\text{d}x^2}} = 6x + \frac{7}{x^2} - \frac{9}{x^4}
\]

OR

\[
\frac{6x^5 + 7x^2 - 9}{x^4}
\]

In some cases, we may want to find all points for which the tangent line to the graph of \( f \) is horizontal or equal to a specified number. Set the derivative equal to the given number and solve for \( x \)

Example 6: Find all x-value(s) on the graph of \( f(x) = x^3 + 5x^2 \) where the tangent line is horizontal.

\[
\frac{f'(x)}{\frac{\text{d}x}{\text{d}x}} = 3x^2 + 10x
\]

\[
\frac{f''(x)}{\frac{\text{d}^2x}{\text{d}x^2}} = 0
\]

\[
3x^2 + 10x = 0
\]

\[
x(3x + 10) = 0
\]

\[
x = 0, \quad 3x + 10 = 0
\]

\[
3x = -10
\]

\[
x = \frac{-10}{3}
\]

\[
\chi = 0, \quad \frac{-10}{3}
\]
Example 7: Find all x-value(s) on the graph of \( f(x) = x^2 + 4x + 1 \) where \( f'(x) = 5 \).

\[
f'(x) = 2x + 4 + 0
\]

\[
f'(x) = 5
\]

\[
2x + 4 = 5
\]

\[
x = \frac{1}{2}
\]

Tangent and Normal Lines

We already known what a tangent line is all about.

A normal line to a curve at a particular point is the line through that point and perpendicular to the tangent.

Recall from Pre-Algebra the slope of any line perpendicular to a line with slope is the negative reciprocal.
**Example 8:** Find the equations of the tangent and normal line to \( f(x) = 3x^2 + 4x + 2 \) at \( x = -1 \)

\[
\begin{align*}
  f'(x) &= 6x + 4 \\
  f''(x) &= 6(-1) + 4 = -2 & \text{Tangent Slope at } x = -1 \\
end{align*}
\]

Normal Slope at \( x = -1 \) is \( \frac{1}{2} \)

\[
\begin{align*}
  y - y_1 &= m(x - x_1) \\
  y - 1 &= -2(x - (-1)) \\
  y - 1 &= -2(x + 1)
end{align*}
\]

**Higher Order Derivatives**

\[
\begin{align*}
  f'(x), & \quad f''(x), & \quad f'''(x), & \quad f^{(4)}(x) \\
  \frac{d}{dx} f(x), & \quad \frac{d^2}{dx^2} f(x), & \quad \frac{d^3}{dx^3} f(x), & \quad \frac{d^4}{dx^4} f(x),
end{align*}
\]

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<th>Derivative</th>
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**Example 9:** Find \( \frac{d^3}{dx^3} f(x) \) for \( f(x) = 2x^5 - 3x^3 + 7x \)

\[
\begin{align*}
  f'(x) &= 10x^4 - 9x^2 + 7 \\
  f''(x) &= 40x^3 - 18x \\
  f'''(x) &= 120x^2 - 18
end{align*}
\]

**Derivatives of the Trigonometric Functions**

\[
\begin{align*}
  (\sin x)' &= \cos x \\
  (\csc x)' &= -\csc x \cot x \\
  (\cos x)' &= -\sin x \\
  (\sec x)' &= \sec x \tan x \\
  (\tan x)' &= \sec^2 x \\
  (\cot x)' &= -\csc^2 x
end{align*}
\]
Example 10: Find the slope of the tangent line to the function $f(x) = 4 \tan x - 6 \cos x$ at $x = \frac{\pi}{4}$.

$$f'(x) = 4 \sec^2 x - 6 (-\sin x) = 4 \sec^2 x + 6 \sin x$$

$$f'(\frac{\pi}{4}) = 4 \sec^2 (\frac{\pi}{4}) + 6 \sin (\frac{\pi}{4}) = 4 \cdot 2 + 6 \cdot (\frac{\sqrt{2}}{2}) = 8 + 3\sqrt{2}$$

Example 11: Find all values of $x$ on $[0, 2\pi)$ where the tangent line is horizontal to $f(x) = 6\sqrt{3} \sin x + 18 \cos x$.

$$f'(x) = 18(-\sin x)$$

$$f'(x) = 0$$

$$18(-\sin x) = 0$$

$$\sin x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Question 18: Evaluate $f(x) = 6\sin x + 10\cos x$ at $x = -\frac{\pi}{4}$.

a. $2\sqrt{2}$

b. $8\sqrt{2}$

c. $-2\sqrt{2}$

d. $4\sqrt{2}$

e. None of the above

Pick C.
Math1431 Section 2.2

Now that we know much easier rules for finding derivatives, let’s revisit differentiability with piecewise functions.

Example 12: Is this function differentiable?

\[
f(x) = \begin{cases} 
3x & x > 3 \\
x^2 + x - 3 & x \leq 3 
\end{cases}
\]

\[
f'(x) = \begin{cases} 
3 & x > 3 \\
2x + 1 & x \leq 3 
\end{cases}
\]

\[
\lim_{{x \to 3^-}} f'(x) = 7 \quad \lim_{{x \to 3^+}} f'(x) = 3
\]

Question 17: Which function do you need to find the slope of tangent lines?

a. 2\textsuperscript{nd} Derivative
b. 3\textsuperscript{rd} Derivative
c. 1\textsuperscript{st} Derivative
d. Original function
e. None of these