Section 5.4: Permutations and Combinations

Definition: n-Factorial
For any natural number \( n \), \( n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 \)
\( 0! = 1 \)

A permutation is an arrangement of a specific set where the order in which the objects are arranged is important.

Formula: Permutations of \( n \) objects, not all distinct

\[
P(n, r) = \frac{n!}{(n-r)!}, \quad r \leq n
\]

where \( n \) is the number of distinct objects and \( r \) is the number of distinct objects taken \( r \) at a time.

Example 1: You are in charge of seating 5 honored guests at the head table of a conference. How many seating arrangements are possible if the 8 chairs are on one side of the head table?

\[
P(8, 5) = \frac{8!}{(8-5)!} = 6,720
\]
Example 2: Find the number of ways 9 people can arrange themselves in a line for a group picture.

\[ n = 9 \]
\[ r = 9 \]

\[ \text{line-up} \rightarrow \text{Perm.} \]
\[ P(9, 9) = 302,880 \]

Example 3: An organization has 30 members. In how many ways can the positions of president, vice-president, secretary, treasurer, and historian be filled if not one person can fill more than one position?

\[ n = 30 \]
\[ r = 5 \]

\[ \text{Positions / Title} \rightarrow \text{Perm.} \]
\[ P(30, 5) = 17,100,720 \]

Example 4: An organization needs to make up a social committee. If the organization has 25 members, in how many ways can a 10 person committee be made?

\[ n = 25 \]
\[ r = 10 \]

\[ \text{Nothing significant} \rightarrow \text{Order DID NOT matter} \]
\[ \text{Combination} \]
\[ C(25, 10) = 3,268,760 \]

Example 5: If there are 40 contestants in a beauty pageant, in how many ways can the judges award 1st prize and 2nd prize if not one person can be awarded 1st and 2nd?

\[ n = 40 \]
\[ r = 2 \]

\[ \text{Positions / Ranking} \rightarrow \text{Perm} \]
\[ P(40, 2) = 1560 \]
Example 6: How many permutations can be formed from all the letters in the word MISSISSIPPI.

\[
\begin{align*}
\text{Letters} & \rightarrow 11 \ total \\
\frac{n!}{n_1! \cdot n_2! \cdots} & = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} \\
& = 34,650
\end{align*}
\]

Example 7: A museum of fine arts owns 8 paintings by a given artist. Another fine arts museum wishes to borrow 3 of these paintings for a special show. How many ways can 3 paintings be selected for shipment out of the 8 available?

\[
\begin{align*}
\text{n} & = 8 \\
\text{r} & = 3 \\
C(8,3) & = \frac{8!}{(8-3)! \cdot 3!} = \frac{8!}{5! \cdot 3!} = 56
\end{align*}
\]

Example 8: A certain company has to transfer 4 of its 10 junior executives to a new location, how many ways can the 4 executives be chosen?

\[
\begin{align*}
\text{n} & = 10 \\
\text{r} & = 4 \\
C(10,4) & = 210
\end{align*}
\]

Example 9: A coin is tossed 5 times.

a. In how many outcomes do exactly 3 heads occur?

\[
\{(H_1H_2H_3TT), (H_1H_2T H_3T), (H_1H_2TT H_3), (H_1TH_3T H_5), (H_1TTH_4H_5), \\
(H_1TH_3H_4T), (TH_2H_3H_4T), (TH_2H_3T H_5), (TH_2TH_4H_5), (TTH_3H_4H_5)\}
\]

Order of Heads didn’t matter

\[
\begin{align*}
\text{Comb.} & \quad C(5,3) = 10
\end{align*}
\]

b. In how many outcomes do at least 4 heads occur?

\[
\{(H_1H_2H_3H_4T), (H_1H_2H_3T H_5), (H_1H_2TH_4H_5), (H_1TH_3H_4H_5), (TH_2H_3H_4H_5)\}
\]

\[
\begin{align*}
\text{4 Heads} & \quad + \quad \text{5 Heads} \\
C(5,4) & \quad + \quad C(5,5) \\
5 & \quad + \quad 1 \\
& \quad = 6
\end{align*}
\]
Example 10: A coin is tossed 20 times.
a. In how many outcomes do exactly 7 tails occur?

\[ c(\text{20 times, \# of tails / 2}) \rightarrow c(20, 7) = 77,520 \]

b. In how many outcomes do at most 18 heads occur?

\[ c(20, 0) + c(20, 1) + c(20, 2) \]

Find Universe/Total \[ 2^{20} = 1,048,576 \]

\[ 0-18 \text{ Heads} = \text{Universe} - \text{Complement} \]

\[ 1,048,576 - 21 \]

= 1,048,555

c. In how many outcomes do at least 19 heads occur?

\[ c(20, 19) + c(20, 20) \]

\[ 20 + 1 = 21 \]

d. In how many ways do at least 3 heads occur?

\[ c(20, 3) + c(20, 4) + c(20, 5) \]

Complement \[ 2^{20} - 2^{21} \]

\[ 1 + 20 + 190 = 211 \]

Answer \[ = \text{Total} - \text{Complement} \]

\[ 2^{20} - 211 = 1,048,576 - 211 = 1,048,365 \]

Example 11: A student belongs to a entertainment club. This month he must purchase 2 DVDs and 3 CDs. If there are 15 DVDs and 10 CDs to choose from, in how many ways can he choose his 5 purchases?

\[ \frac{c(15, 2)}{105} \times \frac{c(10, 3)}{120} = 12,600 \]
Example 12: A committee of 16 people, 7 women and 9 men, is forming a subcommittee that is to be made up of 6 women and 6 men. In how many ways can the subcommittee be formed?

\[
\binom{7}{6} \times \binom{9}{6} = 7 \times 84 = 588
\]

Example 13: A computer store receives a shipment of 35 laser printers, including 6 that are defective. Five of these printers are selected to be displayed in the store.

a. How many of these selections will contain no defective printers?

\[
\binom{29}{5} = 114,755
\]

b. How many of these selections will contain 1 defective printer?

\[
\binom{29}{4} \times \binom{6}{1} = 142,500
\]

c. How many of these selections will contain at least 1 defective printer?

Total \(-\) Complement = Answer

\[
\binom{35}{5} = 114,755
\]

Example 14: A customer at a fruit stand picks a sample of 6 avocados at random from a crate containing 35 avocados of which 8 are rotten. In how many ways can the batch contain at least 2 rotten avocados?

\[
\binom{27}{6} = 941,450
\]

Answer = Total \(-\) Complement

\[
= \binom{35}{6} - 941,450 = 15,623,150 - 941,450 \\
= 10,681,700
\]