Five cards are drawn at random from a well-shuffled complete deck of 52 cards. Find the probability that exactly two of the five cards are hearts.

The number of the pairs of hearts are \( \binom{13}{2} \). The remaining can be everything but hearts, so they must be selected from the remaining \( 39 = 52 - 13 \) cards. So we have

\[
p = \frac{\binom{13}{2} \binom{39}{3}}{\binom{52}{5}}
\]
B

MATH 3338 (SPRING 2001) SECTION 08422
QUIZ 1

NAME:

Last Four digits of your SSN:

Five cards are drawn at random from a well-shuffled complete deck of 52 cards. Find the probability that exactly two of the five cards are red.

26 cards are red. Therefore the remaining cards must be drawn from the group of black cards.

Therefore,

\[ P = \frac{\binom{26}{2} \binom{26}{3}}{\binom{52}{5}}. \]
Five cards are drawn at random from a well-shuffled complete deck of 52 cards. Find the probability that this selection of cards contains exactly one Jack and exactly one Queen.

We 4 Jacks and 4 Queens. The other three cards must not contain any Jacks or Queens; so they must be drawn from a group of 44 cards. So we have

\[ P = \frac{4 \cdot 4 \cdot \binom{44}{3}}{\binom{52}{5}}. \]
A bridge hand consists of 13 cards picked at random from a complete deck of 52 cards. Calculate the probability of having a hand containing at least 3 aces.

Suppose that there are 14 songs on a compact disk and you like 8 of them. When using the random button selector on a CD player, each of the 14 songs is played once in a random order. Find the probability that among the first 3 songs that are played, you like exactly two of them.

When doing blood testing for HIV infections the procedure can be made more cost-effective by combining samples of blood specimens. If samples from three people are combined and the mixture tests negative we know that all three individual samples are negative. Find a probability of a positive result for three samples combined into one mixture, assuming the probability of an individual blood sample testing positive is 0.1.

In a state lottery a three-digit integer is selected at random. If a player bets $1 on a particular number and if that number is selected the payoff is $500 minus the cost of the ticket. Let $X$ equal the payoff for this game. Find the anticipated average payoff of a player who buys a single ticket each time and plays the lottery many times over a long period of time.

Flaws in a certain type of drapery material appear on the average of 1 in 150 square feet. Find the probability of at most one flaw in 225 square feet.

Find the mean, the variance and $P(1 \leq X \leq 2)$ if the mgf of the random variable $X$ is given by

$$(0.3 + 0.7 e^t)^5, \quad t \in \mathbb{R}.$$ Use the definition of the mgf to solve this problem.

Pocket radios come in packages of 40. A shipment is acceptable if a random sample of 5 radios selected from the package contains no more than 2 defective ones. If the package contains 5 defective ones what is the probability that the shipment is accepted.
By a similar passage to the complementary event, one can solve Problems 2.21 and 2.22.

PROBLEMS

2.1 Lottery tickets for a total of $n$ dollars are on sale. The cost of one ticket is $r$ dollars, and $m$ of all tickets carry valuable prizes. Find the probability that a single ticket will win a valuable prize.

2.2 A domino piece selected at random is not a double. Find the probability that the second piece also selected at random, will match the first.

2.3 There are four suits in a deck containing 36 cards. One card is drawn from the deck and returned to it. The deck is then shuffled thoroughly and another card is drawn. Find the probability that both cards drawn belong to the same suit.

2.4 A letter combination lock contains five disks on a common axis. Each disk is divided into six sectors with different letters on each sector. The lock can open only if each of the disks occupies a certain position with respect to the body of the lock. Find the probability that the lock will open for an arbitrary combination of the letters.

2.5 The black and white kings are on the first and third rows, respectively, of a chess board. The queen is placed at random in one of the free squares of the first or second row. Find the probability that the position for the black king becomes checkmate if the positions of the kings are equally probable in any squares of the indicated rows.

2.6 A wallet contains three quarters and seven dimes. One coin is drawn from the wallet and then a second coin, which happens to be a quarter. Find the probability that the first coin drawn is a quarter.

2.7 From a lot containing $m$ defective items and $n$ good ones, $s$ items are chosen at random to be checked for quality. As a result of this inspection, one finds that the first $k$ of $s$ items are good. Determine the probability that the next item will be good.

2.8 Determine the probability that a randomly selected integer $N$ gives as a result of (a) squaring, (b) raising to the fourth power, (c) multiplying by an arbitrary integer, a number ending with a 1.

2.9 On 10 identical cards are written different numbers from 0 to 9. Determine the probability that (a) a two-digit number formed at random with the given cards will be divisible by 18, (b) a random three-digit number will be divisible by 36.

2.10 Find the probability that the serial number of a randomly chosen bond contains no identical digits if the serial number may be any five-digit number starting with 00001.

2.11 Ten books are placed at random on one shelf. Find the probability that three given books will be placed one next to the other.

2.12 The numbers 2, 4, 6, 7, 8, 11, 12 and 13 are written, respectively, on eight indistinguishable cards. Two cards are selected at random from the eight. Find the probability that the fraction formed with these two random numbers is reducible.
A bridge hand consists of 13 cards picked at random from a complete deck of 52 cards. Calculate the probability of having a hand containing exactly 6 spades, 4 clubs and 2 diamonds.

A student is taking an exam consisting of four true-false questions. She is trying to guess the following probabilities:

(a) All four questions are correct provided that she can answer the first question correctly but the remaining three questions are answered at random.

(b) All four questions are answered correctly provided that she knows the correct answers of exactly three of them and only the remaining question is answered at random.

An airplane has 3 engines. A plane can stay aloft and safely land with just one engine. Assuming that any of the three engines has a probability of $10^{-3}$ of shutting down due to a mechanical problem and that all three engines can operate independently of each other, give the probability that the plane crashes due to complete lack of propulsion.

The probability of quadruplets being is 0.0001 per birth. Estimate the probability of at least one quadruplet in the next 500 births. What is the probability of at most two quadruplet births among those 500 (taken from Actuarial 1 exams).

Find the mean, the variance and $P(1 \leq X \leq 2)$ if the mgf of the random variable $X$ is given by

$$M(t) = \frac{0.3e^t}{1 - 0.7e^t}, \quad 0 \leq t < -\ln(0.7).$$

Bowl A contains two red chips; bowl B contains two white chips; and bowl C contains one red chip and one white chip. A bowl is selected at random (with equal probabilities) and one chip is taken from that bowl. If the selected chip is white find the probability that it came from bowl C.

In a state lottery a three-digit integer is selected at random. If a player bets $1 on a particular number and if that number is selected the payoff is $500 minus the cost of the ticket. Let $X$ equal the payoff for this game. Find the anticipated average payoff of a player who buys a single ticket each time and plays the lottery many times over a long period of time.
4. CONDITIONAL PROBABILITY. THE MULTIPLICATION THEOREM FOR PROBABILITIES

**Basic Formulas**

The conditional probability \( P(A | B) \) of the event \( A \) is the probability of \( A \) under the assumption that the event \( B \) has occurred. (It is assumed that the probability of \( B \) is positive.) The events \( A \) and \( B \) are independent if \( P(A | B) = P(A) \). The probability for the product of two events is defined by the formula

\[
P(AB) = P(A)P(B | A) = P(B)P(A | B),
\]

which, generalized for a product of \( n \) events, is

\[
P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2 | A_1)P(A_3 | A_1A_2)\cdots P(A_n | A_1A_2\cdots A_{n-1}).
\]

The events \( A_1, A_2, \ldots, A_n \) are said to be independent if for any \( m \), where \( m = 2, 3, \ldots, n \), and any \( k_j (j = 1, 2, \ldots, n) \), \( 1 \leq k_1 < k_2 < \cdots < k_m \leq n \),

\[
P\left(\bigcap_{j=1}^{m} A_{k_j}\right) = \prod_{j=1}^{m} P(A_{k_j}).
\]

**Solution for Typical Examples**

**Example 4.1** The break in an electric circuit occurs when at least one out of three elements connected in series is out of order. Compute the probability that the break in the circuit will not occur if the elements may be out of order with the respective probabilities 0.3, 0.4 and 0.6. How does the probability change if the first element is never out of order?

**Solution.** The required probability equals the probability that all three elements are working. Let \( A_k \) \((k = 1, 2, 3)\) denote the event that the \( k \)th element functions. Then \( P = P(A_1A_2A_3) \). Since the events may be assumed independent,

\[
p = P(A_1)P(A_2)P(A_3) = 0.7 \cdot 0.6 \cdot 0.4 = 0.168.
\]

If the first element is not out of order, then

\[
p = P(A_2A_3) = 0.24.
\]

Similarly one can solve Problems 4.1 to 4.10.

**Example 4.2** Compute the probability that a randomly selected item is of first grade if it is known that 4 per cent of the entire production is defective, and 75 per cent of the nondefective items satisfy the first grade requirements.

It is given that \( P(A) = 1 - 0.04 = 0.96 \), \( P(B | A) = 0.75 \).

The required probability \( p = P(AB) = (0.96)(0.75) = 0.72 \).

Similarly one can solve Problems 4.11 to 4.19.

**Example 4.3** A lot of 100 items undergoes a selective inspection. The entire lot is rejected if there is at least one defective item in five items checked. What is the probability that the given lot will be rejected if it contains 5 per cent defective items?

**Solution.** Find the probability \( q \) of the complementary event \( A \) consisting of the situation in which the lot will be accepted. The given event is an intersection of five events \( A = A_1A_2A_3A_4A_5 \), where \( A_k \) \((k = 1, 2, 3, 4, 5)\) means that the \( k \)th item checked is good.

The probability of the event \( A_1 \) is \( P(A_1) = 95/100 \) since there are only 100 items, of which 95 are good.

After the occurrence of the event \( A_1 \), there remain 99 items, of which 94 are good and, therefore, \( P(A_2 | A_1) = 94/99 \). Analogously, \( P(A_3 | A_1A_2) = 93/98 \), \( P(A_4 | A_1A_2A_3) = 92/97 \) and \( P(A_5 | A_1A_2A_3A_4) = 91/96 \). According to the general formula, we find that

\[
q = \frac{95}{100} \cdot \frac{94}{99} \cdot \frac{93}{98} \cdot \frac{92}{97} \cdot \frac{91}{96} = 0.77.
\]

The required probability \( p = 1 - q = 0.23 \).

One can solve Problems 4.20 to 4.35 similarly.

**PROBLEMS**

4.1 Two marksmen whose probabilities of hitting a target are 0.7 and 0.8, respectively, fire one shot each. Find the probability that at least one of them will hit the target.

4.2 The probability that the \( k \)th unit of a computer is out of order during a time \( \tau \) equals \( p_k \) \((k = 1, 2, \ldots, n)\). Find the probability that during the given interval of time at least one of \( n \) units of this computer will be out of order if all the units run independently.

4.3 The probability of the occurrence of an event in each performance of an experiment is 0.2. The experiments are carried out successively until the given event occurs. Find the probability that it will be necessary to perform a fourth experiment.

4.4 The probability that an item made on the first machine is of first grade is 0.7. The probability that an item made on the second machine is first grade is 0.8. The first machine makes two items and the second machine three items. Find the probability that all items made will be of first grade.

4.5 A break in an electric circuit may occur only if one element \( K \) or two independent elements \( K_1 \) and \( K_2 \) are out of order with respective probabilities 0.3, 0.2 and 0.2. Find the probability of a break in the circuit.

4.6 A device stops as a result of damage to one tube of a total of \( N \). To locate this tube one successively replaces each tube with a new one. Find the probability that it will be necessary to check \( n \) tubes if the probability is \( P \) that a tube will be out of order.

4.7 How many numbers should be selected from a table of random numbers so that the probability of finding at least one even number among them is 0.9?

4.8 The probability that as a result of four independent trials the event \( A \) will occur at least once is 0.5. Find the probability that the event
will occur in one trial if this probability is constant through all the other trials.

4.9 An equilateral triangle is inscribed in a circle of radius $R$. What is the probability that four points taken at random in the given circle are inside this triangle?

4.10 Find the probability that a randomly written fraction will be irreducible (Chebyshev's problem).\(^a\)

4.11 If two mutually exclusive events $A$ and $B$ are such that $P(A) > 0$ and $P(B) > 0$, are these events independent?

4.12 The probability that the voltage of an electric circuit will exceed the rated value is $p_1$. For an increase in the voltage, the probability that an electric device will stop is $p_2$. Find the probability that the device will stop as a result of an increase in voltage.

4.13 A motorcyclist in a race must pass through 12 obstacles placed along a course $AB$; he will stop at each of them with probability 0.1. Knowing the probability 0.7 with which the motorcyclist passes from $B$ to the final point $C$ without stops, find the probability that no stops will occur on the segment $AC$.

4.14 Three persons play a game under the following conditions: At the beginning, the second and third play in turns against the first. In this case, the first player does not win (but might not lose either) and the probabilities that the second and third win are both 0.3. If the first does not lose, he then makes one move against each of the other two players and wins from each of them with the probability 0.4. After this, the game ends. Find the probability that the first player wins from at least one of the other two.

4.15 A marksman hits a target with the probability 2/3. If he scores a hit on the first shot, he is allowed to fire another shot at another target. The probability of failing to hit both targets in three trials is 0.5. Find the probability of failing to hit the second target.

4.16 Some items are made by two technological procedures. In the first procedure, an item passes through three technical operations, and the probabilities of a defect occurring in these operations are 0.1, 0.2, and 0.3. In the second procedure, there are two operations, and the probability of a defect occurring in each of them is 0.3. Determine which technology ensures a greater probability of first grade production if in the first case, for a good item, the probability of first grade production is 0.9, and in the second case 0.8.

4.17 The probabilities that an item will be defective as a result of a mechanical and a thermal process are $p_1$ and $p_2$, respectively. The probabilities of eliminating defects are $p_3$ and $p_4$, respectively. Find (a) how many items should be selected after the mechanical process in order to be able to claim that at least one of them can undergo the thermal process with a chance of eliminating the defect, (b) the probability that at least one of three items will have a nonremovable defect after passing through the mechanical and thermal processes.

---

\(^a\) Consider that the numerator and denominator are randomly selected numbers from the sequences $1, 2, \ldots, k$, and set $k \to \infty$. 

4.18 Show that if the conditional probability $P(A | B)$ exceeds the unconditional probability $P(A)$, then the conditional probability $P(B | A)$ exceeds the unconditional probability $P(B)$.

4.19 A target consists of two concentric circles of radius $kr$ and $nr$, where $k \leq n$. If it is equally probable that one hits any part of the circle of radius $nr$, estimate the probability of hitting the circle of radius $kr$ in two trials.

4.20 With six cards, each containing one letter, one forms the word talent. The cards are then shuffled and at random cards are drawn one at a time. What is the probability that the arrangement of letters will form the word talent?

4.21 A man has forgotten the last digit of a telephone number and, therefore, he dials it at random. Find the probability that he must dial at most three times. How does the probability change if one knows that the last digit is an odd number?

4.22 Some $m$ lottery tickets out of a total of $n$ are the winners. What is the probability of a winner in $k$ purchased tickets?

4.23 Three lottery tickets out of a total of 40,000 are the big prize winners. Find (a) the probability of getting at least one big prize winner (ticket) per 1000 tickets, (b) how many tickets should be purchased so that the probability of one big winner is at least 0.5.

4.24 Six regular drawings of state bonds plus one supplementary drawing after the fifth regular one take place annually. From a total of 100,000 serial numbers, the winners are 170 in each regular drawing and 270 in each supplementary one. Find the probability that a bond wins after ten years in (a) any drawing, (b) a supplementary drawing, (c) a regular drawing.

4.25 Consider four defective items: one item has the paint damaged, the second has a dent, the third is notched and the fourth has all three defects mentioned. Consider also the event $A$ that the first item selected at random has the paint damaged, the event $B$ that the second item has a dent and the event $C$ that the third item is notched. Are the given events independent in pairs or as a whole set?

4.26 Let $A_1, A_2, \ldots, A_n$ be a set of events independent in pairs. Is it true that the conditional probability that an event occurs, computed under the assumption that other events of the same set have occurred, is the unconditional probability of this event?

4.27 A square is divided by horizontal lines into $n$ equal strips. Then a point whose positions are equally probable in the strip is taken in each strip. In the same way one draws $n - 1$ vertical lines. Find the probability that each vertical strip will contain only one point.

4.28 A dinner party of $2n$ persons has the same number of males and females. Find the probability that two persons of the same sex will not be seated next to each other.

4.29 A party consisting of five males and 10 females is divided at random into five groups of three persons each. Find the probability that each group will have one male member.

4.30 An urn contains $n + m$ identical balls, of which $n$ are white and $m$ black, where $m \geq n$. A person draws balls $n$ times, two balls at a
6. THE TOTAL PROBABILITY FORMULA

Then one ball is taken at random from the second urn and transferred to the third, and so on. Find the probability of drawing a white ball from the last urn.

6.5 There are five guns that, when properly aimed and fired, have respective probabilities of hitting the target as follows: 0.5, 0.6, 0.7, 0.8 and 0.9. One of the guns is chosen at random, aimed and fired. What is the probability that the target is hit?

6.6 For quality control on a production line one item is chosen for inspection from each of three batches. What is the probability that faulty production will be detected if, in one of the batches, 2/3 of the items are faulty and in the other two they are all good?

6.7 A vacuum tube may come from any one of three batches with probabilities \( p_1, p_2, \) and \( p_3 \), where \( p_1 = p_2 = 0.25 \) and \( p_3 = 0.5 \). The probabilities that a vacuum tube will operate properly for a given number of hours are equal to 0.1, 0.2, and 0.4, respectively, for these batches. Find the probability that a randomly chosen vacuum tube will operate for the given number of hours.

6.8 Player A plays two opponents alternately. The probability that he wins from one at the first trial is 0.5 and the probability that he wins from the other at the first trial is 0.6. These probabilities increase by 0.1 each time the opponents repeat the play against A. Assume that A wins the first two games. Find the probability that A will lose the third game if his opponent in the first game is not known and ties are excluded.

6.9 A particular material used in a production process may come from one of six mutually exclusive categories with probabilities 0.09, 0.16, 0.25, 0.25, 0.16 and 0.09. The probabilities that an item of production will be acceptable if it is made from materials in these categories are, respectively, 0.2, 0.3, 0.4, 0.4, 0.3 and 0.2. Find the probability of producing an acceptable item.

6.10 An insulating plate 100 mm. long covers two strips passing perpendicular to its length. Their boundaries are located, respectively, at the distances of 20, 40 mm. and 65, 90 mm. from the edge of the plate. A hole of 10 mm. diameter is made, so that its center is located equiprobably on the plate. Find the probability of an electric contact with any of the strips if a conductor is applied from above to an arbitrary point located at the same distance from the base of the plate as the center of the hole.

6.11 The probability that \( k \) calls are received at a telephone station during an interval of time \( t \) is equal to \( P_k(t) \). Assuming that the numbers of calls during two adjacent intervals are independent, find the probability \( P_k(s) \) that \( s \) calls will be received during an interval \( 2t \).

6.12 Find the probability that 100 light bulbs selected at random from a lot of 1000 will be nondefective if any number of defective bulbs from 0 to 5 per 1000 is equally probable.

6.13 A white ball is dropped into a box containing \( n \) balls. What is the probability of drawing the white ball from this box if all the hypotheses about the initial color composition of the balls are equally probable?
random events

\[ P(H_1) = P(H_2) = 1/2, \quad P(A \mid H_1) = 3/4, \quad P(A \mid H_2) = 1. \]
Thus, using the formula for the total probability, we find that the probability of the event \( A \)
will be \[ P(A) = 1/2(3/4) + 1 = 7/8. \] After the first trial, the probability that
the lot will contain defective items is

\[ P(H_1 \mid A) = \frac{P(H_1)P(A \mid H_1)}{P(A)} = \frac{1/2 \cdot 3/4}{7/8} = \frac{3}{7}. \]

The probability that the lot will contain only good items is given by

\[ P(H_2 \mid A) = \frac{4}{7}. \]

Let \( B \) be the event that the item selected in the first trial turns out to be
defective. The probability of this event can also be found from the formula for
the total probability. If \( p_1 \) and \( p_2 \) are the probabilities of the hypotheses \( H_1 \) and
\( H_2 \) after a trial, then according to the preceding computations \( p_1 = 3/7, \)
\( p_2 = 4/7. \) Furthermore, \( P(B \mid H_1) = 1/4, \quad P(B \mid H_2) = 0. \) Therefore the required
probability is \( P(B) = (3/7) \cdot (1/4) = 3/28. \)

One can solve Problems 7.17 and 7.18 similarly.

problems

7.1 Consider 10 urns, identical in appearance, of which nine contain
two black and two white balls each and one contains five white
and one black ball. An urn is picked at random and a ball drawn at random
from it is white. What is the probability that the ball is drawn from the
urn containing five white balls?

7.2 Assume that \( k_1 \) urns contain \( m \) white and \( n \) black balls each
and that \( k_2 \) urns contain \( m \) white and \( n \) black balls each. A ball drawn
from a randomly selected urn turns out to be white. What is the probability
that the given ball will be drawn from an urn of the first type?

7.3 Assume that 96 per cent of total production satisfies the
standard requirements. A simplified inspection scheme accepts a standard
production with the probability 0.98 and a nonstandard one with the
probability 0.05. Find the probability that an item undergoing this
simplified inspection will satisfy the standard requirements.

7.4 From a lot containing five items one item is selected, which
turns out to be defective. Any number of defective items is equally
probable. What hypothesis about the number of defective items is most
probable?

7.5 Find the probability that among 1000 light bulbs none are
defective if all the bulbs of a randomly chosen sample of 100 bulbs turn
out to be good. Assume that any number of defective light bulbs from 0
to 5 in a lot of 1000 bulbs is equally probable.

7.6 Consider that \( D \) plays against an unknown adversary under
the following conditions: the game cannot end in a tie; the first move

7.7 Consider 18 marksmen, of whom five hit a target with the
probability 0.8, seven with the probability 0.7, four with the probability
0.6 and two with the probability 0.5. A randomly selected marksman
fires a shot without hitting the target. To what group is it most probable
that he belongs?

7.8 The probabilities that three persons hit a target with a dart
are equal to 4/5, 3/4 and 2/3. In a simultaneous throw by all three marksmen,
there are exactly two hits. Find the probability that the third
marksmen will fail.

7.9 Three hunters shoot simultaneously at a wild boar, which is
killed by one bullet. Find the probability that the boar is killed by the first,
second or the third hunter if the probabilities of their hitting the
boar are, respectively, 0.2, 0.4 and 0.6.

7.10 A dart thrown at random can hit with equal probability any
point of a region \( S \) that consists of four parts representing 50 per
cent, 30 per cent, 12 per cent and 8 per cent of the entire region. Which
part of region \( S \) is most likely to be hit?

7.11 In an urn, there are \( n \) balls whose colors are white or black
with equal probabilities. One draws \( k \) balls from the urn, successively,
with replacement. What is the probability that the urn contains only
white balls if no black balls are drawn?

7.12 The firstborn of a set of twins is a boy, what is the probability
that the other is also a boy if among twins the probabilities of
both boys or two girls are \( a \) and \( b \), respectively, and among twins of
different sexes the probabilities of being born first are equal for both sexes?

7.13 Considering that the probability of the birth of twins of the
same sex is twice that of twins of different sexes, that the probabilities
of twins of different sexes are equal in any succession and that the
probabilities of a boy and a girl are, respectively, 0.51 and 0.49, find the
probability of a second boy if the firstborn is a boy.

7.14 Two marksmen fire successively at a target. Their
probabilities of hitting the target on the first shots are 0.4 and 0.5 and
the probabilities of hitting the target in the next shots increase by 0.05 for
each of them. What is the probability that the first shot was fired by the
first marksman if the target is hit by the fifth shot?

7.15 Consider three independent trials, in which the event \( A \)
occurs with the probability 0.2. The probability of the occurrence of the
event \( B \) depends on the number of occurrences of \( A \). If the event \( A \)
occurs once, this probability is 0.1; if \( A \) occurs twice, it is 0.3; if \( A \)