A discrete random variable $X$ with values $0, 1, 2, 3, \ldots$ has the Poisson distribution if its pmf $p$ is given by an equation of the form

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

where $x$ takes the integer values $0, 1, 2, 3, \ldots$.

and $\lambda > 0$ is a parameter.

**Uses of the Poisson distribution.** We use this distribution:

1) To model number of arrivals at queue such as a telephone switchboard or a 911-call center, or a call center in general or flaws in a magnetic tape and similar events. We will return to this case later.

2) We can use the Poisson distribution to avoid tedious calculations with the binomial distribution when $n$ is big but $p$ is small. In this case $n(1-p)$ is very close to $n$. The use of the Poisson distribution in this case is suitable only if $n \geq 100$. When we choose to use the Poisson distribution to approximate probabilities governed by a binomial distribution we set $\lambda = np$. This requires you to check whether the assumptions allowing the use of the binomial hold. An example of this use is the modeling of the number of cell and viral mutations.

Now we return to the most common uses of the Poisson distribution, those described in (1) above. In these cases we use a continuous parameter such as time, length or area, which we can denote by $t$. This parameter is used to monitor the progression of the event described with the variable $X$. Let us introduce this use of the Poisson distribution by an example.

‘Arrivals’ at a call center: Say that $X$ is the number of calls to a 911 center are received between 9-10am at a rate of 3 per minute. To apply the Poisson distribution to model the 911-call numbers we need some additional assumptions to hold.

(a) Calls to the 911-center in two time intervals $[T_1, T_2]$ and $[T_3, T_4]$ with $T_2 < T_3$ are placed independently in each of the two time intervals.

(b) In *any* time interval $[T, T + \Delta t]$ where $\Delta t$ is very small the probability of receiving 2 or more 911-calls is zero and the probability of receiving exactly one call is $\lambda \Delta t$, where $\lambda = 3$ calls/min.

If both of these assumption hold then $X$ has a Poisson distribution with rate $\lambda = 3 \times 60 = 180$ calls, because one hour has 60 minutes and the rate of the 911-calls is 3 per minute. This is where you must be very careful with the units of measurement for the rate. Now, suppose that we want to find the probability that
there will be more that 360 911-calls between 9-10am. Then

\[ P(\text{more that 360 911-calls between 9-10am}) = 1 - P(\text{at most 360 911-calls between 9-10am}) \]

\[ = 1 - P(X \leq 360) = 1 - \sum_{k=0}^{360} e^{-180} \frac{180^k}{k!} = 1 - e^{-180} \sum_{k=0}^{360} \frac{180^k}{k!}. \]

Suppose now that you want to find the probability that there will be no 911-calls for 5 consecutive minutes between 9-10am. In this case \( X \) is the number of 911-calls in any 5-minute time window between 9-10am. Recall that \( X \) takes the values 0, 1, 2, 3, . . . . As long as the duration of the time window remains the same the Poisson distribution model will always give the same probability to this event regardless of the start-time of the 5-minute window. In this case \( \lambda = 15 \). Then

\[ P(X = 0) = e^{-15}. \]