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Analog to Digital, Revisited: Controlling the Accuracy of Reconstruction

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Abstract

In this work, we study analog-to-digital conversion and reconstruction outside of the strict régime of the sampling theorem for band-limited functions. We consider signals that are only essentially band-limited, and allow analysis and synthesis filters to be approximations of ideal filters. Our estimates for the reconstruction error are directly calculated from properties of the analysis and synthesis filters, such as the pass-band and stop-band ripples, the oversampling rate, and the decay properties of the two filters.

Key words and phrases : Sampling, non-bandlimited signals, analog to digital conversion, oversampling

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1 Introduction

The conversion between analog and digital signals is often associated with the classical sampling theorem for band-limited functions ([31, 13, 30]). However, practical implementations of these conversions realize the sampling theorem only in an approximate sense: signals with finite duration are not strictly bandlimited (see, e.g. [8]), and the filters used for analysis and reconstruction are usually only approximations of ideal filters ([27]); see also [22, Chapter 3] and [17, Section 4.4]. The main purpose of this work is to provide a measure for the reconstruction error in a variety of practical applications. The function space we use to model analog signals is the Sobolev space $H_s(\mathbb{R})$, $s > 1/2$, because in many practical situations one may wish to achieve a good accuracy for approximating both a function and its derivatives. This is important for the discretization of numerical solutions of ordinary or partial differential equations (see e.g. [31, 23, 11]).

In this work, we study the error resulting from the inversion of an analog-to-digital conversion of a signal without imposing strict band-limitedness. The sampling theory of non-bandlimited functions has a rather long history starting with [21], (see e.g. [29, 8]). Large parts of multiresolution theory and approximations by shift-invariant subspaces can be placed in this context (e.g. [30, 10]). When a function belongs to a shift-invariant space, it may under certain conditions be reconstructed exactly from its samples (e.g. [3, 15, 16, 7, 1]). If exact reconstruction is not guaranteed, then the best approximation of a given function is obtained by orthogonally projecting to the shift-invariant subspace generated by the sampling kernel. In many applications, the conversion between analog and digital is implemented with filters that do not provide the best approximation. For example, the reconstruction may simply interpolate the sample values of a function by means of (linear) filtering with an interpolating or a quasi-interpolating kernel (e.g [5, 26]). Here, we have calculated estimates for the Sobolev norm of the reconstruction error directly from properties of the analysis and synthesis filters, such as the passband and stop-band ripples, the oversampling rate, and the decay properties of the two filters. These quantities usually appear in engineering applications ([22, Chapter 3], [18, Chapter 6]). The results in this paper contain estimates for the maximal reconstruction error in the Sobolev norm and also a lower bound for the error of the worst-case scenario. Apart from numerical constants, the form of the upper and lower bounds for the maximal reconstruction error are identical, which shows how the properties of the analysis and synthesis filters relate to the quality of reconstruction.

The conversion from analog to digital is accomplished by means of a filter $K_{\mathrm{a}} \in H_s(\mathbb{R})$, $s > 1/2$, to which we refer as the *analysis filter*. The digital-to-analog conversion is accomplished by means of another filter, $K \in H_s(\mathbb{R})$, $s > 1/2$, for which we use the term *synthesis filter*.

To describe the filter specifications, we use the normalized Fourier transform defined on $L^1(\mathbb{R})$ by $\hat{h}(\xi) := \int_{\mathbb{R}} h(x)e^{-2\pi i x\xi}dx$ for all $\xi \in \mathbb{R}$. After selecting a *pass-band/stop-band ripple* $0 < r \leq \frac{1}{2}$, we require that the analysis filter satisfies the inequalities

$$|1 - \widehat{K}_{\rm a}(\xi)| \quad \leq \quad r \quad \text{if} \quad |\xi| \leq \Omega - \delta_1 \tag{1}$$

$$|\widehat{K}_{\rm a}(\xi)| \quad \leq \quad r \quad \text{if} \quad |\xi| \geq \Omega \tag{2}$$

for some $\Omega > 0$ and $0 \leq \delta_1 \leq \frac{\Omega}{2}$. Commonly, Ω is referred to as the *cut-off frequency*, which is related to another concept in our analysis, *the essential frequency band-limit*, $\Omega - \delta_1$. The intervals $\{\Omega - \delta_1 \leq |\xi| \leq \Omega\}$ form the transition band of the analysis filter.

The digitization process is mathematically modeled by the evaluation of the samples of the convolution

$$\tilde{f} := f * \tilde{K}_{\rm a}$$

obtained on the points of a regular grid, where $\tilde{K}_{\rm a} := (\overline{\widehat{K}_{\rm a}})^{\vee}$. The associated *analysis* or *sampling* operator is defined by

$$f \mapsto \left\{ \tilde{f}\left(\frac{m}{2\Omega}\right) \right\}_{m\in\mathbb{Z}} .$$

The latter sequence is square-summable. We will discuss this issue and the boundedness of the sampling operator in the next section. We remark that this sampling rate is higher than the one corresponding to the essential frequency band $[-\Omega + \delta_1, \Omega - \delta_1]$.

To convert a digital signal to an analog one, we use a synthesis filter $K \in H_s(\mathbb{R})$. As before, we require a maximal ripple in pass and stop bands,

$$|1 - \hat{K}(\xi)| \quad \leq \quad r \quad \text{if} \quad |\xi| \leq \Omega' \tag{3}$$

$$|\hat{K}(\xi)| \quad \leq \quad r \quad \text{if} \quad |\xi| \geq \Omega' + \delta_2 \, , \tag{4}$$

for some $\Omega' > 0$ and $\delta_2 \geq 0$. The ripple of the synthesis filter does not need to be the same as that of the analysis filter. However, for the sake of simplicity, whenever we impose the ripple conditions throughout the rest of the paper, we assume that the ripples of both filters are equal. We use the terms transition band and cut-off frequency for the synthesis filter in a similar fashion as we do for the analysis filter. We also assume that for any filter (analysis or synthesis) $\mathcal{K} \in H_s(\mathbb{R})$ there exist $M, c > 0$, and $a > \frac{1}{2}$, such that

$$|\hat{\mathcal{K}}(\xi)| \leq M|\xi|^{-s-a} \, , \quad \text{for every } |\xi| \geq c. \tag{5}$$

This condition ensures that the tails in the frequency domain of both analysis and synthesis filters decay sufficiently fast for our estimates.

The conversion of a digital signal to an analog one can be viewed as "reconstructing the original signal from its samples." This procedure is implemented by the linear mapping $\{\tilde{f}(\frac{m}{2\Omega})\}_{m\in\mathbb{Z}} \mapsto f_{\text{rec}}$, where

$$f_{\text{rec}}(x) := \sum_{m\in\mathbb{Z}} \tilde{f}\left(\frac{m}{2\Omega}\right) K\left(x - \frac{m}{2\Omega}\right) \ . \tag{6}$$

When the sequence of sample values $\{\tilde{f}\left(\frac{m}{2\Omega}\right)\}_{m\in\mathbb{Z}}$ is square-summable, using Minkowski's inequality shows that the above series converges with respect to the Sobolev norm $\|\cdot\|_{s,2}$. The operator defined by

$$\{a_m\}_{m\in\mathbb{Z}} \mapsto \sum_{m\in\mathbb{Z}} a_m K(\cdot - \frac{m}{2\Omega}) \ ,$$

with $\sum_{m\in\mathbb{Z}} |a_m|^2$, is called *the synthesis operator.* For the purposes of this paper, we also refer to this operator as the *reconstruction operator*, especially when $a_m = \tilde{f}(\frac{m}{2\Omega})$.

We now summarize the properties required of analysis and synthesis filters.

Definition 1.1. An analysis filter K_{a} is a function that belongs to $H_s(\mathbb{R})$, $s > 1/2$, its Fourier transform is bounded and satisfies (5), and there exist $\Omega > 0$ and $0 \leq \delta_1 < \frac{\Omega}{2}$ so that $\widehat{K}_{\text{a}}$ satisfies (1) and (2) with some $r \leq 1/2$.

A synthesis filter K is a function that belongs to $H_s(\mathbb{R})$, $s > 1/2$, its Fourier transform is bounded, satisfies (5), and there is $\Omega' > 0$ and $0 \leq \delta_2$ such that $\hat{K}$ satisfies (3) and (4) with some $r \leq 1/2$.

Examples 1.2. Besides the sinc function, which is adjusted for bandwidth 2Ω, $\text{sinc}_\Omega(x) := \frac{\sin(2\pi\Omega x)}{\pi x}$, $x \in \mathbb{R}$, there are many classes of functions that qualify for analysis and synthesis filters, e.g., splines ([2, 27, 9]), fractional splines ([28]), Chebyshev filters ([18]), etc. Apparently, according to (1) and (3), both filters are low pass filters. Pairs of filters for analysis and reconstruction have also been used in frame theory ([19, 20]). We may also use the Gaussian $g_\Omega(x) := \sqrt{\frac{\pi}{2}}\Omega e^{-\pi^2\Omega^2x^2/2}$, $x \in \mathbb{R}$ for digitization. Note that the inflection points of the Fourier transform of this Gaussian filter are at $\xi = \pm\Omega/2$. This motivates us to choose the cut-off frequency to be equal to Ω and $\delta_1 = \Omega/2$ for $r = \frac{1}{2}$. Then, we could pick $g_{2\Omega}$ as a synthesis filter, where again we take $r = 1/2$, $\Omega' = \Omega$ and $\delta_2 = \Omega/2$. We remark that the practical length in the spatial domain of this Gaussian synthesis filter, that is $(\int_{\mathbb{R}} g_{2\Omega}(x)^2x^2dx)^{1/2}$, is smaller than that of the analysis filter, g_Ω.

The above-defined requirements for analysis and synthesis filters are sufficiently general to accomodate typical limitations in the design of acquisition and reconstruction devices. For example, due to the inertia of a sampling apparatus, it may acquire localized averages rather than point values ([27]). In addition, signal acquisition is in practice time/space-limited, and thus analog signals can only be approximately band-limited. For the same reason, real-world analog filters are not ideal, and so aliasing errors must be addressed as part of signal acquisition and reconstruction.

The requirements in the design of analysis and synthesis filters may be even more stringent than the ones in our definition. An additional, common assumption is that of rapid decay of both filters in the spatial domain. One motivation for the use of short synthesis filters is the need to avoid visual artifacts, such as ringing in image applications ([24]). Another reason is that, in real-world applications, one can never use infinitely many samples for signal reconstruction as in Equation (6). Filters with good spatial decay help suppress the error due to truncation of the sampling sequence $\{\tilde{f}\left(\frac{m}{2\Omega}\right)\}_{m\in\mathbb{Z}}$ (see e.g. [25]).

To achieve good spatial decay of a filter while retaining frequency specifications, one might wish to select analysis and synthesis filters as compactly supported C^b (with $b \in \mathbb{Z}^+$ or $b = \infty$) functions in the frequency domain. As b increases, the spatial localization of these filters improves; for $b = \infty$ they belong to the Schwartz space. It is also possible to construct such filters for arbitrarily small values of δ_1 or δ_2, thus enhancing their frequency selectivity at the cost of an increase in filter length.

The trade-off between filter length and transition bandwidth motivated the study of the mathematical properties of a class of filters introduced in [14]. These functions are called Hermite Distributed Approximating Functionals (HDAFs) and have Gaussian decay in both domains ([6]). Furthermore, they have very good time-frequency localization for a given frequency selectivity, as their Heisenberg uncertainty product is asymptotically proportional to $r^{-1/2}$, where r is the pass-band ripple as well as stop-band attenuation, and the order of the HDAF is chosen so that the transition bandwidth is proportional to r. With the general error estimates derived here, we want to provide a tool that helps in evaluating the performance of these and other approximations of the ideal filter for realistic sampling and reconstruction applications.

The main goal of our work is to establish filter properties that are useful to a practicioner and that provide a bound for the reconstruction error in the form $\|f - f_{\mathrm{rec}}\|_{s,2}^2 \leq A\|f\|_{s,2}^2 + B\epsilon^2$ whenever $f \in H^2(\mathbb{R})$ satisfies the condition $\int_{[-\Omega+\delta_1,\Omega-\delta_1]^c} |\hat{f}(\xi)|^2(1+\xi^2)^s d\xi < \epsilon^2$ for its essential frequency support. Moreover, we investigate how the pass and stop band ripples, the transition bandwidth, and the error due to the tails of the filters affect the constants A and B.

This paper is organized as follows. The statements of all our main results are contained in Section 2, together with a discussion of their significance. The proofs of our results have been collected in Section 3.

2 Main results

The specific goals of this work are: First, to obtain an upper bound for the Sobolev-norm of the difference $f - f_{\rm rec}$ referred to as the *reconstruction error* of f; second, to show that this upper bound cannot be improved apart from changing constant factors when considering filters that are useful for practical purposes; third, to show how this upper bound can be made small by tuning certain parameters that determine the performance of a filter in engineering applications.

To state our main result, we need some auxiliary functions. Let $\Omega = \Omega'$.

Given $f \in H_s(\mathbb{R})$ and an analysis filter $K_{\rm a} \in H_s(\mathbb{R})$, we denote

$$A(\xi) := \sum_{m \neq 0} \hat{f}(\xi + 2m\Omega)\overline{\widehat{K_{\rm a}}(\xi + 2m\Omega)},\ A_0(\xi) := A(\xi) + \hat{f}(\xi)\overline{\widehat{K_{\rm a}}(\xi)}\,. \qquad (7)$$

For any $\lambda \leq s$, we write

$$C_{\lambda,a}(\xi) := \sum_{m \neq 0} |\widehat{K_{\rm a}}(\xi + 2m\Omega)|^2 (1 + |\xi + 2m\Omega|^2)^{\lambda},$$

$$C_{0,\lambda,a}(\xi) := C_{\lambda,a}(\xi) + |\widehat{K_{\rm a}}(\xi)|^2(1 + |\xi|^2)^{\lambda}\ .$$

Similarly, we define C_λ and $C_{0,\lambda}$ for a synthesis filter $K \in H_s(\mathbb{R})$ in place of $K_{\rm a}$. We remark that all these series converge absolutely a.e. on $\mathbb{R}$, due to the hypothesis that both filters and the original function belong to the Sobolev space H_s. In addition, we denote

$$G(\xi) := 4|\widehat{K}_{\rm a}(\xi)|^2 \frac{C_s(\xi)}{(1 + |\xi|^2)^s} + 2|1 - \overline{\widehat{K}_{\rm a}(\xi)}\hat{K}(\xi)|^2\ .$$

We recall from Definition 1.1 that $0 \leq \delta_1 \leq \frac{\Omega}{2}$ and $0 \leq \delta_2$. Depending on the value of δ_2, we divide the frequency domain in several regions: If $\delta_2 \leq \Omega$, we define $I_0 := \{\xi : |\xi| \leq \min\{\Omega - \delta_2, \Omega - \delta_1\}\}$, $I_1 := \{\xi : \min\{\Omega - \delta_2, \Omega - \delta_1\} \leq |\xi| \leq \max\{\Omega - \delta_2, \Omega - \delta_1\}\}$ and $I_2 := \{\xi : \max\{\Omega - \delta_2, \Omega - \delta_1\} \leq |\xi| \leq \Omega\}$. Otherwise, if $\delta_2 > \Omega$, we set $I_0 = \{0\}$, $I_1 := \{\xi : 0 \leq |\xi| \leq \Omega - \delta_1\}$, and $I_2 := \{\xi : \Omega - \delta_1 \leq |\xi| \leq \Omega\}$.

For $f \in H_s(\mathbb{R})$, we adopt the notation y_0^2, y_1^2, y_2^2, y_3^2 for the integrals of the restrictions $|\hat{f}(\cdot)|^2(1+|\cdot|^2)^s$ on each one of the regions I_0, I_1, I_2, and $\{\xi : |\xi| \geq \Omega\}$, respectively.

2.1 Upper bounds for the reconstruction error

Theorem 2.1. Let $K_{\rm a}$ and K be analysis and synthesis filters in $H_s(\mathbb{R})$, with some $s > 1/2$. Assume $\Omega = \Omega'$, $0 \leq \delta_1 \leq \frac{\Omega}{2}$ and $0 \leq \delta_2$. Then, for every

$f \in H_s(\mathbb{R})$ the series expression (6) for f_{rec} converges in the Sobolev norm, and the reconstruction error is bounded by the following inequality:

$$\|f - f_{\text{rec}}\|_{s,2}^2 \le \int_{-\Omega}^{\Omega} G(\xi)|\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi + 2\Big(1 + 3 \sup_{|\xi|\le\Omega} C_{-s,a}(\xi) \sup_{|\xi|\le\Omega} C_{0,s}(\xi)\Big) y_3^2. \quad (8)$$

In particular, we have

$$\|f - f_{\text{rec}}\|_{s,2}^2 \le \sup_{I_0} G(\xi) y_0^2 + \sup_{I_1} G(\xi) y_1^2 + \sup_{I_2} G(\xi) y_2^2 + 2\Big(1 + 3 \sup_{|\xi|\le\Omega} C_{-s,a}(\xi) \sup_{|\xi|\le\Omega} C_{0,s}(\xi)\Big) y_3^2. \quad (9)$$

Corollary 2.2. Assume that K_{a} and K are analysis and synthesis filters as in the preceding theorem and satisfy inequalities (1), (2), (3), and (4) for some ripple $r \le 1/2$, $0 \le \delta_1 \le \frac{\Omega}{2}$, and $0 \le \delta_2 < \Omega$. If $f \in H_s(\mathbb{R})$ satisfies

$$\int_{I_0^c} |f(\xi)|^2 (1+|\xi|^2)^s d\xi < \epsilon^2\,, \quad (10)$$

then

$$\|f - f_{\text{rec}}\|_{s,2}^2 \le A\|f\|_{s,2}^2 + B\epsilon^2 \quad (11)$$

with

$$A = 2\left[r^2(2-r)^2 + 2(1+r)^2 \sup_{I_0} \frac{C_s(\xi)}{(1+|\xi|^2)^s}\right]$$

$$B = 2\Big[1 + \sup_{I_1\cup I_2} |1 - \overline{\widehat{K}_{\text{a}}(\xi)}\hat{K}(\xi)|^2 + 2\left\|\widehat{K}_{\text{a}}\right\|_\infty^2 \sup_{I_1\cup I_2} \frac{C_s(\xi)}{(1+|\xi|^2)^s} + 3 \sup_{|\xi|\le\Omega} C_{-s,a}(\xi) \sup_{|\xi|\le\Omega} C_{0,s}(\xi)\Big].$$

Proof. The assumption (10) implies that $y_1^2 + y_2^2 + y_3^2 \le \epsilon^2$. The asserted inequality now follows from separately estimating $G(\xi)$ in each term of the right hand side of (9). For arguments $\xi \in I_0$ we have used $|1 - \overline{\hat{K}_a(\xi)}\hat{K}(\xi)| \le (2-r)r$, whereas for $\xi \in I_0^c$, we have simply taken suprema in each interval. □

Corollary 2.3. If $K_{\text{a}} = \text{sinc}_\Omega$ and the synthesis filter K is as in the preceding corollary, then for every $f \in H_s(\mathbb{R})$ satisfying (10) the inequality stated in Corollary 2.2 can be simplified by replacing B with

$$B = 2\left[1 + r^2 + \frac{2}{(1+(\Omega-\delta_2)^2)^s} \sup_{I_1} C_s(\xi)\right].$$

Proof. Since $\widehat{K}_{\mathrm{a}}(\xi) = 1$ for all $\xi \in [-\Omega, \Omega]$ and $\widehat{K}_{\mathrm{a}}$ vanishes outside $[-\Omega, \Omega]$ we have that $I_2 = \{\Omega\}$, $\sup_{I_1} |1 - \widehat{K}_{\mathrm{a}}(\xi)\hat{K}(\xi)| \leq r$ and $C_{-s,a}(\xi) = 0$ for all $\xi \in [-\Omega, \Omega]$. Now, using $|\xi| \geq \Omega - \delta_2$ in I_1 gives the desired estimate. □

Remarks 2.4.

(i) We briefly comment on the special case when the transition band for the synthesis filter is large. In case $\delta_2 \geq \Omega$, we have $I_0 = \{0\}$ and thus $y_0 = 0$. While the bound in Theorem 2.1 is still meaningful, including this case in the stated corollaries would require the signal norm to be small according to $\|f\|_{s,2} < \epsilon$, which is not of interest for practical purposes.

(ii) The choice of analysis and synthesis filters in Corollary 2.3 has been used in signal processing since it allows to first truncate the Fourier transform of a signal, then sample it, and then reconstruct an approximation of the signal using a synthesis filter with a smooth Fourier transform, thus with good spatial localization. In [27] splines, which are compactly supported in the spatial domain, have been proposed for synthesis filters. For a result similar to Corollary 2.2, see [4]. Linear combinations of modulated Gaussians have also been proposed as synthesis filters ([25]) for reconstructing band-limited signals f from their samples. Strohmer and Tanner ([25]) prove that with synthesis filters of this kind, using only a finite number of samples of f to approximate f_{rec} gives a pointwise reconstruction error that has fractional exponential decay in the number of samples.

(iii) Specializing further, if f belongs to the Paley-Wiener space $W_\Omega := \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } |\xi| \geq \Omega\}$, then we can choose $\epsilon = 0$. Selecting $K_{\mathrm{a}} = K = \mathrm{sinc}_\Omega = (\chi_{[-\Omega,\Omega]})^\vee$ gives $\delta_2 = \delta_1 = 0$, $r = 0$ and $C_s(\xi) = 0$ for $\xi \in [-\Omega, \Omega]$. One can then immediately verify that f_{rec} is the exact reconstruction provided by the Classical Sampling Theorem ([12, Section 6.1]).

We now investigate how the value of the ripple r enters in the estimate of the reconstruction error $\|f - f_{\mathrm{rec}}\|_{s,2}$ when we use analysis and synthesis filters for which (1), (2), (3), and (4) hold. To formulate this result, we define $c(\Omega, s, a) := (\frac{2^s M^2}{\Omega(2a-1)} + 5^s \frac{2}{\Omega})$ and $p := \frac{2a-1}{a+s}$, where $a > \frac{1}{2}$.

Theorem 2.5. Let f be an arbitrary function in $H_s(\mathbb{R})$, $\epsilon > 0$, $\Omega > 0$ and $0 < \delta \leq \Omega/2$ such that

$$\int_{|\xi| \geq \Omega - \delta} |f(\xi)|^2 (1 + |\xi|^2)^s d\xi < \epsilon^2 . \tag{12}$$

Now, we assume that the following conditions hold:

(1) The analysis and synthesis filters K_{a} and K satisfy (1), (2), (3), and (4) with $\Omega = \Omega'$ and some $\delta_1, \delta_2 > 0$ such that $\max\{\delta_1, \delta_2\} = \delta$.

(2) There exists $a \geq s + 1$ such that $|\hat{K}(\xi)| \leq |\xi|^{-(a+s)}$ for all $|\xi| \geq \Omega + \delta_2$. In addition, we assume $r < (2\Omega)^{-(a+s)}$.

Then,

$$\|f - f_{\mathrm{rec}}\|_{s,2}^2 \le Ar \, \|f\|_{s,2}^2 + B\epsilon^2 \, , \tag{13}$$

where A and B are positive constants depending only on a, Ω, s, $\left\| \widehat{K}_{\mathrm{a}} \right\|_\infty$, and $\left\| \hat{K} \right\|_\infty$. More precisely, we can choose

$$A = r(2-r)^2 + 4(1+r)^2 c(\Omega, s, a) r^{p-1}$$

and

$$\begin{aligned} B =& 2\Big(1 + 2\|K_{\mathrm{a}}\|_\infty^2 \frac{4^s[3(1+9\Omega^2)^s \|\hat{K}\|_\infty^2 + c(\Omega, s, a)]}{[4+\Omega^2]^s} \\ &+ 3\frac{2r^2}{\Omega}[3(1+9\Omega^2)^s \|\hat{K}\|_\infty^2 + c(\Omega, s, a) r^p] + (1 + (1+r)\|K_{\mathrm{a}}\|_\infty)^2\Big) \, . \end{aligned}$$

Remark 2.6. The hypotheses in the previous theorem may seem technical, but in fact each of them relates to the digitization process. The maximum allowable reconstruction error (in terms of the Sobolev norm) is prescribed. In most cases the bandwidth Ω is also predetermined, e.g., in digital scanners or audio streaming devices. What we actually propose is not to sample the original signal at the critical rate, but rather to sample it at a higher rate according to inequality (12). The merit of this oversampling technique will be discussed in Remark 2.8(ii), because it requires the statement of Theorem 2.7. Inequality (13) can be used to determine the requirements that the analysis and synthesis filters must meet, so that for an original analog signal satisfying (12) the reconstruction error will not exceed the maximum allowable level.

2.2 Lower bound for the error of the worst case scenario

The last result we present shows that under certain additional mild conditions, namely $\sup_{|\xi|\le\Omega} C_{-s,a}(\xi) \; C_{0,s}(\xi) < 1/4$ and piecewise continuity of $\widehat{K}_{\mathrm{a}}$ and $\hat{K}$, the conclusion of Theorem 2.1 provides an upper bound of the reconstruction error that cannot be improved, apart from changing constant factors. The additional conditions imply that the analysis and synthesis filters generally provide a good reconstruction of f. For the purposes of this, study a function h defined on $\mathbb{R}$ is piecewise continuous, if there exist $t_1 < t_2 < \cdots < t_q$, so that the restrictions of h in the subintervals $(-\infty, t_1)$, (t_i, t_{i+1}) and (t_q, ∞), where $2 \le i \le q-1$, are continuous, the one-sided limits of h exist and are finite at every t_i, and h is right or left continuous at every t_i. The most classical representative of the class of filters whose Fourier transforms are piecewise continuous but not continuous is the ideal filter or sinc function. We remark that filters with such Fourier transforms are hardly useful to implementations due to their length in the spatial/time domain. Therefore, the piecewise continuity assumption for $\hat{K}$

does not practically reduce the scope of the following result. As the reader can directly verify from the proof of this theorem, its conclusion is still valid with essential suprema of G replacing the suprema of G on the intervals I_0, I_1, and I_2, respectively, if the hypothesis of the piecewise continuity of $\hat{K}$ is dropped.

Theorem 2.7. Assume that K_{a} and K are analysis and synthesis filters, respectively, satisfying the hypotheses of Theorem 2.1 and $\sup_{|\xi|\leq\Omega} C_{-s,a}(\xi)C_{0,s}(\xi) < 1/4$. Furthemore, assume that $\widehat{K}_{\mathrm{a}}$ and $\hat{K}$ are piecewise continuous. Then, for every selection of $y_i \in \mathbb{R}$, $i = 1,2,3,4$ and $0 < \eta < 1$, there exists a real valued $f \in H_s(\mathbb{R})$ with

$$y_0^2 = \int_{I_0} |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi \qquad y_1^2 = \int_{I_1} |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi$$
$$y_2^2 = \int_{I_2} |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi \qquad y_3^2 = \int_{|\xi|\geq\Omega} |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi \qquad (14)$$

such that its reconstruction error is bounded below by

$$\|f - f_{\mathrm{rec}}\|_{s,2}^2 \geq \frac{1-\eta}{8}\left[\sup_{I_0} G(\xi)y_0^2 + \sup_{I_1} G(\xi)y_1^2 + \sup_{I_2} G(\xi)y_2^2 + 2y_3^2\right]. \qquad (15)$$

The previous theorem gives rise to some observations on how the properties of the analysis and synthesis filters affect the magnitude of the reconstruction error.

Remarks 2.8.

(i) Comparing (15) in the previous theorem with (9) and in particular with (11) we conclude that the effectiveness of the reconstruction of a function $f \in H_s(\mathbb{R})$ depends on the coefficients of y_i^2, $i = 1,2,3,4$, provided, according to Inequality (22), that a small enough ripple is selected to ensure that $\sup_{|\xi|\leq\Omega} C_{-s,a}(\xi)C_{0,s}(\xi) < 1/4$ holds. Then, for a given tolerance $\epsilon > 0$, choosing $\sum_{i=2}^4 y_i^2 < \epsilon^2$, we can obtain a small reconstruction error only if $\sup_{I_0} G(\xi)$ is small. The reader may recall that we called I_0 the effective frequency band of f. To minimize $\sup_{I_0} G(\xi)$, both $\widehat{K}_{\mathrm{a}}$ and $\hat{K}$ must have a "plateau" very close to 1 throughout I_0; this "plateau" yields the term $r^2(r+2)^2$ in the right hand side of (11), demonstrating how important it is to keep the ripple small. However, this is not enough: $\sup_{I_0} C_s(\xi)$ must be small as well. The latter quantity has a bound that is proportional to r^p. In conclusion, the factors that determine the magnitude of the reconstruction error are the level of the tolerance $\epsilon > 0$ and the size of the ripple r. With $\Omega = \Omega'$, condition (3) implies that $\hat{K}$'s "plateau" must at least extend from $-\Omega$ to Ω. In fact, the same condition suggests that $\hat{K}$'s "plateau" may extend further out than $[-\Omega, \Omega]$. Choosing a large δ_1 decreases the effective bandwidth for functions; on the other hand, large values of δ_1 and

δ_2 allow to improve the decay of both the analysis and synthesis filters in the spatial domain.

(ii) So far we used the term "oversampling" when we referred to Inequality (12). Now we need to clarify why sampling at a rate of $\frac{1}{2\Omega}$ when (12), or more generally (10), are satisfied amounts to oversampling. Consider an analysis filter $K_{\rm a}$ whose Fourier transform vanishes outside $[-\Omega, \Omega]$. Then, (9) and (15) imply that, for every $f \in H_s(\mathbb{R})$

$$\|f - f_{\rm rec}\|_{s,2}^2 \leq \sup_{I_0} G(\xi) y_0^2 + \sup_{I_1} G(\xi) y_1^2 + \sup_{I_2} G(\xi) y_2^2 + 2y_3^2$$

but for some $f_0 \in H_s(\mathbb{R})$

$$\|f_0 - (f_0)_{\rm rec}\|_{s,2}^2 \geq \frac{1}{16} (\sup_{I_1} G(\xi) y_1^2 + \sup_{I_2} G(\xi) y_2^2) \ .$$

Unless we suppress the frequency content in I_1 and I_2, we address the worst-case estimate giving rise to the lower bound of $\|f - f_{\rm rec}\|_{s,2}$. On the other hand, if we require that $\int_{I_0^c} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \epsilon^2$, then according to (13) the reconstruction error can be made as small as we wish because it is bounded by terms linear in r and in ϵ^2. This is the motivation for sampling the function at a rate which is higher than that corresponding to the effective frequency band I_0.

3 Proofs of the main results

3.1 Upper bounds for the reconstruction error

Proof of Theorem 2.1. The 2Ω-periodization of $\hat{f}\overline{\widehat{K_{\rm a}}}$ is A_0, which belongs to $L^1(\mathbb{R})$. So taking the samples of $\tilde{f}$ at the points $\frac{m}{2\Omega}$ is a meaningful process. Furthermore, using the Cauchy Schwarz inequality gives

$$\begin{aligned} |A_0(\xi)| &= \left| \sum_{m \in \mathbb{Z}} \hat{f}(\xi + 2m\Omega)(1 + |\xi + 2m\Omega|^2)^{s/2} \frac{\overline{\widehat{K_{\rm a}}(\xi + 2m\Omega)}}{(1 + |\xi + 2m\Omega|^2)^{s/2}} \right| \\ &\leq \left(\sum_{m \in \mathbb{Z}} |\hat{f}(\xi + 2m\Omega)|^2 (1 + |\xi + 2m\Omega|^2)^s \right)^{1/2} (C_{0,-s,a}(\xi))^{1/2} \ . \end{aligned}$$

Since $\widehat{K_{\rm a}}$ is bounded and $s > 1/2$ we obtain that $C_{0,-s,a}$ is uniformly bounded. Therefore, A_0 belongs to $L^2([-\Omega, \Omega])$ and

$$\int_{-\Omega}^{\Omega} |A_0(\xi)|^2 d\xi \leq \sup_{\xi \in [-\Omega, \Omega]} C_{0,-s,a}(\xi) \|f\|_{s,2} \ ,$$

so the sequence $\{\tilde{f}(\frac{m}{2\Omega})\}_{m\in\mathbb{Z}}$ is square-summable. Consequently, the series

$$f_{\text{rec}} = \sum_{m\in\mathbb{Z}} \tilde{f}(\frac{m}{2\Omega})K(\cdot - \frac{m}{2\Omega})$$

converges in the norm $\|\cdot\|_{s,2}$.

Applying the Fourier transform on both sides of Equation (6) and periodizing gives

$$\begin{aligned}\hat{f}_{rec}(\xi) &= \left[\sum_{m\in\mathbb{Z}} \tilde{f}\left(\frac{m}{2\Omega}\right) e^{-2\pi i \frac{m}{2\Omega}\xi}\right] \hat{K}(\xi) = \left[\sum_{m\in\mathbb{Z}} (\hat{f}\overline{\widehat{K_{\text{a}}}})^{\vee}\left(\frac{m}{2\Omega}\right) e^{-2\pi i \frac{m}{2\Omega}\xi}\right] \hat{K}(\xi) \\ &= \sum_{m\in\mathbb{Z}} \hat{f}(\xi + 2m\Omega)\overline{\widehat{K_{\text{a}}}(\xi + 2m\Omega)}\hat{K}(\xi) = A_0(\xi)\hat{K}(\xi) \ \text{ a.e. in } \mathbb{R}\ .\end{aligned}$$

The key observation for the calculation of the reconstruction error is given by the following equation:

$$\begin{aligned}\|f - f_{\text{rec}}\|_{s,2}^2 &= \int_{-\Omega}^{\Omega} |\hat{f}(\xi) - A_0(\xi)\hat{K}(\xi)|^2(1 + |\xi|^2)^s d\xi \\ &+ \sum_{l\neq 0} \int_{-\Omega}^{\Omega} |\hat{f}(\xi + 2l\Omega) - A_0(\xi)\hat{K}(\xi + 2l\Omega)|^2(1 + |\xi + 2l\Omega|^2)^s d\xi.\end{aligned} \tag{16}$$

Since Ω is the cut-off frequency, we can view the second sum in the right hand side of (16) as the overall *aliasing error* due to the conversion of f from an analog to a digital signal and the inversion of this process.

Next, we obtain upper estimates for each one of the terms in the right hand side of the previous equation. Observe that

$$A_0(\xi)\hat{K}(\xi) = A(\xi)\hat{K}(\xi) + \hat{f}(\xi)\overline{\widehat{K_{\text{a}}}(\xi)}\hat{K}(\xi) \quad \text{a.e.}$$

Combining the previous identity with $|a \pm b|^2 \leq 2|a|^2 + 2|b|^2$, for every $a, b \in \mathbb{C}$, we infer the following upper bound for the first summand of the right hand side of (16):

$$\begin{aligned}&\int_{-\Omega}^{\Omega} |\hat{f}(\xi) - A_0(\xi)\hat{K}(\xi)|^2(1 + |\xi|^2)^s d\xi \quad \leq \\ 2\int_{-\Omega}^{\Omega} |\hat{f}(\xi)|^2|1 - \overline{\widehat{K}_{\text{a}}(\xi)}\hat{K}(\xi)|^2(1 + |\xi|^2)^s d\xi \quad &+ \quad 2\int_{-\Omega}^{\Omega} |A(\xi)\hat{K}(\xi)|^2(1 + |\xi|^2)^s d\xi\ .\end{aligned} \tag{17}$$

Using the Cauchy-Schwarz inequality we obtain

$$|A(\xi)|^2 \leq C_{-s,a}(\xi) \sum_{l\neq 0} \left|\hat{f}(\xi + 2l\Omega)\right|^2 \left(1 + |\xi + 2l\Omega|^2\right)^s \quad |\xi| \leq \Omega \text{ a.e.} \tag{18}$$

With (18) and $|\hat{K}(\xi)|^2(1+|\xi|^2)^s \le \sup\{C_{0,s}(\xi) : |\xi| \le \Omega\}$ we have

$$\int_{-\Omega}^{\Omega} |A(\xi)\hat{K}(\xi)|^2(1+|\xi|^2)^s d\xi \le \sup_{|\xi|\le\Omega} C_{-s,a}(\xi) \sup_{|\xi|\le\Omega} C_{0,s}(\xi) y_3^2 \,. \tag{19}$$

Now we estimate the second term in the right hand side of (16):

The periodicity of A_0 implies that this term is bounded above by the sum

$$2\sum_{l\neq 0}\int_{-\Omega}^{\Omega}\Big(|\hat{f}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s + |A_0(\xi)|^2|\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s\Big)d\xi \le 2y_3^2 + 2\int_{-\Omega}^{\Omega}|A_0(\xi)|^2 C_s(\xi)d\xi \,. \tag{20}$$

However,

$$\int_{-\Omega}^{\Omega}|A_0(\xi)|^2C_s(\xi)d\xi \le 2\int_{-\Omega}^{\Omega}|\hat{f}(\xi)|^2|\widehat{K}_{\mathrm{a}}(\xi)|^2C_s(\xi)d\xi + 2\int_{-\Omega}^{\Omega}|A(\xi)|^2C_s(\xi)d\xi \le$$
$$2\int_{-\Omega}^{\Omega}|\widehat{K}_{\mathrm{a}}(\xi)|^2\frac{C_s(\xi)}{(1+|\xi|^2)^s}|\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi + 2\int_{-\Omega}^{\Omega}|A(\xi)|^2C_s(\xi)d\xi \,. \tag{21}$$

Using (18) again in estimating the second integral in (21), (19), and (20), the definition of G, and (16) we deduce

$$\begin{aligned}\|f-f_{\mathrm{rec}}\|_{s,2}^2 \le &\int_{I_0} G(\xi)|\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi + \int_{I_1} G(\xi)|\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi \\ &+ \int_{I_2} G(\xi)|\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi + 2\sup_{|\xi|\le\Omega} C_{-s,a}(\xi)\sup_{|\xi|\le\Omega} C_{0,s}(\xi)y_3^2 \\ &+ 2\left(1+2\sup_{|\xi|\le\Omega} C_{-s,a}(\xi)\sup_{|\xi|\le\Omega} C_s(\xi)\right)y_3^2\end{aligned}$$

which immediately implies (8) and (9). □

Next we investigate how the decay properties of $\widehat{K}_{\mathrm{a}}$ and $\hat{K}$, the ripple r, and the cut-off frequency Ω of the analysis filter impact the bounds for $C_{-s,a}$ and C_s.

Proof of Theorem 2.5. First observe that (2) implies that for every $|\xi| \le \Omega$ we have

$$\begin{aligned}C_{-s,a}(\xi) \le r^2\sum_{l\neq 0}(1+|\xi+2l\Omega|^2)^{-s} &\le \frac{r^2}{2\Omega}\int_{|u|\ge\Omega}(1+(\xi+u)^2)^{-s}du \\ &\le \frac{r^2}{\Omega}\int_0^{\infty}(1+u^2)^{-1}du < \frac{2r^2}{\Omega}\,.\end{aligned} \tag{22}$$

We now turn to C_s. We begin with $\xi \in I_0$. For these ξ and $l \neq 0$ we have $|\xi + 2l\Omega| \geq \Omega + \delta_2$. Due to (5), there exists a $c, M > 0$ such that $|\hat{K}(\xi)| \leq M|\xi|^{-s-a}$, for every $|\xi| \geq c$, with $a > 1/2$. Without any loss of generality we can assume $c = \Omega + \delta_2$, for if $c > \Omega + \delta_2$; then we can choose a constant M to be greater than rc^{a+s} and have (5) valid for all $|\xi| \geq \Omega + \delta_2$. Thus, for all $N \in \mathbb{Z}^+$ we have

$$\begin{aligned} C_s(\xi) &\leq \sum_{l \neq 0} \min\{r^2, M^2|\xi + 2l\Omega|^{-2s-2a}\}(1 + |\xi + 2l\Omega|^2)^s \\ &\leq \sum_{1 \leq |l| \leq N} \min\{r^2, M^2|\xi + 2l\Omega|^{-2s-2a}\}(1 + |\xi + 2l\Omega|^2)^s \\ &+ \sum_{|l| \geq N+1} \min\{r^2, M^2|\xi + 2l\Omega|^{-2s-2a}\}(1 + |\xi + 2l\Omega|^2)^s. \qquad (23) \end{aligned}$$

We want to find N such that $\min\{r^2,\ M^2|\xi + 2l\Omega|^{-2s-2a}\} = M^2|\xi + 2l\Omega|^{-2s-2a}$ for all $\xi \in I_0$, provided $|l| \geq N + 1$. Choosing N bigger than that would unnecessarily increase the first of the two sums in the right hand side of (23). So we conclude that for every N with the property prescribed above we have

$$\begin{aligned} &\sum_{1 \leq l \leq N} \min\{r^2, M^2|\xi + 2l\Omega|^{-2s-2a}\}(1 + |\xi + 2l\Omega|^2)^s \\ &\leq \sum_{1 \leq |l| \leq N} r^2(1 + |\xi + 2l\Omega|^2)^s \leq 2Nr^2(1 + [(2N + 1)\Omega]^2)^s. \end{aligned}$$

These inequalities show that the only way to control the growth of the first term in the right hand side of (23) is to choose r sufficiently small. On the other hand, $t \mapsto t^{-2s-2a}(1 + t^2)^s$ is decreasing, so we can obtain an integral estimate for the second sum in (23). Thus,

$$\begin{aligned} &\sum_{|l| \geq N+1} \min\{r^2, M^2|\xi + 2l\Omega|^{-2s-2a}\}(1 + |\xi + 2l\Omega|^2)^s \\ &\leq M^2 \sum_{|l| \geq N+1} |\xi + 2l\Omega|^{-2s-2a}(1 + |\xi + 2l\Omega|^2)^s \\ &\leq \tfrac{M^2}{2\Omega}\left[\int_{|u| \geq (2N+1)\Omega} |u|^{-2s-2a}(1 + |u|^2)^s du\right] \leq \tfrac{M^2}{\Omega}\left[\int_{(2N+1)\Omega}^{\infty} u^{-2a}\left(1 + \tfrac{1}{u^2}\right)^s du\right] \\ &\leq \tfrac{M^2}{\Omega}\left[\int_{(2N+1)\Omega}^{\infty} u^{-2a}\left(1 + \tfrac{1}{u^2}\right)^s du\right] \leq \tfrac{2^s M^2}{\Omega}\left[\int_{(2N+1)\Omega}^{\infty} u^{-2a} du\right] \\ &\leq \tfrac{2^s M^2}{\Omega(2a-1)}\left((2N + 1)\Omega\right)^{1-2a} \qquad (24) \end{aligned}$$

Let us assume that things worked so that a sufficiently large N with the prescribed properties is selected also satisfying $(2N + 1)\Omega > 1$. Since $a > 1/2$ the second of the two terms in the right hand side of (23) is bounded above by a quantity, which decreases as N increases. At this point the reader must recall that our goal is to find a sufficiently small upper bound for $C_s|_{I_0}$. This can be accomplished if both summands in the right hand side of (23) are small. To this effect we choose $r < (2\Omega)^{-(a+s)}$, equivalently $2\Omega < r^{-\frac{1}{a+s}}$. Then,

$$\frac{1}{2} \leq \frac{1}{2}\left(\frac{r^{-\frac{1}{a+s}}}{\Omega} - 1\right)$$

Now, define

$$N := \left\lfloor \left[\frac{1}{2}\left(\frac{r^{-\frac{1}{a+s}}}{\Omega} - 1\right)\right] + 1 \right\rfloor .$$

Thus, $(2N-1)\Omega \leq r^{-\frac{1}{a+s}}$ which gives

$$(2N+1)\Omega < r^{-\frac{1}{a+s}} + 2\Omega < 2r^{-\frac{1}{a+s}}.$$

This inequality allows us to derive upper bounds for both terms of (23). The first of these two terms is bounded above by

$$\begin{aligned} 2Nr^2(1+[(2N+1)\Omega]^2)^s &\leq \frac{1}{\Omega}(2N+1)\Omega r^2(1+[(2N+1)\Omega]^2)^s \\ &\leq \frac{1}{\Omega}2r^{-\frac{1}{a+s}}r^2(1+[(2N+1)\Omega]^2)^s . \end{aligned}$$

Now, recall, $r \leq \frac{1}{2} < 1$, so $1 < r^{-\frac{2}{a+s}}$. Then,

$$2Nr^2(1+[(2N+1)\Omega]^2)^s \leq \frac{2}{\Omega}5^s r^{\frac{2a-1}{a+s}} . \tag{25}$$

Also

$$\frac{2^s M^2}{\Omega(2a-1)}\left((2N+1)\Omega\right)^{1-2a} \leq \frac{2^s M^2}{\Omega(2a-1)} r^{\frac{2a-1}{a+s}} .$$

The last inequality, (23), (24), and (25) yield

$$C_s(\xi) \leq \left(\frac{2^s M^2}{\Omega(2a-1)} + 5^s\frac{2}{\Omega}\right) r^{\frac{2a-1}{a+s}}. \tag{26}$$

Our next goal is to find an upper estimate of $C_{0,s}(\xi)$ for all $|\xi| \leq \Omega$. Notice that

$$C_{0,s}(\xi) = \sum_{l=\pm1,0} |\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s + \sum_{|l|\geq 2} |\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s .$$

To estimate the second we use the upper bound we obtained for $C_s|_{I_0}$. We observe that this estimate is valid for all ξ such that $|\hat{K}(\xi)| \leq \min\{r^2, M^2|\xi|^{-2s-2a}\}$. But the latter inequality is valid for all $|\xi| \geq 2\Omega$. Therefore,

$$\sum_{|l|\geq 2} |\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s \leq \left(\frac{2^s M^2}{\Omega(2a-1)} + 5^s\frac{2}{\Omega}\right) r^{\frac{2a-1}{a+s}} .$$

On the other hand,

$$\sum_{l=\pm 1,0} |\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s \leq 3(1+9\Omega^2)^s\|\hat{K}\|_\infty^2,$$

so

$$C_{0,s}(\xi) \leq 3(1+9\Omega^2)^s\|\hat{K}\|_\infty^2 + \left(\frac{2^s M^2}{\Omega(2a-1)} + 5^s\frac{2}{\Omega}\right) r^{\frac{2a-1}{a+s}} \ . \tag{27}$$

We recall that $c(\Omega, s, a) := (\frac{2^s M^2}{\Omega(2a-1)} + 5^s\frac{2}{\Omega})$ and $p := \frac{2a-1}{a+s}$. Since $\delta = \max\{\delta_1, \delta_2\}$ inequality (11) applies. Using, (26) it is not hard to see that the coefficient of $\|f\|_{s,2}^2$ is bounded above by $r^2(r+2)^2 + 4(1+r)^2c(\Omega,s,a)r^p$. Since $r \leq 1/2$ and $0 \leq p-1 < 1$, one can see that $r^{p-1} < 1$, if $a \geq s+1$. This yields the form of the coefficient of $\|f\|_{s,2}^2$. To see that the coefficient of ϵ^2 is bounded by a constant that does not depend on r, first observe that the coefficient of the corresponding term in (11) contains a constant term. Furthermore,

$$\sup_{|\xi|\leq\Omega} C_{-s,a}(\xi) \sup_{|\xi|\leq\Omega} C_{0,s}(\xi) \leq \frac{2r^2}{\Omega}[3(1+9\Omega^2)^s\|\hat{K}\|_\infty^2 + c(\Omega,s,a)r^p] \tag{28}$$

and

$$\sup_{I_1\cup I_2} \frac{C_s(\xi)}{(1+|\xi|^2)^s} \leq \frac{\sup_{|\xi|\leq\Omega} C_{0,s}(\xi)}{(1+|\Omega-\delta|^2)^s} \leq \frac{4^s[3(1+9\Omega^2)^s\|\hat{K}\|_\infty^2 + c(\Omega,s,a)]}{[4+\Omega^2]^s} \ .$$

Finally, the remaining contribution to the coefficient of ϵ^2 in (11) is bounded by $(1+(1+r)\left\|\widehat{K}_{\rm a}\right\|_\infty)^2$. □

3.2 Lower bound for the error of the worst case scenario

Proof of Theorem 2.7. Now, let $f^* \in H_s(\mathbb{R})$ be real-valued and such that $\hat{f}^*$ is essentially bounded and equal to zero a.e. outside $[-\Omega, \Omega]$. We also select f^* satisfying all but the first of conditions (14). Then (16) implies

$$\begin{aligned} \|f^* - (f^*)_{\rm rec}\|_{s,2}^2 &= \frac{1}{2}\int_{|\xi|\leq\Omega} |\widehat{f^*}(\xi)|^2 2|1-\overline{\hat{K}_a(\xi)}\hat{K}(\xi)|^2(1+|\xi|^2)^s d\xi \\ &\quad + \frac{1}{4}\int_{|\xi|\leq\Omega} |\widehat{f^*}(\xi)|^2 4|\hat{K}_a(\xi)|^2 \frac{C_s(\xi)}{(1+|\xi|^2)^s}(1+|\xi|^2)^s d\xi \ . \\ &\geq \frac{1}{4}\int_{|\xi|\leq\Omega} |\widehat{f^*}(\xi)|^2 G(\xi)(1+|\xi|^2)^s d\xi. \end{aligned}$$

The facts that this synthesis filter satisfies (5), $\hat{K}$ is piecewise continuous, and $a > 1/2$ imply that C_s is piecewise continuous as a function defined on the interval $[-\Omega, \Omega]$. It also may have a finite number of discontinuity points in

$[-\Omega,\Omega]$ with finite side limits and left or right side continuity at each one of these points. The same properties are also true for the restriction $\widehat{K}_{\mathrm{a}}$ on $[-\Omega,\Omega]$ and thus for G. Therefore, G achieves a maximum value at each one of the intervals I_0, I_1, and I_2. Now, let ξ_0 be a point where G achieves its maximum in I_0. Now, for any given $0<\eta<1$, there exists a closed interval $A:=\{\xi : G(\xi)>(1-\eta)\sup_{I_0} G(\xi)\}$ with positive measure containing ξ_0. Of course, the same is true for both I_1 and I_2. So, there exist closed intervals B_1 and B_2 of I_1 and I_2, respectively, with positive measure such that $B_i:=\{\xi : G(\xi)>(1-\eta)\sup_{I_i} G(\xi)\}$, with $i=1,2$. Select f^* to vanish outside $(A\cup -A)\cup(B_1\cup -B_1)\cup(B_2\cup -B_2)$. Denote this particular choice of f^* by f^*_ϵ. Then, taking into account that $|\widehat{f^*}(\xi)|$ is even, we obtain

$$\begin{aligned}\|f^*_\epsilon-(f^*_\epsilon)_{\mathrm{rec}}\|^2_{s,2} &\geq \frac{1}{4}\int_{|\xi|\leq\Omega}|\widehat{f^*_\epsilon}(\xi)|^2 G(\xi)(1+|\xi|^2)^s d\xi\\ &\geq \frac{1-\eta}{8}\left[\sup_{I_0} G(\xi)y_0^2+\sup_{I_1} G(\xi)y_1^2+\sup_{I_2} G(\xi)y_2^2\right]. \quad (29)\end{aligned}$$

Now, let $f^{**}\in H_s(\mathbb{R})$ be real-valued such that $\widehat{f^{**}}$ vanishes a.e. on $[-\Omega,\Omega]$ and satisfies $y_3^2=\int_{|\xi|\geq\Omega}|\widehat{f^{**}}(\xi)|^2(1+|\xi|^2)^s d\xi$. Also, let A^{**} be the function A corresponding to f^{**} defined by (7). Then, using (16) and $|a+b|\geq\frac{1}{2}|a|^2-|b|^2$ we have

$$\begin{aligned}&\|f^{**}-(f^{**})_{\mathrm{rec}}\|^2_{s,2}=\int_{-\Omega}^{\Omega}|A^{**}(\xi)|^2|\hat{K}(\xi)|^2(1+|\xi|^2)^s d\xi\\ &+\sum_{l\neq 0}\int_{-\Omega}^{\Omega}|\widehat{f^{**}}(\xi+2l\Omega)-A^{**}(\xi)\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s d\xi\\ &\geq\frac{1}{2}\sum_{l\neq 0}\int_{-\Omega}^{\Omega}|\widehat{f^{**}}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s d\xi\\ &-\int_{-\Omega}^{\Omega}|A^{**}(\xi)|^2\sum_{l\neq 0}|\hat{K}(\xi+2l\Omega)|^2(1+|\xi+2l\Omega|^2)^s d\xi\\ &\geq\frac{1}{2}y_3^2-\sup_{|\xi|\leq\Omega}C_{-s,a}(\xi)C_{0,s}(\xi)y_3^2\geq\frac{1}{4}y_3^2\end{aligned}$$

Now, define $f_1:=f^*_\epsilon-f^{**}$ and $f_2:=f^*_\epsilon+f^{**}$. Both f_1 and f_2 are real-valued and belong to $H_s(\mathbb{R})$. Taking in acount that the reconstruction operator is linear and using the parallelogram law we obtain

$$\begin{aligned}\|f_1-(f_1)_{\rm rec}\|_{s,2}^2+\|f_2-(f_2)_{\rm rec}\|_{s,2}^2 &= 2(\|f_\epsilon^*-(f_\epsilon^*)_{\rm rec}\|_{s,2}^2+\|f^{**}-f_{\rm rec}^{**}\|_{s,2}^2)\\ &\geq \frac{1}{2}y_3^2+\frac{1-\eta}{4}\left(\sup_{I_0}G(\xi)y_0^2+\sup_{I_1}G(\xi)y_1^2+\sup_{I_2}G(\xi)y_2^2\right)\\ &\geq \frac{1-\eta}{4}\left(\sup_{I_0}G(\xi)y_0^2+\sup_{I_1}G(\xi)y_1^2+\sup_{I_2}G(\xi)y_2^2+2y_3^2\right).\end{aligned}$$

The last inequality implies that at least one of the terms $\|f_1-(f_1)_{\rm rec}\|_{s,2}^2$ and $\|f_2-(f_2)_{\rm rec}\|_{s,2}^2$ must be greater than the sum $\frac{1-\eta}{8}\left(\sup_{I_0}G(\xi)y_0^2+\sup_{I_1}G(\xi)y_1^2+\sup_{I_2}G(\xi)y_2^2+y_3^2\right)$. The proof is completed once the reader notices that both f_1 and f_2 satisfy conditions (14). □

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