GENERALIZED FRAME MULTIRESOLUTION ANALYSIS OF ABSTRACT HILBERT SPACES

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Abstract. We define a very generic class of multiresolution analysis of abstract Hilbert spaces. Their core subspaces have a frame produced by the action of an abelian unitary group on a countable frame multiscaling vector set, which may be infinite. We characterize all the associated frame multiwavelet vector sets and we generalize the concept of low and high pass filters. We also prove a generalization of the Quadratic (Conjugate) Mirror filter condition and we give two algorithms for the construction of the high pass filters associated to a given low pass filter.

1. Introduction and preliminaries

Let H be a complex Hilbert space. A *unitary system* $\mathcal{U}$ is a set of unitary operators acting on H which contains the identity operator I on H ([16]). Now, let D be the *(dyadic) Dilation operator*

$$(Df)(t) = \sqrt{2} f(2t) \quad f \in L^2(\mathbb{R}) \tag{1}$$

and T the *Translation operator*

$$(Tf)(t) = f(t-1) \quad f \in L^2(\mathbb{R}). \tag{2}$$

The unitary system $\mathcal{U}_{D,T} := \{D^n T^m : n, m \in \mathbb{Z}\}$ called the *Affine system* has been extensively used in wavelet analysis.

A Riesz basis of a Hilbert space is a basis similar to an orthonormal basis, i.e. there exists a bounded and invertible operator defined on the Hilbert space mapping every element of an orthonormal basis of the Hilbert space to exactly one element of the Riesz basis and vice-versa ([13, 19]). The set $\{\psi_k : k = 1, 2, \ldots, n\}$ is an orthonormal (Riesz) multiwavelet set if $\{D^j T^m \psi_k : k = 1, 2, \ldots, n,\ j, m \in \mathbb{Z}\}$ is an orthonormal (Riesz) basis of $L^2(\mathbb{R})$. Dai and Larson were the first who used operator-theoretic tools to formulate and study an abstract wavelet theory for Hilbert spaces ([16]). Following [16] C is an *orthonormal (Riesz) multiwavelet vector set* of H with respect to the unitary system $\mathcal{U}$ if $\{U\psi : U \in \mathcal{U},\ \psi \in C\}$ is an orthonormal (Riesz) basis for the Hilbert space H.

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The family $\{x_i : i \in I\}$ is a *frame* for the Hilbert space H if there exist constants $A, B > 0$ such that for every $x \in H$ we have

$$A \|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B \|x\|^2 \ .$$

We refer to the optimal positive constants A, B satisfying the previous inequalities as the *lower and upper frame bounds* of $\{x_i : i \in I\}$ respectively. We refer to the frame as *tight* if $A = B$ and as *Parseval frame* (PF) if $A = B = 1$. A term widely used in the past for Parseval frames was normalised tight. A frame $\{x_i : i \in I\}$ of H is called *exact* if each one of its proper subsets is not a frame for H. Riesz bases are exact frames and vice-versa.

For the frame $\{x_i : i \in I\}$ of H we define the operator S called the *analysis operator*[1] corresponding to the frame $\{x_i : i \in I\}$ by the following equation:

$$Sx = \{\langle x, x_i \rangle\}_{i \in I} \ , \quad x \in H.$$

Using this operator we can construct the canonical dual frame of $\{x_i : i \in I\}$. In what follows, for a given frame we will only be concerned for its canonical dual, so from now on we use the term "dual frame" to indicate the canonical dual of a frame. The significance of the dual frame is that its elements define the coefficient functionals which give the expansion of every element of H in terms of the original frame. This expansion is known as the *reconstruction formula*. Therefore, frames can be as useful as Riesz and orthonormal bases. Among frames the most desirable ones are perhaps the PFs, because they are identical with their duals. For more details on the general theory of frames we encourage the interested reader to refer to [8, 13, 19].

Signals, which are modelled as vectors, can be expanded with respect to a Riesz basis. Such expansions correspond to the exact sampling of signals. Sampling is necessary because it allows us to convert a signal from analog into digital and also because the available computational hardware and software requires that all input signals are digital. Sampling is performed by a variety of devices such as scanners and digital cameras. Nevertheless exact sampling is not always the most favorable type of sampling. In fact, in practice we oversample. This allows us to deal more effectively with certain deficiencies of communication channels, such as noise (see [10]). On the other hand, oversampling intuitively corresponds to an expansion of a signal with respect to a frame, because frames may not be exact. After this brief engineering intermezzo, which illustrates the potential of frame theory for applications, let us continue our discussion on the preliminaries of the Generalized Frame Multiresolution Analysis.

In the abstract Hilbert space setting we can define wavelet frames with respect to a unitary system $\mathcal{U}$ ([19]). Let C be a subset of H and $\mathcal{U}$ a unitary system acting on H. If $\{U\psi : U \in \mathcal{U}, \ \psi \in C\}$ is a frame (resp. tight, Parseval) for the Hilbert space H we call the set C a *frame multiwavelet vector set (resp. tight, Parseval)* and the

[1] Han and Larson in [19] call the same operator *frame transform*.

family $\{U\psi : U \in \mathcal{U},\ \psi \in C\}$ a *multiwavelet vector frame (resp. tight, Parseval).* If C is a singleton, we refer to the single element of C as a *frame wavelet vector.* If $H = L^2(\mathbb{R}^n)$ we refer to frame multiwavelet vector sets as frame multiwavelets.

In the rest of the present paper, D will denote an arbitrary unitary operator defined on H and not the Dyadic dilation operator, unless it is otherwise stated. We are interested in unitary systems $\mathcal{U}$ acting on H of the form $\mathcal{U} = \mathcal{U}_0 G$, where $\mathcal{U}_0 = \{D^n : n \in \mathbf{Z}\}$ and G is an abelian unitary group. We will often refer to G as a *translation group.* Obviously unitary systems of this form generalize the Affine system.

Definition 1. *A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of an abstract Hilbert space H is a Generalized Frame Multiresolution Analysis of H (GFMRA) if it is increasing, i.e. $V_j \subseteq V_{j+1}$ for every $j \in \mathbb{Z}$ and satisfies the following properties:*
(a) $V_j = D^j(V_0),\ j \in \mathbb{Z}$
(b) $\bigcap_j V_j = \{0\},\ \overline{\bigcup_j V_j} = H$
(c) There exists a subset B of V_0 such that the set $G(B) = \{g\phi : g \in G, \phi \in B\}$ is a frame of V_0.
Every such set B is called a frame multiscaling vector set for $\{V_j\}_j$. Every subset C of V_1 such that $G(C) = \{g\psi : g \in G, \psi \in C\}$ is a frame of $W_0 := V_1 \cap V_0^{\perp}$ is called a semiorthogonal frame multiwavelet vector set associated with $\{V_j\}_j$.

If B is a singleton we refer to its unique element as a *frame scaling vector* or, if $H = L^2(\mathbb{R}^n)$, as a *frame scaling function.* Note, that if C is a semiorthogonal frame multiwavelet vector set associated with the GFMRA $\{V_j\}_j$ then C is a frame multiwavelet vector set for H, because $\{D^j g\psi : j \in \mathbb{Z}, g \in G, \psi \in C\}$ is a frame for H with the same frame bounds as $G(C)$. We also refer to the subspace V_0 as a *core* subspace. In this paper we study only semiorthogonal frame multiwavelet vector sets. Thus, for convenience, we will not make any further use of the term "semiorthogonal" when we refer to frame multiwavelet vector sets.

The goal of the present paper is to generalize the theory of multiresolution analysis (MRA) by introducing a very generic MRA structure called Generalized Frame MRA. Moreover, we characterize all frame multiwavelet vector sets associated with a Generalized Frame Multiresolution analysis of an abstract Hilbert space and we give two algorihtms for the construction of these sets (section 2), proving so that such sets always exist. We also give some examples of Generalized Frame MRAs (section 3), which by no means should be considered exhaustive. The construction of each frame multiwavelet vector set associated with a given GFMRA requires only a frame multiscaling vector set of this GFMRA. The examples in section 3 establish that frame multiscaling vector sets for the same GFMRA may have various cardinalities. This particular fact reveals some of the capabilities of the GFMRA theory.

In order to accomplish the construction of the frame multiwavelet sets associated with a GFMRA $\{V_j\}_j$ we need the following additional hypotheses, which we assume that are satisfied throughout the rest of this paper.

- There exists a mapping $\sigma : G \to G$ satisfying
$$gD = D\sigma(g)\ , \quad \text{for every } g \in G.$$
This particular assumption implies that σ is an injective homomorphism and $\sigma(G)$ is a subgroup of G. (See [20] for proofs)
- $|G : \sigma(G)| = n < +\infty$ and that $\{V_j\}_j$, where $|G : \sigma(G)|$ is the index of the subgroup $\sigma(G)$.
- The GFMRA $\{V_j\}_j$ has a countable frame multiscaling vector set.

We will refer to a GFMRA satisfying the last property as *countably generated*.

Before proceeding we wish to add a few comments regarding these extra hypotheses. Let D and T be the operators defined by eqs. (1) and (2). It is not hard to check that $TD = DT^2$. In fact this non-commutation relation is the key for the production of wavelets associated with Multiresolution Analyses of $L^2(\mathbb{R})$. In this particular case we have $G = \{T^n : n \in \mathbb{Z}\}$ and $\sigma(T^n) = T^{2n}$, for every $n \in \mathbb{Z}$, thus the index of $\sigma(G)$ equals 2. This shows that the first two hypotheses are neither restrictive nor technical. The third hypothesis is very general. In subsection 3.2 we give an example (Example 2) of a countably infinitely generated GFMRA, which can also be generated by a single scaling function.

If the core subspace of a GFMRA admits an orthonormal or a Riesz basis of the form $G(B)$, where B is a subset of the core subspace, then we refer to this GFMRA as a *Multiresolution analysis* (MRA) of H and to B as a *multiscaling vector set*.

If we do not specify the unitary system then either it is clearly defined from the context, or we assume that it is the Affine system acting on $L^2(\mathbb{R})$.

GFMRAs of $L^2(\mathbb{R})$ were introduced in [26], where it was proved that every orthonormal wavelet of $L^2(\mathbb{R})$ is associated with a GFMRA. However, the primary contribution of [26] was the method developed in the proof of the aforementioned result for the construction a PF multiscaling set of functions defining the GFMRA with whom the given orthonormal wavelet is associated with. The existence of such a PF multiscaling set was also proved later in [7].

MRAs of abstract Hilbert spaces were studied in [5, 6, 20, 37]. Among the classes of MRAs that were studied in these papers, the most general ones are the Generalized MRAs of Baggett and co-workers ([5, 6, 7]). These Generalized MRAs satisfy properties (a) and (b) of definition 1, the first two of the additional hypotheses following definition 1 and the following: G is countable and for every $g \in G$ we have $g(V_0) = V_0$. Obviously GFMRAs are Generalized MRAs. On the other hand, Generalized MRAs of $L^2(\mathbb{R}^n)$ ([7, 15]) defined with respect to a translation group G, which is discrete and isomorphic to $\mathbb{Z}^n$ are GFMRAs, because, in this case, there exists a countable subset B of V_0 such that all the translations of the elements of B with respect to G form a PF of V_0 (see Theorem 3.1. [40], see also [12]). Therefore, we can conclude that the GFMRA is indeed the most general class of multiresolution analysis in $L^2(\mathbb{R}^n)$ at least. Baggett and co-workers study only orthonormal multiwavelet vector sets of abstract Hilbert spaces and they focus mostly on establishing the existence of such sets associated with a certain class of their Generalized MRAs. They also prove that every orthonormal multiwavelet vector set is associated with a Generalized MRA from the same class (Theorem 1.3, [6]), thus generalizing the aforementioned result of Papadakis in [26]. Their main tool is an abstract integer-valued function called multiplicity function and not the frame multiscaling vector sets. Construction of multiwavelets in $L^2(\mathbb{R}^n)$ based on their techniques are primarily due to Courter ([15]). Further examples can be found in [5, 6, 7]. The techniques developed by the Baggett group do not seem to lead directly to constructions of multiwavelet vector sets from frame multiscaling vector sets even in the case of $L^2(\mathbb{R}^n)$.

Our paper is supplemented by another paper ([27]), where, in a four-page survey, we give further details concerning the connection of our work with the work of others.

It becomes apparent from definition 1 that most of the MRAs studied in the literature, such as MRAs of $L^2(\mathbb{R}^n)$ whose dilations are defined by arbitrary expanding matrices and their translations are induced by lattices similar to $\mathbb{Z}^n$, are special cases of GFMRAs. On the other hand, every MRA admits a frame multiscaling vector set. A very trivial, yet generic, example illustrating this fact is the following: Let ϕ be a scaling function of an MRA of $L^2(\mathbb{R})$. Then $\{\phi, T\phi, \ldots T^n\phi\}$ $(n \in \mathbb{Z}^+)$ is a tight frame multiscaling set of functions for the same MRA. However, not every GFMRA is an MRA (see example 1 in subsection 3.2).

Singly generated GFMRAs of $L^2(\mathbb{R})$ have been introduced and studied by Benedetto and Li ([9]) and later, but independently, by Kim and Lim ([23]). We refer to such GFMRAs as FMRAs. A recent paper by Benedetto and Trieber ([10]) generalizes some of the results of ([9]), contains some further examples and facilitates an interesting connection between FMRAs, subband encoding and denoising of signals. In all these papers frame wavelets are constructed from frame scaling functions and the same frame wavelets can be derived in a different way from our more general results (subsection 3.1). Comparing our techniques with those in [9, 10, 23], one can see that the use of Von Neumann algebra theory yields more geometrically transparent proofs even in the abstract Hilbert space setting. It also allows us to avoid a lot of technicalities, which would otherwise clutter our arguments.

Now, consider the Hilbert space $\ell^2(G)$. If $g \in G$ let ℓ_g be the unitary operator acting on $\ell^2(G)$ defined by the equation $\ell_g(\delta_k) = \delta_{gh}$, where δ_h is the Kronecker's delta function defined on G. The set $\{\delta_h : h \in G\}$ is an orthonormal basis of $\ell^2(G)$ and the set $G^* := \{\ell_g : g \in G\}$ is a unitary group acting on $\ell^2(G)$. In fact the mapping

$$g \to \ell_g, \quad g \in G$$

known as the left regular representation of G is a group isomorphism between G and G^*. Thus G^* is abelian. Moreover, it is not hard for someone to verify that if $h, g \in G$, and $a \in \ell^2(G)$, then $\ell_g a(h) = a(g^{-1}h)$. Furthermore, G^* is discrete with respect to the SOT (Strong operator topology). From now on we will work on G^* instead of G. By $\widehat{G^*}$ we denote the dual group of G^* and by μ the normalized Haar measure on $\widehat{G^*}$.

Let $g \in G$. We define $f_g : \widehat{G^*} \to \widehat{G^*}$ by the formula

$$f_g(\gamma) = \gamma(l_g), \quad \gamma \in \widehat{G^*}.$$

It is not hard to see that $\{f_g : g \in G\}$ is an orthonormal basis of $L^2(\widehat{G^*})$.

The homomorphism σ induces the mapping σ^* defined on G^*, by the following equation:

$$\sigma^*(\ell_g) = \ell_{\sigma(g)}, \quad g \in G.$$

Obviously σ^* is an injective homomorphism and $|G^* : \sigma^*(G^*)| = |G : \sigma(G)|$. Following [20] we define ρ on $\widehat{G^*}$ by the equation

$$\rho(\gamma)(\ell_g) = \gamma \circ \sigma^*(\ell_g), \quad g \in G,\ \gamma \in \widehat{G^*}.$$

Next we give a brief account of the properties of mapping ρ, which plays an instrumental role in section 2.

Proposition 2. *([20, 37]) The following are true:*

(a) *The mapping ρ is a continuous and open homomorphism. In addition, ρ is measure preserving, i.e. $\mu(\rho^{-1}(A)) = \mu(A)$, for every measurable subset A of $\widehat{G^*}$.*

(b) *The kernel of ρ has exactly n elements and is group isomorphic to $\widehat{G/\sigma(G)}$.*

(c) *There exists a measurable mapping $\tilde{\rho} : \widehat{G^*} \to \widehat{G^*}$ associated with ρ such that $\rho(\tilde{\rho}(\gamma)) = \gamma$, for every $\gamma \in \widehat{G^*}$.*

(d) *$\rho^{-1}(A) = \bigcup_{i=0}^{n-1} \gamma_i \tilde{\rho}(A)$, for every subset A of $\widehat{G^*}$. Moreover, $\gamma_i \tilde{\rho}(A)$ and $\gamma_j \tilde{\rho}(A)$ are disjoint, if $i \neq j$.*

For the definition of $\tilde{\rho}$ and proofs of the properties of ρ and $\tilde{\rho}$ the reader should refer to [20] and follow the convention that the mapping σ in [20] is the mapping σ^* defined in the previous paragraph. Both mappings were introduced in [37].

In section 2 we will use the Hilbert space $L^2(\widehat{G^*}, \ell^2)$. This space consists of all weakly measurable functions $\omega : \widehat{G^*} \to \ell^2$ such that $\int_{\widehat{G^*}} \| \omega(\gamma) \|_{\ell^2}^2 \, d\gamma$ is finite[2]. The norm of ω is defined by $\| \omega \| := \left(\int_{\widehat{G^*}} \| \omega(\gamma) \|_{\ell^2}^2 \, d\gamma \right)^{1/2}$. If $g \in G$, M_g is the multiplicative operator defined by

$$M_g f = f_g f, \quad f \in L^2(\widehat{G^*}, \ell^2).$$

Obviously M_g is unitary for every $g \in G$, and $M_g^* = M_{g^{-1}}$. Let us close the present section with a few clarifying remarks on our notation.

If $\mathcal{A}$ is a set of bounded operators defined on H, then $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. Let f be a measurable function. We refer to the set $\{t : f(t) \neq 0\}$ as the support of function and we denote this set by s*uppf*. In several cases we find convenient and more accurate to denote the function f by $f(\cdot)$, where the $\cdot$ replaces the variable,

[2]All integrals on $\widehat{G^*}$ are defined with respect to μ.

rather than using the inaccurate notation $f(x)$ for f. Finally, we use $|A|$, A^- and A^c to denote the cardinality of a set A, its closure in the appropriate topology and its set-theoretic complement, respectively. If, in particular, A is a subset of a vector space $[A]$ stands for the linear span of A. Last but not least, we reserve the term *subspace* for closed linear manifolds.

2. The construction and the characterization of the frame multiwavelet vector sets associated with a $GFMRA$

2.1. The characterization of the frame multiwavelet vector sets associated with $\{V_j\}_j$. Assume that $\{\phi_k, k \in \mathbb{N}\}$ is a set of frame multiscaling vectors for $\{V_j\}_j$. Define $S : V_0 \to L^2(\widehat{G^*}, \ell^2)$ by the following equation

$$Sx := \sum_{g\in g} \sum_{k\in\mathbb{N}} \langle x, g\phi_k\rangle f_g \delta_k \ , \quad x \in V_0. \tag{3}$$

Those who are familiar with frame theory will immediately recognise that S is the frame or analysis operator corresponding to the frame $\{g\phi_k : g \in G, k \in \mathbb{Z}\}$ of V_0. Thus S is well-defined, bounded and

$$AI_{V_0} \leq S^*S \leq BI_{V_0}$$

where A, B are frame bounds for $\{g\phi_k : g \in G, \ k \in \mathbb{N}\}$.

Remark 1. The first who observed that a projection applied on a Riesz basis gives a frame was Aldroubi ([1]). Han and Larson, completely independently, prove in Chapter 1 of [19], that every frame can be dilated to a Riesz basis, establishing, thus, the converse of Aldroubi's observation. Remarkably enough the Han and Larson "dilation" idea is in the heart of our construction, as the following argument will facilitate.

Assume that we have a GFMRA generated by a (single) PF scaling function, say ϕ. Consider the analysis operator corresponding to the PF $\{g\phi : g \in G\}$ mapping V_0 into $\ell^2(G)$. The "dilation" of $\{g\phi : g \in G\}$ is the orthonormal basis $\{\ell_g\delta_I : g \in G\}$ of $\ell^2(G)$, where I is the identity operator on H. This naturally induces the group G^*, which now acts on $\ell^2(G)$. It now becomes clear that we do not only "dilate" the frame $\{g\phi : g \in G\}$ of V_0 to a Riesz basis, but we also "dilate" the group $G\mid_{V_0}$, by creating G^*. Even more is true; if P is the range projection of the analysis operator corresponding to $\{g\phi : g \in G\}$, then the group $PG^* \mid_{P(\ell^2(G))}$ is isomorphically homeomorphic to $G\mid_{V_0}$.

On the other hand, we actually have $\ell^2(G) = \ell^2(G^*)$. Since we need a natural substitute of the Fourier transform we work on $L^2(\widehat{G^*})$ instead of $\ell^2(G)$. If we work with an infinite frame multiscaling vector set, we use $L^2(\widehat{G^*}, \ell^2)$ instead of $L^2(\widehat{G^*})$. If we work with a frame multiscaling vector set with r elements, where r is finite, we use $L^2(\widehat{G^*}, \mathbb{C}^r)$. Without any loss of generality throughout section 2 we will exclusively use $L^2(\widehat{G^*}, \ell^2)$.

Taking the polar decomposition of S we obtain $S = Y|S|$, where $|S| = (S^*S)^{1/2}$. The previous inequality implies that both S^*S and $|S|$ are invertible, i.e. their inverses are well-defined and bounded as well. Since S is injective, Y is an isometry. It is also not hard to verify that S^*S belongs to the commutant of G and that the same is true for $|S|$ as well. If we define

$$\tilde{\phi}_k := |S|^{-1}\phi_k, \quad k \in \mathbb{N}$$

then $\{g\tilde{\phi}_k : k \in \mathbb{N}, g \in G\}$ is a PF of V_0, because if x belongs to V_0, then we obtain

$$\begin{aligned} Yx &= S|S|^{-1}x = \sum_g \sum_k \langle |S|^{-1}x, g\phi_k\rangle f_g\delta_k \\ &= \sum_g \sum_k \langle x, g\tilde{\phi}_k\rangle f_g\delta_k \end{aligned} \tag{4}$$

so $\|x\|^2 = \|Yx\|^2 = \sum_g \sum_k \left|\langle x, g\tilde{\phi}_k\rangle\right|^2$.

Since $D^*\phi_k \in V_{-1}$ which is contained in V_0 we can consider $SD^*\phi_k$. We define

$$m_k := SD^*\phi_k \text{ and } \tilde{m}_k := YD^*\phi_k.$$

Definition 3. *If m_k are the functions defined by the previous pair of equations we call the set $\{m_k : k \in \mathbb{N}\}$ low pass filter set associated with the multiscaling set $\{\phi_k : k \in \mathbb{N}\}$.*

Let P be the orthogonal projection onto the range of S. Notice that due to the lower frame inequality the range of S is closed. Polar decomposition also implies $Y(V_0) = S(V_0)$.

Lemma 2.1. *The following are true:*
(a) $Yg\tilde{\phi}_k = P(f_g\delta_k) \quad g \in G, k \in \mathbb{N}$
(b) $YgY^* = M_gP, \quad g \in G.$
(c) $P \in \{M_g : g \in G\}'.$

Proof. (a) Let $h \in G$. Since Y is an isometry we have $\langle Y(g\tilde{\phi}_K),\ Y(h\tilde{\phi}_\ell)\rangle = \langle g\tilde{\phi}_k, h\tilde{\phi}_\ell\rangle = \langle Y(g\tilde{\phi}_k), f_h\delta_\ell\rangle, k,\ \ell \in \mathbb{N}$ due to equation (4). On the other hand we obtain

$$\langle Y(g\tilde{\phi}_k), f_h\delta_\ell\rangle = \langle Y(g\tilde{\phi}_k), P(f_h\delta_\ell)\rangle.$$

Thus for every $k, \ell \in \mathbb{N}$ and $g, h \in G$ we have

$$\langle P(f_h\delta_\ell), Y(g\tilde{\phi}_k)\rangle = \langle Y(h\tilde{\phi}_\ell), Y(g\tilde{\phi}_k)\rangle$$

which implies that $P(f_h\delta_\ell) = Y(h\tilde{\phi}_\ell)$, because $\{Y(g\tilde{\phi}_k) :\ g \in G,\ k \in \mathbb{N}\}$ spans $Y(V_0)$.

(b) Equation (4) implies that for every $g \in G$ and $k \in \mathbb{N}$

$$Y^*(f_g\delta_k) = g\tilde{\phi}_k \ .$$

Therefore, if h is in G and k is in $\mathbb{N}$, we obtain

$$\begin{aligned} YgY^*(f_h\delta_k) &= Y(gh\tilde{\phi}_k) \\ &= \sum_{h',\ell}\langle gh\tilde{\phi}_k, h'\tilde{\phi}_\ell\rangle f_{h'}\delta_\ell = \sum_{h',\ell}\langle gh\tilde{\phi}_k, gg^{-1}h'\tilde{\phi}_\ell\rangle f_{h'}\delta_\ell \\ &= M_gY(h\tilde{\phi}_k) = M_gP(f_h\delta_k). \end{aligned}$$

This establishes (b). Now for $g, h \in G, k \in \mathbb{N}$ we have

$$\begin{aligned} M_gP(f_h\delta_k) &= YgY^*(f_h\delta_k) = Y(gh\tilde{\phi}_k) \\ &= P(f_gf_h\delta_k) = PM_g(f_h\delta_k). \end{aligned}$$

This establishes (c). □

Lemma 2.1(a) shows that $\{g\tilde{\phi}_k : g \in G,\ k \in \mathbb{N}\}$ is unitarily equivalent to the PF $\{Pf_g\delta_k : g \in G,\ k \in \mathbb{N}\}$. This fact is crucial for our study, because it helps us to prove the remaining statements in lemma 2.1, which play a key role in the characterization of the frame multiwavelet vector sets associated with GFMRAs and in the proof of the Generalized Quadratic Mirror filter condition[3] (see theorem 7). The proof of statement (a) of lemma 2.1 was adopted from [19], but was included here for the sake of completeness.

The space $L^2(\widehat{G^*}, \ell^2)$ is isometrically isomorphic to the space $L^2(\widehat{G^*})\otimes\ell^2$. Under the same isomorphism M_g can be identified with $\mu_g \otimes I_{\ell^2}$, where μ_g is the multiplicative operator defined by $\mu_g\omega = f_g\omega$, $\omega \in L^2(\widehat{G^*})$ and I_{ℓ^2} is the identity operator on ℓ^2. Therefore $\{M_g : g \in G\}'$ can be identified with $L^\infty(\widehat{G^*}) \otimes \mathcal{B}(\ell^2)$. The latter Von Neumann algebra is spatially isomorphic to the Von Neumann $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$. This particular Von Neumann algebra consists of all essentially bounded WOT-measurable functions $F : \widehat{G^*} \to \mathcal{B}(\ell^2)$. Therefore, $\{M_g : g \in G\}'$ is the algebra of all bounded operators $\mathbf{F}$, defined in the following way: For every $\mathbf{F}$ there exists $F \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ such that

$$\mathbf{F}\omega(\gamma) = F(\gamma)\omega(\gamma), \quad \omega \in L^2(\widehat{G^*}, \ell^2)$$

and vice versa. For more details the reader may refer to [18] and in particular to examples 4.3.5 and 4.3.10.

Thus for $P \in \{M_g : g \in G\}'$ there exists a function $P \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ such that

$$(Pf)(\gamma) = P(\gamma)f(\gamma), \gamma \in \widehat{G^*}.$$

Notice that we used the same notation for the projection P and the operator-valued function P in $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$. Throughout the present paper we adopt this convention in order to simplify our notation and we hope that this will not confuse the reader.

[3]The Quadratic Mirror filter condition is also known as the Smith-Barnwell condition and as Conjugate Mirror filter condition.

In addition, we set $K = L^2(\widehat{G^*}, l^2)$. Since P is an orthogonal projection it turns out that $P(\gamma)$ is an orthogonal projection for a.e. γ in $\widehat{G^*}$.

Since we want to characterize the sets of frame multiwavelet vectors associated with $\{V_j\}_j$ it is reasonable to work with Y instead of S, because Y preserves inner products. This motivates us to define

$$\tilde{m}_k := YD^*\phi_k, \quad k \in \mathbb{N}.$$

Our next goal is to find the relationship between $\tilde{m}_k$ and m_k. For every $k \in \mathbb{N}$ we obtain

$$m_k = SD^*\phi_k = Y|S|Y^*(YD^*\phi_k) = Y|S|Y^*\tilde{m}_k, \tag{5}$$

because $Y^*Y = I_{V_0}$.

Lemma 2.2. *Both S^*S and $|S|$ belong to G' and $Y|S|Y^*$ belongs to $\{M_g : g \in G\}'$.*

Proof. The fact that g belongs to G gives us

$$\begin{aligned} gS^*Sx &= \sum_{h\in G}\sum_{k\in\mathbb{N}}\langle x, h\phi_k\rangle gh\phi_k \\ &= \sum_{h}\sum_{k}\langle gx, gh\phi_k\rangle gh\phi_k = S^*Sgx, \quad x \in V_0. \end{aligned}$$

This implies that $S^*S \in G'$ and so $|S| \in G'$. On the other hand, for every $g \in G$ we have

$$\begin{aligned} M_gY|S|Y^* &= M_gPY|S|Y^* = YgY^*Y|S|Y^* = Yg|S|Y^* \\ &= Y|S|gY^* = Y|S|Y^*(YgY^*) = Y|S|Y^*M_gP. \end{aligned}$$

So far we used Lemma 2.1 (b) and the previous conclusion. Since P belongs to $\{M_g : g \in G\}'$ and $Y^*P = Y^*$ we obtain

$$Y^*M_g = Y^*PM_g = Y^*M_gP.$$

This completes the proof of the lemma. □

The final conclusion of Lemma 2.2 implies the existence of a function $A : \widehat{G^*} \to \mathcal{B}(l^2)$, which is WOT-measurable such that $essup\{\|A(\gamma)\| : \gamma \in \widehat{G^*}\} = \||S|\|$, and

$$(Y|S|Y^*\omega)(\gamma) = A(\gamma)\omega(\gamma), \quad \omega \in K$$

Since $Y|S|Y^*$ belongs to $\{M_g : g \in G\}'$, is positive and

$$SS^* = Y|S||S|Y^* = (Y|S|Y^*)^2 \tag{6}$$

we obtain $SS^* \in \{M_g : g \in G\}'$ and that $Y|S|Y^*$ is the square root of SS^*. Therefore,

$$SS^*\omega(\gamma) = A(\gamma)^2\omega(\gamma), \quad \omega \in K.$$

At this point, the reader should recall that $|S|$ is a positive operator, so $A(\gamma)$ is a positive operator a.e. Now, let $k \in \mathbb{N}$. Then

$$SS^*\delta_k = S\phi_k = \sum_g \sum_l \langle \phi_k, g\phi_l \rangle f_g \delta_l. \tag{7}$$

Equation (7) implies that $A(\cdot)^2$ is the *autocorrelation function corresponding to the frame multiscaling vector set* $\{\phi_k : k \in \mathbb{N}\}$. Since $A(\gamma)$ is positive, it follows that $A(\gamma)$ is the square root of this autocorrelation matrix, for a.e. γ in $\widehat{G^*}$.

The range space of SS^* is the same with the range space of S (see proposition 2.5.13 of [22]) and the range of S is closed, because S is the analysis operator corresponding to the frame $\{g\phi_k : g \in G,\ k \in \mathbb{N}\}$. Let us denote the range of S by $\mathcal{R}(S)$. Then

$$SS^* \mid_{\mathcal{R}(S)} : \mathcal{R}(S) \to \mathcal{R}(S)$$

is an invertible bounded operator, because $SS^* \mid_{\mathcal{R}(S)}$ is injective, since $ker(SS^*) = ker(S^*) = \mathcal{R}(S)^{\perp}$. Note that P is the range projection of SS^* as well. Summarizing the arguments in the preceding two paragraphs we obtain the following proposition.

Proposition 4. *The following are true:*

(i) *For every* $k \in \mathbb{N}$

$$A(\gamma)^2\delta_k = \sum_{g\in G} \sum_{l=1}^{\infty} \langle \phi_k, g\phi_l \rangle f_g \delta_l, \quad \gamma \in \widehat{G^*}.$$

(ii) *There exist* $B_1, B_2 > 0$ *such that, first,* $\|A(\gamma)\| \le B_2$ *a.e.; second, for a.e.* γ *such that* $P(\gamma) \neq 0$ *and* $x \in P(\gamma)(l^2)$ *we have*

$$B_1 \|x\| \le \|A(\gamma)x\| \le B_2 \|x\|.$$

(iii) $(SS^*f)(\gamma) = A(\gamma)^2 f(\gamma)$ *and* $(Y\,|S|\,Y^*f)(\gamma) = A(\gamma)f(\gamma)$ *for every* $f \in K$.

(iv) $P(\gamma)A(\gamma)^2P(\gamma) = A(\gamma)^2P(\gamma) = A(\gamma)^2$ *and* $P(\gamma)A(\gamma)P(\gamma) = A(\gamma)P(\gamma) = A(\gamma)$ *for a.e.* $\gamma \in \widehat{G^*}$.

Proposition 4 and equation (5) imply

$$\tilde{m}_k(\gamma) = A(\gamma)^{-1} m_k(\gamma),\ \gamma \in \widehat{G^*}. \tag{8}$$

where $A(\gamma)^{-1}$ is defined only for these γ in the $suppP$. In this case the domain of $A(\gamma)^{-1}$ is $P(\gamma)(\ell^2)$. We adopt the convention that $A(\gamma)^{-1} = 0$ if γ does not belong to $suppP$. Our next goals for this section are to derive the Generalized QMF condition; characterize the frame multiwavelets vector sets associated with a given frame multiscaling vector set and finally show that such sets do exist by giving two algorithms for their construction. This will complete a first project on the study of the general theory of GFMRAs of abstract Hilbert spaces. Yet several and interesting problems have to be studied in order to have a complete and in depth study of the topic. Other results and further examples can be found in [27] and [28].

The equation $gD = D\sigma(g)$ implies $D^*g = \sigma(g)D^*$, $g \in G$. Since D is a unitary operator, $D^*(V_0) = V_{-1}$ and $\{g\phi_k : g \in G, k \in \mathbb{N}\}$ is a frame of V_0 with frame bounds, say $A, B > 0$, then $\{D^*g\phi_k : g \in G, k \in \mathbb{N}\} = \{\sigma(g)D^*\phi_k : g \in G, k \in \mathbb{N}\}$ is a frame of V_{-1} with frame bounds $A, B > 0$. Using the fact that Y is an isometry and the second conclusion of Lemma 2.1 we get that $\{M_{\sigma(g)}\tilde{m}_k : g \in G, k \in \mathbb{N}\}$ is a frame of $Y(V_{-1})$ with frame bounds $A, B > 0$. Indeed,

$$Y\sigma(g)D^*\phi_k = Y\sigma(g)Y^*(YD^*\phi_k) = M_{\sigma(g)}P\tilde{m}_k = M_{\sigma(g)}\tilde{m}_k$$

because $\tilde{m}_k \in Y(V_{-1}) \subseteq Y(V_0) = \mathcal{R}(S) = P(K)$.

Now let $\{\psi_i : i \in I\}$ be a multiwavelet frame vector set associated with $\{V_j\}_j$. Therefore, $\{g\psi_i : i \in I\}$ is a frame of W_0 with frame bounds A', B'. We use the index set I instead of $\mathbb{N}$ although I is countable, because in the construction of frame multiwavelet vector sets, which we will present in subsection 2.2, we use the index set $\mathbb{N} \times \{0, 1, \ldots, n-1\}$. Arguing as in the preceding paragraph we obtain that $\{M_{\sigma(g)}\tilde{h}_i : i \in I, g \in G\}$ where $\tilde{h}_i := YD^*\psi_i$, is a frame of $Y(W_{-1})$. Since Y is an isometry and $V_0 = W_{-1} \oplus V_{-1}$ we have that

$$\{M_{\sigma(g)}\tilde{h}_i : i \in I, g \in G\} \cup \{M_{\sigma(g)}\tilde{m}_k : k \in \mathbb{N}, g \in G\}$$

is a frame of $Y(V_0) = P(K)$ with frame bounds $A'' = \min\{A, A'\}$ and $B'' = \max\{B, B'\}$ and $\{M_{\sigma(g)}\tilde{h}_i : i \in I, g \in G\}$ is orthogonal to $\{M_{\sigma(g)}\tilde{m}_k : k \in \mathbb{N}, g \in G\}$.

Definition 5. *Every subfamily $\{h_i : i \in I\}$ of K such that $\{M_{\sigma(g)}h_i : i \in I, g \in G\} \cup \{M_{\sigma(g)}\tilde{m}_k : k \in \mathbb{N}, g \in G\}$ is a frame of $P(K)$ and $\{M_{\sigma(g)}h_i : i \in I, g \in G\}$ is orthogonal to $\{M_{\sigma(g)}\tilde{m}_k : k \in \mathbb{N}, g \in G\}$ is a high pass filter set associated with the low pass filter set $\{m_k : k \in \mathbb{N}\}$.*

The previous definition and the arguments in the paragraph preceding it imply that $\{\tilde{h}_i : i \in I\}$ is a high pass filter set associated with the low pass filter set $\{m_k : k \in \mathbb{N}\}$.

We will adopt the tensor product notation to denote the orthogonal direct sum of a countable number of copies of a certain Hilbert space or of a bounded operator.

Let $X : K \to K \otimes n$ be the linear operator defined by the equation

$$(Xf)(\gamma) = \frac{1}{\sqrt{n}}(f(\tilde{\rho}(\gamma)), f(\gamma_1\tilde{\rho}(\gamma)), \ldots, f(\gamma_{n-1}\tilde{\rho}(\gamma))),\ f \in K, \gamma \in \widehat{G^*} \tag{9}$$

where $\{1 = \gamma_0, \gamma_1, \ldots, \gamma_{n-1}\} = \ker\rho$. Recall from proposition 2 that $\{\gamma_j\tilde{\rho}(\widehat{G^*}), j = 0, 1, \ldots, n-1\}$ is a measurable partition of $\widehat{G^*}$.

Lemma 2.3. *The operator X defined by equation (9) is a well defined isometric isomorphism and $XM_{\sigma(g)}X^* = M_g \otimes n$, for every $g \in G$.*

Proof. Obviously X is linear. We will show that X is an isometry. First notice that the equation $\rho(\tilde{\rho}(\gamma)) = \gamma$, for a.e. γ in $\widehat{G^*}$ and the fact that $\gamma_0, \gamma_1, \ldots, \gamma_{n-1} \in \ker\rho$ imply

$$\tilde{\rho}(\rho(\gamma)) = \gamma_j^{-1}\gamma, \tag{10}$$

for every $\gamma \in \gamma_j \tilde{\rho}(\widehat{G^*})$ and $j = 0, 1, \ldots, n-1$.

Now we have

$$\int_{\widehat{G^*}} \|f(\gamma_j \tilde{\rho}(\gamma))\|^2 \, d\gamma = \int_{\widehat{G^*}} \|f(\gamma_j \tilde{\rho}(\rho(\gamma)))\|^2 \, d\gamma \,, \tag{11}$$

because ρ is a continuous surjection and $\widehat{G^*}$ is a compact group. Equations (10) and (11) we imply

$$\begin{aligned} \int_{\widehat{G^*}} \|f(\gamma_{j_0} \tilde{\rho}(\gamma))\|^2 \, d\gamma &= \sum_{j=0}^{n-1} \int_{\gamma_j \tilde{\rho}(\widehat{G^*})} \|f(\gamma_{j_0} \tilde{\rho}(\rho(\gamma)))\|^2 \, d\gamma \\ &= \sum_{j=0}^{n-1} \int_{\gamma_j \tilde{\rho}(\widehat{G^*})} \left\|f(\gamma_{j_0} \gamma_j^{-1} \gamma)\right\|^2 d\gamma = n \int_{\tilde{\rho}(\widehat{G^*})} \|f(\gamma_{j_0} \gamma)\|^2 \, d\gamma \\ &= n \int_{\gamma_{j_0} \tilde{\rho}(\widehat{G^*})} \|f(\gamma)\|^2 \, d\gamma \,, \end{aligned}$$

for every $j_0 \in \{0, 1, \ldots, n-1\}$. On the other hand,

$$\begin{aligned} \|Xf\|^2 &= \int_{\widehat{G^*}} \|Xf(\gamma)\|^2 \, d\gamma = \frac{1}{n} \sum_{j=0}^{n-1} \int_{\widehat{G^*}} \|f(\gamma_j \tilde{\rho}(\gamma))\|^2 \, d\gamma \\ &= \int_{\widehat{G^*}} \|f(\gamma)\|^2 \, d\gamma = \|f\|^2 \,. \end{aligned}$$

Therefore, Xf belongs to $K \otimes n$ and X is an isometry. Now let $f_0, f_1, \ldots, f_{n-1} \in K$ and f be defined by the formula

$$f(\gamma) := \sqrt{n} \sum_{j=0}^{n-1} f_j(\rho(\gamma)) \chi_{\gamma_j \tilde{\rho}(\widehat{G^*})}(\gamma), \quad \gamma \in G^* \tag{12}$$

Then, it is not hard for someone to verify that $f \in K$ and $Xf = (f_0, \ldots, f_{n-1})$.

If $g \in G$, $\gamma \in \widehat{G^*}$ and $f \in K$, then

$$XM_{\sigma(g)} f(\gamma) = \frac{1}{\sqrt{n}} (f_{\sigma(g)}(\tilde{\rho}(\gamma)) f(\tilde{\rho}(\gamma)), \ldots, f_{\sigma(g)}(\gamma_{n-1} \tilde{\rho}(\gamma)) f(\gamma_{n-1} \tilde{\rho}(\gamma)) \,.$$

If $j = 0, 1, 2, \ldots, n-1$, then

$$\begin{aligned} f_{\sigma(g)}(\gamma_j \tilde{\rho}(\gamma)) &= \gamma_j(\ell_{\sigma(g)}) \tilde{\rho}(\gamma)(\ell_{\sigma(g)}) = \gamma_j(\sigma^*(\ell_g)) \tilde{\rho}(\gamma)(\sigma^*(\ell_g)) \\ &= \rho(\gamma_j)(\ell_g) \rho(\tilde{\rho}(\gamma))(\ell_g) = \gamma(\ell_g), \quad \gamma \in \widehat{G^*} \end{aligned}$$

because $\gamma_j \in \ker \rho$ and $\rho(\tilde{\rho}(\gamma)) = \gamma$ for every $\gamma \in \widehat{G^*}$. Hence,

$$f_{\sigma(g)}(\gamma_j \tilde{\rho}(\gamma)) = f_g(\gamma) \quad \text{a.e. in } \widehat{G^*},$$

therefore, $XM_{\sigma(g)} f(\gamma) = f_g(\gamma) X f(\gamma)$, $\gamma \in \widehat{G^*}$. This finaly establishes that $XM_{\sigma(g)} X^* = M_g \otimes n$, for every $g \in G$. □

Define $\tilde{P}(\gamma) := \sum_{j=0}^{n-1} \oplus P(\gamma_j\gamma)$. Obviously the projection valued function $\tilde{P}$ defines a bounded operator on $K \otimes n$. In fact it not difficult for someone to verify that

$$XPX^*(\omega_0, \omega_1, \dots, \omega_{n-1})(\gamma) = \tilde{P}(\tilde{\rho}(\gamma))(\omega_0(\gamma), \omega_1(\gamma), \dots, \omega_{n-1}(\gamma)) \text{ a.e. in } \widehat{G^*}, \tag{13}$$

where $\omega_0, \omega_1, \dots, \omega_{n-1}$ belong to K. We also have we get that

$$XY(V_0) = XP(K) = XPX^*X(K) = XPX^*(K \otimes n).$$

The intuitive meaning of the latter equalities is that V_0 is mapped isometrically into $K \otimes n$ and its image under XY is the "bundle" of the closed subspaces $\tilde{P}(\tilde{\rho}(\gamma))(l^2 \otimes n)$ as γ runs through $\widehat{G^*}$. We refer to those subspaces of $l^2 \otimes n$ as the *fibers* of $XY(V_0)$. Note that all the fibers of $K \otimes n$ are identical with $l^2 \otimes n$ for a.e. $\gamma \in \widehat{G^*}$.

The terms "fiber" and "fiberization" were introduced by Ron and Shen. They used fiberaization to study affine frames in $L^2(\mathbb{R}^n)$ ([32]). A subsequent paper by Bownik utilizes the same technique to study the structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ ([12]). In both papers, however, the main results rely on the fact that certain subspaces are generated by the action of an abelian unitary group of translations defined on $L^2(\mathbb{R}^n)$ on a certain set of generators. Thus the projections onto these subspaces as well as certain of their properties can be derived from the fact that these projections are in the commutant of the aforementioned group of translations, which is a completely characterized Von Neumann algebra, e.g $L^\infty([0,1)^n, \mathcal{B}(\ell^2)))$. In fact, in most proofs in this section we use such characterizations, which were known since the early years of the development of the Von Neumann algebra theory.

The following lemma summarizes some of the main arguments, which will lead us to the characterization of all frame multiwavelet vector sets associated with a given countably generated GFMRA.

Lemma 2.4. *Let $\{m_k : k \in \mathbb{N}\}$ be the low pass filter set corresponding to the frame multiscaling vector set $\{\phi_k : k \in \mathbb{N}\}$ and $\tilde{m}_k$ be given by eq. (8). Then $\{h_i : i \in I\}$ is a high pass filter set associated with $\{m_k : k \in \mathbb{N}\}$ if and only if*

$$\{(M_g \otimes n)X\tilde{m}_k : k \in \mathbb{N}, g \in G\} \cup \{(M_g \otimes n)Xh_i : g \in G, i \in I\},$$

is a frame of $XY(V_0)$ and $\{(M_g \otimes n)X\tilde{m}_k : k \in \mathbb{N}, g \in G\}$ is orthogonal to $\{(M_g \otimes n)Xh_i : g \in G, i \in I\}$.

Proof. The conclusion of the lemma follows immediately from lemma 2.3 and definition 5. □

The next lemma characterizes frames of abstract Hilbert spaces in terms of the preframe operator introduced by Casazza ([13]), which is also known as the *synthesis operator.*

Definition 6. *Let $\mathcal{H}$ be a Hilbert space and $\{x_r : r \in I\}$ be a Bessel family of $\mathcal{H}$. Assume that $\mathcal{K}$ is another Hilbert space, such that* $\dim \mathcal{K} = |I|$ *and $\{e_r : r \in I\}$ is an* *orthonormal basis of $\mathcal{K}$. The linear operator defined by*

$$Te_r = x_r, \quad r \in I$$

is called the preframe operator corresponding to $\{x_r : r \in I\}$ *and* $\{e_r : r \in I\}$.

It can easily be seen that T is well-defined and bounded because it is the adjoint of the analysis operator S corresponding to the Bessel family $\{x_r : r \in I\}$, which is defined by the following equation:

$$Sx = \sum_{r \in I} \langle x, x_r \rangle e_r, \; x \in \mathcal{H}. \tag{14}$$

It can be proved, by means of the concept of the dual frame, that T is always surjective, if $\{x_r : r \in I\}$ is a frame of $\mathcal{H}$.

Lemma 2.5. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\{x_r : r \in J\}$ be a complete family of vectors in $\mathcal{H}$, i.e. $[x_r : r \in J]^- = \mathcal{H}$. Assume that $\dim \mathcal{K} = |J|$ and that $\{e_r : r \in J\}$ is an orthonormal basis of $\mathcal{K}$. Then the following are true:*

(a) *If there exists $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $Te_r = x_r$, $r \in J$, then $\{x_r : r \in J\}$ is a Bessel family of $\mathcal{H}$. Conversely, let $T : [e_r : r \in J] \to \mathcal{H}$ be the linear operator such that for every $r \in J$ we have that $Te_r = x_r$. If $\{x_r : r \in J\}$ is a Bessel family of $\mathcal{H}$, then T can be extended to a bounded operator defined on $\mathcal{K}$.*

(b) *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Te_r = x_r$ for every $r \in J$. If V is the range space of T^* and $T^*T \mid_V : V \to V$ is bounded and invertible (resp. T^*T is an orthogonal projection), then $\{x_r : r \in J\}$ is a frame (resp. a PF) of $\mathcal{H}$.*

Conversely, if $\{x_r : r \in J\}$ is a frame (resp. a PF) of $\mathcal{H}$, then there exists a unique $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying $Te_r = x_r$ for every $r \in J$. Furthermore, $V := \mathcal{R}(T^*)$ is closed and $T^*T \mid_V : V \to V$ is bounded and invertible (resp. T^*T is an orthogonal projection).

(c) *If $\{x_r : r \in J\}$ is a frame of $\mathcal{H}$, then the set*

$$\{T((T^*T) \mid_V)^{-1/2} P_V e_r \; : \; r \in J\} ,$$

where P_V is the orthogonal projection onto V, is a PF of $\mathcal{H}$.

(d) *If $\{x_r : r \in J\}$ is a Riesz basis of $\mathcal{H}$, then $V = \mathcal{K}$ and vice-versa.*

Proof. *(a)* If T is bounded it is easy to check that

$$T^*x = \sum_{r \in I} \langle x, x_r \rangle e_r, \; x \in \mathcal{H}.$$

Set $S = T^*$. Then for every $x \in \mathcal{H}$ we have that $< Sx, Sx > \leq \|S\|^2 \|x\|^2$ and by using the previous equations we get

$$< S^*Sx, x > = < TSx, x > = < \sum_r < x, x_r > x_r, \; x > = \sum_r | < x, x_r > |^2 \; ,$$

which finaly implies $\sum_j | < x, x_r > |^2 \leq \|T\|^2 \|x\|^2$, for every $x \in \mathcal{H}$. Therefore $\{x_r : r \in J\}$ is a Bessel family of $\mathcal{H}$.

Conversely, if $\{x_r : r \in J\}$ is a Bessel family of $\mathcal{H}$, it is well-known from the abstract frame theory that the analysis operator defined by equation (14) is bounded and the restriction of the adjoint of the analysis operator on $[e_r : r \in J]$ is equal to T. Therefore T can be extended to a bounded oparator defined on $\mathcal{K}$. For further details the reader may refer to [8, 13, 19].

(b) Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $Te_r = x_r$,for every r. We will prove that T is surjective. Since $\{x_r : r \in J\}$ is complete in $\mathcal{H}$ and is contained in the range of T, we have that T^* is injective. Now, let $y \in \mathcal{H}$. Then T^*y is in V. If we set

$$x := ((T^*T) \mid_V)^{-1} T^* y \ , \tag{15}$$

then $((T^*T) \mid_V)x = T^*y$, which, in turn, implies that $Tx = y$, because T^* is injective. Therefore T is surjective.

On the other hand, the restriction of T on V is injective, hence, by the Open mapping theorem, we obtain that $T \mid_V$ is bounded and invertible. On the other hand for every $r \in J$ we have that $TP_V e_r = x_r$, where P_V is the orthogonal projection onto V (recall that $kerT = (I - P_V)(\mathcal{K})$). Since $\{x_r : r \in J\}$ is similar (through $T \mid_V$) to a PF, it is a frame of $\mathcal{H}$.

If T^*T is a projection, then $T^*T = P_V$. Let y be in $\mathcal{H}$. We define $x := T^*y$. Then $x = T^*Tx$, because x belongs to V. Since T^* is injective we obtain that $y = Tx$, hence, T is surjective. Thus

$$(TT^*)(TT^*) = TP_V T^* = TT^*.$$

This implies that TT^* is a selfadjoint idempotent operator, thus it is also an orthogonal projection onto the closure of the range of T, which is equal to $\mathcal{H}$. This, in turn implies that T^* is an isometry, therefore $\{x_r : r \in J\}$ is a PF of $\mathcal{H}$.

In order to prove the converse, first note that $\{x_r : r \in J\}$ is a Bessel family because it is a frame. Let $T : [e_r : r \in J] \to \mathcal{H}$ be the linear operator such that for every $r \in J$ we have that $Te_r = x_r$. Then (a) implies that T can be extended to a bounded operator defined on $\mathcal{K}$, *which, for convenience, we denote by T as well.* Obviously every $T' \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying $T'e_r = x_r$, for every $r \in J$ is equal to T. Using the lower frame inequality and following arguments that can be found in the first chapter of [19] we prove that $\mathcal{R}(T^*)$ is closed. Since $\overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(T^*T)}$ (see Proposition 2.5.13 of [22]) and $(T^*T) \mid_V$ is injective (because both T^* and $T \mid_V$ are injective) we conclude that $(T^*T) \mid_V : V \to V$ is bounded and bijective and, thus, invertible.

If $\{x_r : r \in J\}$ is a PF of $\mathcal{H}$, then T^* is an isometry, hence, T^*T is a projection.

(c) Let x be in $\mathcal{H}$. Then

$$\begin{aligned}\sum_r | < x, T((T^*T)\mid_V)^{-1/2}P_V e_r > |^2 &= \sum_r | < ((T^*T)\mid_V)^{-1/2}T^*x,\ P_V e_r > |^2 = \\ \| ((T^*T)\mid_V)^{-1/2}T^*x \|^2 &= < ((T^*T)\mid_V)^{-1/2}T^*x,\ ((T^*T)\mid_V)^{-1/2}T^*x >= \\ &= < T((T^*T)\mid_V)^{-1}T^*x, x > \ .\end{aligned}$$

Using again eq. (15) and the argument following this equation we obtain that $x = T((T^*T)\mid_V)^{-1}T^*x$, which completes that proof of (c).

(d) Assume that $\{x_r : r \in J\}$ is a Riesz basis of $\mathcal{H}$. Then (b) implies that $T\mid_V$ is bounded and invertible. If V is a proper subspace of $\mathcal{K}$, then the Riesz basis $\{x_r : r \in J\}$ would be isomoprhic to a PF, which is not a Riesz basis. Thus $V = \mathcal{K}$. The converse is trivial. □

Remark 2. Note that in the proof of (a) of the previous proposition we did not use the hypothesis $[x_r : r \in J]^- = \mathcal{H}$.

Let us, now, utilize the previous lemma. Recall our hypothesis that $\{\psi_i : i \in I\}$ is a frame multiwavelet set associated with the GFMRA $\{V_j\}_j$. Also recall that we have defined $\tilde{h}_i = YD^*\psi_i$ $(i \in I)$ and so $\{\tilde{h}_i : i \in I\}$ is a high pass filter set associated with the low pass filter set $\{m_k : k \in \mathbb{N}\}$. Now lemma 2.4 implies that

$$(M_g \otimes n)X\tilde{m}_k : g \in G, k \in \mathbb{N}\} \cup \{(M_g \otimes n)X\tilde{h}_i : g \in G, i \in I\}$$

is a frame of $XY(V_0) = XPX^*(K \otimes n)$. Let $\{i_1, i_2, \ldots\}$ be an enumeration of the countable set I. We define the preframe operator $T : K \oplus K \to K \otimes n$ by the following formulas:

$$T(f_g(\delta_k \oplus 0)) = f_g X\tilde{m}_k, \quad g \in G, \quad k \in \mathbb{N} \tag{16}$$

$$T(f_g(0 \oplus \delta_k)) = f_g X\tilde{h}_{i_k}, \quad g \in G, \quad k \in \mathbb{N}. \tag{17}$$

The previous lemma implies that there exists an orthogonal projection P_0 defined on $K \oplus K$, such that $(T^*T)\mid_{P_0(K\oplus K)}$ is bounded and invertible. From the same lemma we have that $P_0(K \oplus K)$ is the range of the analysis operator corresponding to the frame $\{(M_g \otimes n)X\tilde{m}_k : g \in G, k \in \mathbb{N}\} \cup \{(M_g \otimes n)X\tilde{h}_i : g \in G, i \in I\}$ of $XPX^*(K \otimes n)$.

On the other hand equations (16) and (17) imply that, if $(T_{l,p})_{l=0,p=1,2}^{n-1}$ is the matrix representation of T with respect to the decompositions $K \oplus K$ and $K \otimes n$, then every $T_{l,p}$ is bounded and belongs to the commutant of $\{M_g : g \in G\}$. Therefore there exist operator valued functions $T_{l,p} : \widehat{G^*} \to \mathcal{B}(\ell^2)$, satisfying the following

(a) If $\omega_1, \omega_2 \in K$ then

$$T(\omega_1, \omega_2)(\gamma) = (T_{0,1}(\gamma)\omega_1(\gamma) + T_{0,2}(\gamma)\omega_2(\gamma), \ldots, T_{n-1,1}(\gamma)\omega_1(\gamma) + T_{n-1,2}\omega_2(\gamma)),$$

for a.e. γ in $\widehat{G^*}$.

(b) *essup*$\{\|T_{l,p}(\gamma)\| : \gamma \in \widehat{G^*}, l = 0, \ldots, n-1, p = 1, 2\} \leq \|T\|$.

Let us now introduce a convenient notation, which we will use for the rest of the present paper. If A is a matrix then by $[A]_r$ we will denote the r-th column of A.

Once again using equations (16) and (17) we obtain

$$[T_{j,1}(\gamma)]_k = \frac{1}{\sqrt{n}}\tilde{m}_k(\gamma_j\tilde{\rho}(\gamma)),\ \ \gamma \in \widehat{G^*}, k \in \mathbb{N}.$$

and

$$[T_{j,2}(\gamma)]_k = \frac{1}{\sqrt{n}}\tilde{h}_{i_k}(\gamma_j\tilde{\rho}(\gamma)),\ \ \gamma \in \widehat{G^*},\ \ k \in \mathbb{N}.$$

Next define $\tilde{M}_0(\gamma)$ and $\tilde{H}(\gamma)$ by the formulas

$$[\tilde{M}_0(\gamma)]_k = \tilde{m}_k(\gamma),\ \ \ \gamma \in \widehat{G^*},\ \ k \in \mathbb{N}, \tag{18}$$

$$[\tilde{H}(\gamma)]_k = \tilde{h}_{i_k}(\gamma)\ \ \gamma \in G^*,\ \ k \in \mathbb{N}. \tag{19}$$

From the preceding discussion for $j = 0, 1, \dots, n-1$ and for a.e. $\gamma \in \widehat{G^*}$ we have

$$T_{j,1}(\gamma) = \frac{1}{\sqrt{n}}\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))$$

and

$$T_{j,2}(\gamma) = \frac{1}{\sqrt{n}}\tilde{H}(\gamma_j\tilde{\rho}(\gamma))\ .$$

These two equations combined with (b) above, imply that $\tilde{H}$ and $\tilde{M}_0$ belong to $L^\infty(\widehat{G^*}, \mathcal{B}(l^2))$. Now (a) combined with the latter pair of equations yields the following matrix representation of $T(\gamma)$

$$T(\gamma) = \frac{1}{\sqrt{n}}\begin{pmatrix} \tilde{M}_0(\tilde{\rho}(\gamma)) & \tilde{H}(\tilde{\rho}(\gamma)) \\ \tilde{M}_0(\gamma_1\tilde{\rho}(\gamma)) & \tilde{H}(\gamma_1\tilde{\rho}(\gamma)) \\ \vdots & \vdots \\ \tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma)) & \tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma)) \end{pmatrix} \quad \text{a.e.}$$

Next, note that

$$\langle f_g X\tilde{m}_k, X\tilde{h}_{i_\ell}\rangle = 0 \quad \text{for every } g \in G, k, \ell \in \mathbb{N}.$$

This implies that for a.e. γ in $\widehat{G^*}$ each one of the columns of the matrix

$$(\tilde{M}_0(\tilde{\rho}(\gamma))^T, \tilde{M}_0(\gamma_1\tilde{\rho}(\gamma))^T \dots \tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma))^T)^T$$

is orthogonal to every column of the matrix

$$(\tilde{H}(\tilde{\rho}(\gamma))^T, \tilde{H}(\gamma_1\tilde{\rho}(\gamma))^T, \dots, \tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma))^T)^T\ .$$

Indeed, the orthogonality between $\{f_g X\tilde{m}_k\}_{g,k}$ and $\{X\tilde{h}_{i_\ell}\}_{\ell\in\mathbb{N}}$ implies that for every $g \in G$ we have

$$\begin{aligned} 0 &= \langle f_g X\tilde{m}_k, X\tilde{h}_{i_\ell}\rangle = \frac{1}{n}\sum_{j=0}^{n-1}\langle f_g\tilde{m}_k(\gamma_j\tilde{\rho}(\cdot)), \tilde{h}_{i_\ell}(\gamma_j\tilde{\rho}(\cdot))\rangle \\ &= \frac{1}{n}\int_{\widehat{G^*}} f_g(\gamma)\left[\sum_{j=0}^{n-1}\langle \tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))\delta_k, \tilde{H}(\gamma_j\tilde{\rho}(\gamma))\delta_\ell\rangle\right] d\gamma \\ &= \frac{1}{n}\int_{\widehat{G^*}} f_g(\gamma)\left[\sum_{j=0}^{n-1}\langle \tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))\delta_k, \delta_\ell\rangle\right] d\gamma\ . \end{aligned}$$

Thus we get that for a.e. γ in $\widehat{G^*}$

$$\langle\sum_{j=0}^{n-1}\tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))\delta_k,\ \delta_\ell\rangle = 0,\ \ k,\ell\in\mathbb{N},$$

because $\{f_g : g \in G\}$ is an orthonormal basis for $L^2(\widehat{G^*})$. This establishes the previous claim and, in particular, implies

$$\sum_{j=0}^{n-1}\tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma)) = 0 \quad \text{a.e. in } \widehat{G^*}$$

which, in turn, gives us

$$\sum_{j=0}^{n-1}\tilde{H}(\gamma_j\gamma)^*\tilde{M}_0(\gamma_j\gamma) = 0 \quad \text{a.e. in } \widehat{G^*}.$$

Now we can conclude that T^*T, which belongs to the commutant of $\{M_g \oplus M_g : g \in G\}$, is multiplicative and that is induced by the following operator-valued function:

$$\frac{1}{n}\begin{pmatrix} \sum_{j=0}^{n-1}\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma)) & 0 \\ & \\ 0 & \sum_{j=0}^{n-1}\tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{H}(\gamma_j\tilde{\rho}(\gamma)) \end{pmatrix}.$$

Moreover, $T^*T\mid_{P_0(K\oplus K)}: P_0(K\oplus K)\to P_0(K\oplus K)$ is bounded invertible. Note that P_0 is also the range projection of T^*T, thus $P_0 \in \{M_g\oplus M_g : g\in G\}'$. Therefore, there exist projection-valued functions $P_1(\cdot)$ and $P_2(\cdot)$, which belong to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$, such that for a.e. $\gamma \in \widehat{G^*}$ the range projections of

$$\sum_{j=0}^{n-1}\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))$$

and

$$\sum_{j=0}^{n-1} \tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{H}(\gamma_j\tilde{\rho}(\gamma))$$

are $P_1(\gamma)$ and $P_2(\gamma)$ respectively. Now, if we consider the restriction of T^*T on $P_0(K\oplus K)$ we obtain that for a.e. γ in $\widehat{G^*}$

$$\{\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))\} \mid_{P_1(\gamma)(\ell^2)}: P_1(\gamma)(\ell^2) \to P_1(\gamma)(\ell^2)$$

and

$$\{\sum_{j=0}^{n-1} \tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{H}(\gamma_j\tilde{\rho}(\gamma))\} \mid_{P_2(\gamma)(\ell^2)}: P_2(\gamma)(\ell^2) \to P_2(\gamma)(\ell^2)$$

are both bounded and invertible. Moreover the functions

$$\gamma \to \left\|(\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))^*\tilde{M}_0(\gamma_j\tilde{\rho}(\gamma)) \mid_{P_1(\gamma)(\ell^2)})^{-1}\right\|$$

and

$$\gamma \to \left\|(\sum_{j=0}^{n-1} \tilde{H}(\gamma_j\tilde{\rho}(\gamma))^*\tilde{H}(\gamma_j\tilde{\rho}(\gamma)) \mid_{P_2(\gamma)(\ell^2)})^{-1}\right\|$$

are essentially bounded and are defined on suppP_1 and suppP_2 respectively. Using the properties of ρ and $\tilde{\rho}$ one can easily verify that all the properties the previous two operator-valued functions are inherited by the following operator-valued functions

$$\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\gamma)^*\tilde{M}_0(\gamma_j\gamma) \mid_{P_1(\rho(\gamma))(\ell^2)}: P_1(\rho(\gamma))(\ell^2) \to P_1(\rho(\gamma))(\ell^2)$$

and

$$\sum_{j=0}^{n-1} \tilde{H}(\gamma_j\gamma)^*\tilde{H}(\gamma_j\gamma) \mid_{P_2(\rho(\gamma))(\ell^2)}: P_2(\rho(\gamma))(\ell^2) \to P_2(\rho(\gamma))(\ell^2)\ ,$$

which are defined a.e. on the supports of $P_1 \circ \rho$ and $P_2 \circ \rho$ respectively and vanish elsewhere.

The range of T is $XPX^*(K \otimes n)$. Again by applying Proposition 2.5.13 of [22] we obtain that the range projection of T is equal to the range projection of TT^*, which is equal to XPX^*. Since TT^* is multiplicative induced by the function $T(\cdot)T(\cdot)^*$, it follows that the range projection of $T(\gamma)$ is equal to $\tilde{P}(\tilde{\rho}(\gamma))$ a.e., because, due to eq. (13), $\tilde{P}(\tilde{\rho}(\gamma))$ is the range projection of $T(\gamma)T(\gamma)^*$ a.e. Thus for a.e. $\gamma \in \widehat{G^*}$ the columns of the matrix representation of the operator $T(\gamma)$ form a complete set in the range of $\tilde{P}(\tilde{\rho}(\gamma))$, namely

$$\{[\tilde{M}_0(\tilde{\rho}(\gamma))^T, \ldots, \tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma))^T)^T]_k : k \in \mathbb{N}\} \cup \{[(\tilde{H}(\tilde{\rho}(\gamma))^T, \ldots, \tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma))^T)^T]_{i_k} : k \in \mathbb{N}\}.$$

Once again applying lemma 2.5 we obtain that for a.e. γ in $\widehat{G^*}$ the latter set is also a frame of $\tilde{P}(\tilde{\rho}(\gamma))(\ell^2 \otimes n)$.

Now, let us find a pair of frame bounds for this particular frame. Recall that A'', B'' are the frame bounds for the frame $\{M_{\sigma(g)}\tilde{m}_k : g \in G, k \in \mathbb{N}\} \cup \{M_{\sigma(g)}\tilde{h}_i : i \in I, g \in G\}$. Therefore, A'', B'' are frame bounds for $\{f_g X \tilde{m}_k : g \in G, k \in \mathbb{N}\} \cup \{f_g X \tilde{h}_i : g \in G, i \in I\}$, because X is unitary. Thus

$$\left\|(T\mid_{P_0(K\oplus K)})^{-1}\right\| \leq \frac{1}{\sqrt{A''}} \text{ and } \left\|T\mid_{P_0(K\oplus K)}\right\| \leq \sqrt{B''}.$$

Since T is induced by multiplication with $T(\gamma)$ we obtain that the function $\gamma \to T(\gamma)$ satisfies the following properties:

$$essup\{\|T(\gamma)\| : \gamma \in \widehat{G^*}\} \leq \sqrt{B''}$$

and

$$essup\{\left\|(T(\gamma)\mid_{P_1(\gamma)(\ell^2)\oplus P_2(\gamma)(\ell^2)})^{-1}\right\| : \gamma \in \widehat{G^*}\} \leq \frac{1}{\sqrt{A''}}.$$

Now, define

$$\tilde{Q}_1(\gamma) := \sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\gamma)^* \tilde{M}_0(\gamma_j\gamma) \tag{20}$$

and

$$\tilde{Q}_2(\gamma) := \sum_{j=0}^{n-1} \tilde{H}(\gamma_j\gamma)^* \tilde{H}(\gamma_j\gamma). \tag{21}$$

Since $\tilde{M}_0$, $\tilde{H}$ belong to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ we have that $\tilde{Q}_1$ and $\tilde{Q}_2$ belong to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ as well. We are now ready to state our first main result which summarizes the preceding discussion.

Theorem 7. *Let $\{\phi_k : k \in \mathbb{N}\}$ be a set of frame multiscaling vectors for the GFMRA $\{V_j\}_j$. Then, M_0 and $\tilde{M}_0$ belong to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$.*

Let $\tilde{Q}_1$ is the operator-valued function defined by eq. (21). Let $P_1(\gamma)$ be the range projection of $\tilde{Q}_1(\gamma)$, $\gamma \in \widehat{G^*}$. Then $\tilde{Q}_1(\gamma)\mid_{P_1(\gamma)(\ell^2)}$: $P_1(\gamma)(\ell^2) \to P_1(\gamma)(\ell^2)$ is invertible for a.e. $\gamma \in suppP_1$. Moreover both functions $\gamma \to \left\|\tilde{Q}_1(\gamma)\mid_{P_1(\gamma)(\ell^2)}\right\|$ and $\gamma \to \left\|(\tilde{Q}_1(\gamma)\mid_{P_1(\gamma)(\ell^2)})^{-1}\right\|$ are essentially bounded.

Furthermore, if $\{\psi_i : i \in I\}$, where I is countable, is a set of frame multiwavelet vectors associated with $\{V_j\}_j$, $\tilde{H}$ is defined by equation (19), $\tilde{Q}_2$ by eq. (21) and $P_2(\gamma)$ is the range projection of $\tilde{Q}_2(\gamma)$ $(\gamma \in \widehat{G^*})$, then the following are true:

(a) $\tilde{H}$ *belongs to* $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ *and for a.e.* $\gamma \in$ *supp*P_2 *the operator* $\tilde{Q}_2(\gamma)\mid_{P_2(\gamma)(\ell^2)}$: $P_2(\gamma)(\ell^2) \to P_2(\gamma)(\ell^2)$ *is invertible and the functions* $\gamma \to \left\|\tilde{Q}_2(\gamma)\mid_{P_2(\gamma)(\ell^2)}\right\|$, $\gamma \to \left\|(\tilde{Q}_2(\gamma)\mid_{P_2(\gamma)(\ell^2)})^{-1}\right\|$ *are essentially bounded.*

(b) For a.e. $\gamma \in \widehat{G^*}$ *the closed linear span of the set*

$\{[(\tilde{M}_0(\gamma)^T, \tilde{M}_0(\gamma_1\gamma)^T, \ldots, \tilde{M}_0(\gamma_{n-1}\gamma)^T)^T]_k : k \in \mathbb{N}\} \cup \{[(\tilde{H}(\gamma)^T, \tilde{H}(\gamma_1\gamma)^T, \ldots, \tilde{H}(\gamma_{n-1}\gamma)^T)^T]_i : i \in I\}$
is equal to $\tilde{P}(\gamma)(\ell^2 \otimes n)$ *and*

$$\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\gamma)^* \tilde{H}(\gamma_j\gamma) = 0 \quad \text{a.e.}$$

Apparently $\tilde{Q}_l(\gamma) = 0$ outside of s*upp*P_l $(l = 1, 2)$. Conditions (a) and (b) of the previous theorem are necessary for all frame multiwavelet vector sets associated with $\{V_j\}_j$. Later we will prove that these conditions are sufficient as well.

It is worthy to note that the first statement of the theorem generalizes the Quadratic Mirror Filter (QMF) conditions, so we will refer to them as *Generalized QMF* conditions.

Our next goal is to construct the set of the multiwavelet frame vectors in terms of the set $\{\phi_k : k \in \mathbb{N}\}$ which should be considered as the only available information for $\{V_j\}_j$.

We have already proved that $S^*S \in G'$ (see lemma 2.2). Thus if we define

$$\phi'_k := (S^*S)^{-1}\phi_k, \quad k \in \mathbb{N} \tag{22}$$

then $\phi'_k \in V_0$. But $(S^*S)^{-1}$ commutes with G, so

$$g\phi'_k = g(S^*S)^{-1}\phi_k = (S^*S)^{-1}(g\phi_k), \; g \in G, k \in \mathbb{N},$$

which implies that $\{g\phi'_k : g \in G, k \in \mathbb{N}\}$ is the dual frame of $\{g\phi_k : g \in G, k \in \mathbb{N}\}$ in V_0. Thus, for every ψ_i which belongs to a frame multiwavelet set of vectors associated with $\{V_j\}_j$, we have

$$D^*\psi_i := \sum_g \sum_k \langle D^*\psi_i, g\phi'_k\rangle g\phi_k, \quad i \in I.$$

Therefore, we must compute the coefficient families $\{\langle D^*\psi_i, g\phi'_k\rangle\}_{g,k}$ for every $i \in I$. Note that

$$\langle D^*\psi_i, g\phi'_k\rangle = \langle D^*\psi_i, (S^*S)^{-1}g\phi_k\rangle = \langle S(S^*S)^{-1}D^*\psi_i, f_g\delta_k\rangle \;\; g \in G, k \in \mathbb{N}. \tag{23}$$

But $S(S^*S)^{-1} = Y\,|S|\,(|S|^2)^{-1} = Y\,|S|^{-1}$. Recall that $P(K)$ is the range of the isometry Y. Therefore,

$$S(S^*S)^{-1}D^*\psi_i = Y\,|S|^{-1}\,Y^*YD^*\psi_i = (Y\,|S|^{-1}\,Y^*)\tilde{h}_i = (Y\,|S|\,Y^*\mid_{P(K)})^{-1}\tilde{h}_i \tag{24}$$

Proposition 4 and equations (24) and (23) imply

$$\langle D^*\psi_i, g\phi'_k\rangle = \langle A(\cdot)^{-1}\tilde{h}_i(\cdot), f_g\delta_k\rangle \ .$$

If $\{a^{(i)}_{g,k}\}_{i,g,k}$ is the square summable family of scalars satisfying

$$A(\cdot)^{-1}\tilde{h}_i(\cdot) = \sum_{g,k} a^{(i)}_{g,k} f_g\delta_k \ , \tag{25}$$

then

$$\psi_i = \sum_{g,k} a^{(i)}_{g,k} Dg\phi_k, \quad i \in I \ . \tag{26}$$

The preceding argument proves the following proposition.

Proposition 8. *Let $\{\phi_k : k \in \mathbb{N}\}$ be a frame multiscaling vector set and $\{\phi'_k : k \in \mathbb{N}\}$ be defined by equation (22). Then the family $\{\phi'_k : k \in \mathbb{N}\}$ is the dual frame multiscaling vector set corresponding to the set $\{\phi_k : k \in \mathbb{N}\}$, in the sense that $\{g\phi'_k : g \in G, k \in \mathbb{N}\}$ is the dual frame of $\{g\phi_k : g \in G, k \in \mathbb{N}\}$ in V_0. Moreover, if $\{\psi_i : i \in I\}$ is a frame multiwavelet vector set associated with $\{V_j\}_j$ and $\{\tilde{h}_i : i \in I\}$ is the high pass filter set corresponding to $\{\psi_i : i \in I\}$, then every ψ_i is given by the equations (25) and (26).*

So far we derived the necessary conditions for the the high-pass filters corresponding to a given set of frame multiwavelet vectors associated with $\{V_j\}_j$. We have also obtained concrete equations giving $\{\psi_i : i \in I\}$ in terms of the frame $\{g\phi_k : g \in G, k \in \mathbb{N}\}$. All these results are summarized in theorem 7 and proposition 8.

We previously defined the concept of a low pass filter set corresponding to a set of frame multiscaling vectors. It will be very convenient though to introduce the following generalization of the concept of a low pass filter corresponding to a Riesz scaling function.

Definition 9. *If $\{\phi_k : k \in \mathbb{N}\}$ is a frame multiscaling vector set we call the operator valued function $M_0 \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ defined by the equation*

$$[M_0(\gamma)]_k := m_k(\gamma) \ \text{ a.e.}$$

the low pass filter corresponding to $\{\phi_k : k \in \mathbb{N}\}$. We will also refer to $\tilde{M}_0$ as the normalized low pass filter corresponding to $\{\phi_k : k \in \mathbb{N}\}$.

Recall $M_0(\gamma) = A(\gamma)\tilde{M}_0(\gamma)$ a.e.

Our next goal in this section is to state and prove the converse of theorem 7, which will establish that the necessary conditions for the high pass filter sets are also sufficient. The reader will notice that the proof of the next theorem is easier than the proof of theorem 7.

Theorem 10. *Let $\{\phi_k : k \in \mathbb{N}\}$ be a frame multiscaling vector set for the GFMRA $\{V_j\}_j$. Assume that M_0 is the low pass filter corresponding to the set $\{\phi_k : k \in \mathbb{N}\}$. Assume that there exist essentially bounded measurable functions $\tilde{Q}_2, \tilde{H} : \widehat{G^*} \to \mathcal{B}(\ell^2)$, and a projection-valued measurable function $P_2 : \widehat{G^*} \to \mathcal{B}(\ell^2)$, such that $P_2(\gamma)$ is the range projection of $\tilde{Q}_2(\gamma)$ a.e. satisfying also conditions (a), (b) of theorem 7. If we define*

$$\psi_i := \sum_{g,k} a^{(i)}_{g,k} Dg\phi_k,$$

where $\{a^{(i)}_{g,k} : i \in I, g \in G, k \in \mathbb{N}\}$ are defined by the equation $A(\cdot)^{-1}[\tilde{H}(\cdot)]_i = \sum_{g,k} a^{(i)}_{g,k} f_g \delta_k$,, then the set $\{\psi_i : i \in I\}$ is a set of frame multiwavelet vectors associated with the GFMRA $\{V_j\}_j$.

Before proceeding to the proof of theorem 10 we must stress the fact that theorems 7 and 10 give a complete characterization of all frame multiwavelet vector sets associated with a given GFMRA. Moreover the techniques used in the proofs of these theorems and conditions (a) and (b) of theorem 7 show how to construct these sets. We will construct frame multiwavelet vector sets in the next subsection.

Proof of Theorem 10. By theorem 7 implies that there exist a measurable function $\tilde{Q}_1 : \widehat{G^*} \to \mathcal{B}(\ell^2)$ and a projection-valued function $P_1 : \widehat{G^*} \to \mathcal{B}(\ell^2)$ satisfying the following hypotheses:

- (A): $\tilde{Q}_1(\gamma) \mid_{P_1(\gamma)(\ell^2)}: P_1(\gamma)(\ell^2) \to P_1(\gamma)(\ell^2)$ is bounded and invertible for a.e. $\gamma \in \operatorname{supp} P_1$ and vanishes outside of $\operatorname{supp} P_1$.
- (B): For a.e. $\gamma \in \widehat{G^*}$ we have that $\tilde{Q}_1(\gamma) := \sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\gamma)^* \tilde{M}_0(\gamma_j\gamma)$, where $\tilde{M}_0$ is defined by equation (18).
- (C): The functions $\gamma \to \left\| \tilde{Q}_1(\gamma) \mid_{P_1(\gamma)(\ell^2)} \right\|, \gamma \to \left\| (\tilde{Q}_1(\gamma) \mid_{P_1(\gamma)(\ell^2)})^{-1} \right\|$ are essentially bounded.

Now, for every γ in $\widehat{G^*}$ define $T(\gamma) : \ell^2 \oplus \ell^2 \to \ell^2$ to be the linear operator whose matrix representation with respect to the standard orthonormal basis $\{\delta_k : k \in \mathbb{N}\}$ of ℓ^2 is the following:

$$T(\gamma) := \frac{1}{\sqrt{n}} \begin{pmatrix} \tilde{M}_0(\tilde{\rho}(\gamma)) & \tilde{H}(\tilde{\rho}(\gamma)) \\ \tilde{M}_0(\gamma_1\tilde{\rho}(\gamma)) & \tilde{H}(\gamma_1\tilde{\rho}(\gamma)) \\ \dots & \dots \\ \tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma)) & \tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma)) \end{pmatrix}$$

Condition (a) of theorem 7 and the fact that $\tilde{M}_0 \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ imply that for a.e. $\gamma \in \widehat{G^*}$, $T(\gamma)$ is a bounded well-defined linear operator on $\ell^2 \oplus \ell^2$. Next, condition

(b) of theorem 7 gives us

$$T(\gamma)^*T(\gamma) = \frac{1}{n}\begin{pmatrix} \tilde{Q}_1(\tilde{\rho}(\gamma)) \mid_{P_1'(\gamma)(\ell^2)} & 0 \\ 0 & \tilde{Q}_2(\tilde{\rho}(\gamma)) \mid_{P_2'(\gamma)(\ell^2)} \end{pmatrix}$$

where $P_1' = P_1 \circ \tilde{\rho}$ and $P_2' = P_2 \circ \tilde{\rho}$. Hypothesis (A) and condition (a) imply that for a.e. γ in $\widehat{G^*}$ the operator $T(\gamma)^*T(\gamma) \mid_{P_1'(\gamma)(\ell^2)\oplus P_2'(\gamma)(\ell^2)}$ is bounded and invertible. Moreover the function $\gamma \to T(\gamma)$ defines the bounded operator T, which commutes with $\{M_g \otimes n : g \in G\}$. Note that T is bounded, because the functions $\gamma \to \left\|\tilde{M}_0(\gamma)\right\|$, $\gamma \to \left\|\tilde{H}(\gamma)\right\|$ are essentially bounded. Let $\{i_1, i_2, \ldots\}$ be an enumeration of I. Since $T \in \{M_g \otimes n : g \in G\}'$ we have that

$$\begin{aligned} T(f_g(\delta_k \oplus 0)) &= \frac{1}{\sqrt{n}} f_g(\gamma)([\tilde{M}_0(\tilde{\rho}(\gamma))]_k^T, [\tilde{M}_0(\gamma_1\tilde{\rho}(\gamma))]_k^T, \ldots, [\tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma))]_k^T) \\ T(f_g(0 \oplus \delta_k)) &= \frac{1}{\sqrt{n}} f_g(\gamma)([\tilde{H}(\tilde{\rho}(\gamma))]_{i_k}^T,\ [\tilde{H}(\gamma_1\tilde{\rho}(\gamma))]_{i_k}^T, \ldots, [\tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma))]_{i_k}^T) \end{aligned}$$

Moreover, due to hypotheses (A),(C) and condition (a) of theorem 7 the restriction of T^*T on the range of the projection $P_1' \oplus P_2'$ is invertible.

Before proceeding recall the following property of $\tilde{\rho}$: If $\mu(A) = 0$, then $\mu(\tilde{\rho}^{-1}(A)) = 0$. Note that $\tilde{\rho}^{-1}(A) = \rho(\tilde{\rho}(\widehat{G^*}) \cap A)$ and that $\mu(\rho(B)) = 0$ whenever $\mu(B) = 0$. The latter property of ρ can easily be proved using on (a) and (d) of proposition 2.

By condition (b) of theorem 7 we have that for a.e. γ in G^* the closed linear span of the set

$$\{([\tilde{M}_0(\tilde{\rho}(\gamma))]_k^T, [\tilde{M}_0(\gamma_1\tilde{\rho}(\gamma))]_k^T, \ldots, [\tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma))]_k^T) : k \in \mathbb{N}\}$$

$$\cup\{([\tilde{H}(\tilde{\rho}(\gamma))]_{i_k}^T, [\tilde{H}(\gamma_1\tilde{\rho}(\gamma))]_{i_k}^T, \ldots, [\tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma))]_{i_k}^T) : k \in \mathbb{N}\}$$

is equal to $\tilde{P}(\tilde{\rho}(\gamma))(\ell^2 \otimes n)$. Now, $T(f_g(\delta_k \oplus 0)), T(f_g(0 \oplus \delta_k)) \in XPX^*(K \otimes n)$. On the other hand, if we assume that $(\omega_0, \omega_1 \ldots, \omega_{n-1})$ belongs to $XPX^*(K \otimes n)$ and is orthogonal to the family $\{T(f_g(\delta_k \oplus 0)), T(f_g(0 \oplus \delta_k)) : g \in G, k \in \mathbb{N}\}$, then it is not hard to check that for a.e. γ the vector $(\omega_0(\gamma), \omega_1(\gamma) \ldots, \omega_{n-1}(\gamma))$ is orthogonal to $([\tilde{M}_0(\tilde{\rho}(\gamma))]_k^T, [\tilde{M}_0(\gamma_1\tilde{\rho}(\gamma))]_k^T, \ldots, [\tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma))]_k^T)$ and $([\tilde{H}(\tilde{\rho}(\gamma))]_{i_k}^T, [\tilde{H}(\gamma_1\tilde{\rho}(\gamma))]_{i_k}^T, \ldots,$ $[\tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma))]_{i_k}^T)$ for every $k \in \mathbb{N}$. This implies that $\omega_j = 0$ for every $j = 0, 1, \ldots, n-1$. Therefore, the family $\{T(f_g(\delta_k \oplus 0)),\ T(f_g(0 \oplus \delta_k)) : g \in G,\ k \in \mathbb{N}\}$ is complete in the range of XPX^*. Furthermore, lemma 2.5 implies that this particular family is also a frame for the range of XPX^*.

Now notice that equation

$$\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\tilde{\rho}(\gamma))^*\tilde{H}(\gamma_j\tilde{\rho}(\gamma)) = 0 \quad \text{a.e.}$$

implies that the closed linear span of $\{T(f_g(\delta_k \oplus 0)) : g \in G, k \in \mathbb{N}\}$ is the orthogonal complement of the closed linear span of $\{T(f_g(0 \oplus \delta_k)) : g \in G, k \in \mathbb{N}\}$. Since the former set spans $XY(V_{-1})$, we have that the latter spans $XY(W_{-1})$. Therefore $\{T(f_g(0 \oplus \delta_k)) : g \in G, k \in \mathbb{N}\}$ is a frame for $XY(W_{-1})$. Since X is a surjective isometry between K and $K \otimes n$ we obtain that $\{X^*T(f_g(0 \oplus \delta_k)) : g \in G, k \in \mathbb{N}\}$ is a frame for $Y(W_{-1})$ with the same frame bounds. But

$$X^*T(f_g(0 \oplus \delta_k)) = X^*(M_g \otimes n)T(0 \oplus \delta_k) = X^*(M_g \otimes n)XX^*T(0 \oplus \delta_k)$$
$$= M_{\sigma(g)}[\tilde{H}(\cdot)]_{i_k}.$$

Recall that Y is an isometry mapping V_0 onto $Y(V_0)$. Therefore, if we define $\psi_{i_k} := DY^*[\tilde{H}(\cdot)]_{i_k}, k \in \mathbb{N}$, then $\{\psi_i : i \in I\}$ is a frame multiwavelet vector set associated with $\{V_j\}_j$. Indeed, this follows from the fact that Y is an isometry and from (b) of lemma 2.1. The final arguments of the proof of proposition 8 give us

$$\psi_i := \sum_{g,k} a^{(i)}_{g,k} g\phi_k, \quad i \in I$$

where $\{a^{(i)}_{g,k}\}_{g,k}$ are the coefficients of the expansion of $A(\cdot)^{-1}[\tilde{H}(\cdot)]_i$ with respect to the orthonormal basis $\{f_g\delta_k : g \in G, k \in \mathbb{N}\}$ of K. □

Definition 11. *Let $\{h_i : i \in I\}$ be a high pass filter set associated with the low pass filter set $\{m_k : k \in \mathbb{N}\}$. We call the operator valued function $H \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ defined by the equation $[H(\gamma)]_i := h_i(\gamma)$ a.e. a high pass filter corresponding to the low pass filter M_0.*

We will complete this subsection with a corollary of theorems 7 and 10 regarding the case where $\{\phi_k : k \in J\}$, is a finite Riesz multiscaling set of vectors for $\{V_j\}_j$. Note that in view of Proposition 10 of [20], a set of Riesz multiwavelet vectors associated with $\{V_j\}_j$ may not even exist, if J is infinite. However, in the next subsection we will prove that a set of frame multiwavelet vectors associated with $\{V_j\}_j$ can always be constructed.

Theorem 12. *Let $\{\phi_k : k \in J\}$, where J is a finite subset of $\mathbb{N}$ be a set of Riesz multiscaling vectors for the GFMRA $\{V_j\}_j$. Then M_0, $\tilde{M}_0 \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$. Let $\tilde{Q}_1$, be the operator-valued function defined on K by the eq. (21). Then $\tilde{Q}_1(\gamma)$ is invertible a.e. and both functions $\gamma \to \left\|\tilde{Q}_1(\gamma)\right\|$ and $\gamma \to \left\|(\tilde{Q}_1(\gamma))^{-1}\right\|$ are essentially bounded.*

Furthermore, $\{\psi_i : i \in I\}$, where $|I| = |J|(n-1)$, is a set of Riesz multiwavelet vectors associated with $\{V_j\}_j$ then $\tilde{H}$ and $\tilde{Q}_2$ defined by equations (19) and (21), respectively, satisfy the following properties:

(a)$\tilde{H} \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$, $\tilde{Q}_2(\gamma)$ are invertible a.e. and the functions $\gamma \to \left\|\tilde{Q}_2(\gamma)\right\|$, $\gamma \to \left\|(\tilde{Q}_2(\gamma))^{-1}\right\|$ are essentially bounded.

(b)

$$\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\gamma)^* \tilde{H}(\gamma_j\gamma) = 0 \quad \text{a.e.}$$

Conversely, if $\tilde{H}$ and $\tilde{Q}_2$ satisfy conditions (a) and (b), then $\{\psi_i : i \in I\}$, where $|I| = |J|\,(n-1)$ and ψ_i are defined by eq. (19), is a set of Riesz multiwavelet vectors associated with $\{V_j\}_j$.

We omit the proof of this theorem because it follows easily from (d) of lemma 2.5 and the arguments in the proofs of theorems 7 and 10. However, it is worthy to stress the difference between condition (b) of the previous corollary and condition (b) of theorems 7 and 10. This difference can be better understood if we use the concepts of wandering vectors and subspaces (see [20]) and of wandering frame collections ([39]).

2.2. **Existence and construction of frame multiwavelet sets of vectors associated with GFMRAs.** Let us now proceed to the proof of the existence of frame multiwavelet sets associated with a GFMRA. This result and its is short, elegant proof were shown to us by Eric Weber (see also [39] for a generalization of the following theorem).

Theorem 13. *For every GFMRA there exists a set of frame multiwavelet vectors associated with it.*

Proof. Let $\{\phi_k : k \in \mathbb{N}\}$ be a frame multiscaling set of vectors associated with the GFMRA $\{V_j\}_j$. Fix $g_0, g_1, \dots, g_{n-1}$ such that $G/\sigma(G) = \{g_i\sigma(G) : i = 0, 1, \dots, n-1\}$. As we have already noted $\{g_i\sigma(g)\tilde{\phi}_k : k \in \mathbb{N}, g \in G, i = 0, 1, \dots, n-1\}$ is a N.T. frame for V_0. Now, we claim that the orthogonal projection $Q : V_0 \to V_{-1}$ belongs to $(\sigma(G) \mid_{V_0})'$. Indeed, since for every $g \in G$ we have $\sigma(g)(V_{-1}) = V_{-1}$ and $\sigma(g)(V_0) = V_0$, it follows that $Q(\sigma(g) \mid_{V_0})Q = (\sigma(g) \mid_{V_0})Q$; since $\sigma(G)$ is a group we obtain that Q belongs to $(\sigma(G) \mid_{V_0})'$, thus $I \mid_{V_0} - Q$ belongs to $(\sigma(G) \mid_{V_0})'$ as well.

But the image of a PF under an orthogonal projection is a PF for the range of the projection. Therefore,

$$\{(I \mid_{V_0} - Q)\sigma(g)g_i\tilde{\phi}_k : g \in G, i = 0, 1, \dots, n-1, k \in \mathbb{N}\} =$$
$$\{\sigma(g)(I \mid_{V_0} - Q)g_i\tilde{\phi}_k = g \in G, i = 0, 1, \dots, n-1, k \in \mathbb{N}\}$$

is a PF for W_{-1}. This finally implies that

$$\{D(I \mid_{V_0} - Q)g_i\tilde{\phi}_k : i = 0, 1, \dots, n-1, k \in \mathbb{N}\}$$

is a frame multiwavelet set vectors associated with $\{V_j\}_j$. □

Remark 3. The preceding proof is simple but does not guarantee the existence of orthonormal MRA wavelets in the special case of a singly generated orthonormal MRAs. Indeed, assume that ϕ is the orthonormal scaling function of an MRA of $L^2(\mathbb{R})$. Then $\{T^n\phi : n \in \mathbb{Z}\}$ is an orthonormal basis for V_0, where T is the translation operator defined by eq. (4) and the dilation operator is given by eq. (3). Following the arguments of the proof of the previous proposition $\{D(I\mid_{V_0} - Q)\phi, D(I\mid_{V_0} - Q)T\phi\}$ is a set of frame multiwavelet vectors associated with this MRA. If both $(I\mid_{V_0} - Q)\phi$ and $(I\mid_{V_0} - Q)T\phi$ are non-zero, then the arguments of the proof of the previous theorem cannot guarantee the existence of orthonormal wavelets associated with MRAs. Let us give an example to illustrate this case. Assume that ϕ is the Haar scaling function. If $(I\mid_{V_0} - Q)\phi = 0$, then ϕ belongs to V_1, which as we know is not true for the Haar MRA. Similarly we prove that $(I\mid_{V_0} - Q)T\phi \neq 0$ as well.

Remark 3 suggests that we have to invent another construction of frame multiwavelet sets of vectors to reduce the redundancy of these sets and consequently the redundancy of the frames these sets produce. Next we will present another algorithm for the construction of frame multiwavelet sets of vectors associated with a given GFMRA. If we apply this second algorithm to the case of singly generated orthonormal MRAs we will get a formula for an orthonormal wavelet associated with the given MRA, in the sense that the integral translations of this wavelet is an *orthonormal* basis for W_0 of the given MRA ([29]).

A very natural question that arises is whether the argument of the proof of theorem 13 can lead to an algorithm for the construction of frame multiwavelet sets of vectors associated with a GFMRA. A careful look at the proof reveals that this can be achieved only if we can write Q with an explicit formula. In fact this is true. We will show next how to produce this formula.

First Algorithm for the construction of high pass filter sets. The key result in the proof of theorem 13 is that Q is in the commutant of $\sigma(G)\mid_{V_0}$. From this fact we can conclude that the projection onto $XY(V_{-1})$, which we denote by $Q_{0,0}$, commutes with $\{M_g \otimes n : g \in G\}$, because $Q_{0,0} = XYQY^*X^*$. Therefore, there exists a projection valued function in $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2 \otimes n))$ implementing $Q_{0,0}$. To avoid introducing extra notation we will denote this projection valued function by $Q_{0,0}$ as well. Therefore $Q_{0,0}(\gamma)$ are a.e. projections whose range we denote by $\mathcal{M}_{0,0}(\gamma)$. Heuristicaly speaking we can say that each $\mathcal{M}_{0,0}(\gamma)$ is the fiber of $XY(V_{-1})$ at γ. From the proof theorem 7 we can obtain that for a.e. $\gamma \in \widehat{G^*}$ the set $\{([\tilde{M}_0(\tilde{\rho}(\gamma))]_k^T, [\tilde{M}_0(\gamma_1\tilde{\rho}(\gamma))]_k^T, \dots, [\tilde{M}_0(\gamma_{n-1}\tilde{\rho}(\gamma))]_k^T) : k \in \mathbb{N}\}$ is a frame for $\mathcal{M}_{0,0}(\gamma)$.

Therefore, if we find a formula for $Q_{0,0}(\gamma)$, then, as it was previously mentioned, we can construct a high pass filter set which will, in turn, define a frame multiwavelet vector set.

Now recall the following fact, which is easy to prove. If H is a Hilbert space and $\{x_r : r \in J\}$ is a PF of a subspace N of H, then the orthogonal projection P_N onto N is given by $P_N = \sum_r x_r \otimes x_r$, where $x \otimes y$ is the rank one operator defined by $x \otimes y(w) = < w, x > y$, $w \in H$. Define

$$\tilde{\tilde{M}}_0(\gamma) := \tilde{M}_0(\gamma)\tilde{Q}_1(\gamma)^{-1/2}P_1(\gamma) \quad \gamma \in \widehat{G^*} \ .$$

and

$$v_k(\gamma) := ([\tilde{\tilde{M}}_0(\tilde{\rho}(\gamma))]_k^T, [\tilde{\tilde{M}}_0(\gamma_1\tilde{\rho}(\gamma))]_k^T, \ldots, [\tilde{\tilde{M}}_0(\gamma_{n-1}\tilde{\rho}(\gamma))]_k^T) \quad k \in \mathbb{N}.$$

Using (c) of lemma 2.5 and eq. (20) is a PF of $\mathcal{M}_{0,0}(\gamma)$, for a.e. γ such that $\mathcal{M}_{0,0}(\gamma) \neq 0$. Therefore,

$$Q_{0,0}(\gamma) = \sum_{k=1}^{\infty} v_k(\gamma) \otimes v_k(\gamma) \ .$$

The preceding argument is significant, because it gives us an explicit formula for $Q_{0,0}$, in terms of the low pass filter $\tilde{M}_0$, and thus ultimately in terms of the given frame multiscaling set of vectors $\{\phi_k : k \in \mathbb{N}\}$.

Let $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}\}$ be the standard orthonormal basis of $\mathbb{C}^n$. Recall that by $\delta_k : k \in \mathbb{N}\}$ we denoted the standard orthonormal basis of ℓ^2. Thus $\{\delta_k \otimes \epsilon_r : k \in \mathbb{N},\ r = 0, 1, \ldots, n-1\}$ is an orthonormal basis of $\ell^2 \otimes n$. According to the proof of theorem 13, the set $\{(XPX^* - Q_{0,0})f_g\delta_k \otimes \epsilon_r : g \in G,\ k \in \mathbb{N},\ r = 0, 1, \ldots, n-1\}$ is a PF of $XY(W_{-1})$, because $\{XPX^*(f_g\delta_k \otimes \epsilon_r) : g \in G,\ k \in \mathbb{N},\ r = 0, 1, \ldots, n-1\}$ is a PF for $XY(V_0)$ and $Q_{0,0}$ is the projection onto $XY(V_{-1})$. But $(M_g \otimes n)(XPX^* - Q_{0,0}) = (XPX^* - Q_{0,0})(M_g \otimes n)$ for every $g \in G$, thus

$$(XPX^* - Q_{0,0})f_g\delta_k \otimes \epsilon_r = f_g\{(XPX^* - Q_{0,0})\delta_k \otimes \epsilon_r\} \ ,$$

where $g \in G$, $k \in \mathbb{N}$ and $r = 0, 1, \ldots, n-1$. Lemma 2.4 implies that $\{h_{k,r} : k \in \mathbb{N},\ r = 0, 1, \ldots, n-1\}$ is a high pass filter set associated with the low pass filter M_0, where

$$h_{k,r} := X^*(\tilde{P}(\tilde{\rho}(\gamma)) - Q_{0,0}(\gamma))\delta_k \otimes \epsilon_r \ , \quad k \in \mathbb{N},\ r = 0, 1, \ldots, n-1.$$

This completes the proof of the first algorithm for the construction of high pass filter sets.

Let us now present another algorithm for the construction of the associated high pass filter set $\{\tilde{h}_i : i \in I\}$, which reduces the redundancy of the resulting frame multiwavelet vector set.

Second Algorithm for the construction of high pass filter sets. We will follow the notation we used in the proof the first algorithm. Set

$$s_{1,0}(\gamma) := \tilde{P}(\tilde{\rho}(\gamma))\delta_1 \otimes \epsilon_0 - Q_{0,0}(\gamma)\delta_1 \otimes \epsilon_0 \ \ \gamma \in \widehat{G^*}$$

and

$$t_{1,0}(\gamma) := \begin{cases} 0 & \text{if} \quad s_{1,0}(\gamma) = 0 \\ \frac{s_{1,0}(\gamma)}{||s_{1,0}(\gamma)||} & \text{if} \quad s_{1,0}(\gamma) \neq 0 \ . \end{cases}$$

Now, let $\mathcal{M}_{1,0}(\gamma) := \mathcal{M}_{0,0}(\gamma) \oplus [t_{1,0}(\gamma)]$ and $Q_{1,0}(\gamma)$ be the orthogonal projection onto $\mathcal{M}_{1,0}(\gamma)$. Therefore,

$$Q_{1,0}(\gamma) = Q_{0,0}(\gamma) + t_{1,0}(\gamma) \otimes t_{1,0}(\gamma) \ .$$

Notice that $s_{1,0}$ and thus $t_{1,0}$ are measurable functions; thus $Q_{1,0}$ is measurable as well and $\mathcal{M}_{1,0}(\gamma)$ is contained in $\tilde{P}(\tilde{\rho}(\gamma))(\ell^2 \otimes n)$, for every γ in $\widehat{G^*}$. Next we continue in the same manner until we exhaust all vectors $\delta_1 \otimes \epsilon_r$. Then we use all $\delta_2 \otimes \epsilon_r$ and so forth. Let us present rigorously the inductive step of our algorithm.

Assume that $\mathbb{N} \times n$ is well-ordered by the lexicographical order. Assume that we have defined $t_{k,r}$ for every $(k,r) \leq (k_0, r_0)$. Now, we want to define $t_{(k_0,r_0)^+}$, where $(k_0, r_0)^+$ is the immediate successor of (k_0, r_0). Let Q_{k_0,r_0} be the orthogonal projection onto $\mathcal{M}_{k_0,r_0} := XY(V_{-1}) \oplus [f_g t_{k,r} : (k,r) \leq (k_0, r_0),\ g \in G]^-$ and $(k_0, r_0)^+ = (k', r')$. Define

$$s_{k',r'}(\gamma) := \tilde{P}(\tilde{\rho}(\gamma))\delta_{k'} \otimes \epsilon_{r'} - Q_{k_0,r_0}(\gamma)\delta_{k'} \otimes \epsilon_{r'} \tag{27}$$

and

$$t_{k',r'}(\gamma) := \begin{cases} 0 & \text{if} \quad s_{k',r'}(\gamma) = 0 \\ \frac{s_{k',r'}(\gamma)}{||s_{k',r'}(\gamma)||} & \text{if} \quad s_{k',r'}(\gamma) \neq 0. \end{cases}$$

Now, define $\mathcal{M}_{k',r'}(\gamma) := \mathcal{M}_{k_0,r_0}(\gamma) \oplus [t_{k',r'}(\gamma)]$, so $Q_{k',r'}(\gamma) = Q_{k_0,r_0}(\gamma) + t_{k',r'}(\gamma) \otimes t_{k',r'}(\gamma)$.

Observe that $\{Q_{k,r} : (k,r) \in \mathbb{N} \times n\}$ is a nest of projections which belong to the Von Neumann algebra $\{M_g \otimes n : g \in G\}'$ and their range is contained in $XY(V_0)$. Recall that the projection onto $XY(V_0)$ is XPX^*, which is a multiplicative operator induced by $\tilde{P} \circ \tilde{\rho}$. We claim that

$$XPX^* = \bigvee_{(k,r)\in\mathbb{N}\times n} Q_{k,r} \ . \tag{28}$$

Note that eq. (27) implies that for every $(k,r) \in \mathbb{N} \times n$ we have that $\tilde{P}(\tilde{\rho}(\gamma))\delta_k \otimes \epsilon_r$ belongs to the range of $Q_{k',r'}(\gamma)$. On the other hand, $\{\tilde{P}(\tilde{\rho}(\gamma))\delta_k \otimes \epsilon_r :\ (k,r) \in \mathbb{N} \times n\}$ is a PF for $\tilde{P}(\tilde{\rho}(\gamma))(\ell^2 \otimes n)$ a.e., which (heuristically speaking) is the fiber of $XY(V_0)$ at γ. Therefore,

$$\tilde{P}(\tilde{\rho}(\gamma)) = \bigvee_{(k,r)\in\mathbb{N}\times n} Q_{k,r}(\gamma) \quad a.e. \tag{29}$$

which implies eq. (28).

Define $\tilde{h}_{k,r} := X^{-1} t_{k,r}$, where $(k,r) \in \mathbb{N} \times n$, and $\tilde{H}$ by the equation

$$[\tilde{H}(\gamma)]_{k,r} := \tilde{h}_{k,r}(\gamma) \quad a.e.$$

The previous equation establishes the measurability of $\tilde{H} : \widehat{G^*} \to \mathcal{B}(\ell^2 \otimes n, \ell^2)$. Also $\tilde{H}$ belongs to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2 \otimes n, \ell^2))$, because for a.e. γ the non zero $t_{k,r}(\gamma)$ form an orthonormal set. Moreover, we have

$$\frac{1}{\sqrt{n}}[\tilde{H}(\gamma_j \tilde{\rho}(\gamma))]_{k,r} = \langle t_{k,r}(\gamma), \epsilon_j \rangle \ \ j = 0, 1, \ldots, n-1.$$

The latter equation combined with eq. (29) establishes that $\tilde{H}$ and $\tilde{\tilde{M}}_0$ and, thus, $\tilde{H}$ and $\tilde{M}_0$ satisfy the completeness part of Condition (b) of theorem 7.

The orthogonality of $t_{k,r}(\gamma)$ to $\mathcal{M}_{0,0}(\gamma)$ a.e. implies

$$\sum_{j=0}^{n-1} \tilde{M}_0(\gamma_j\gamma)^* \tilde{H}(\gamma_j\gamma) = 0 \text{ a.e.}$$

Our next step is to prove that

$$P_2(\gamma) := \frac{1}{n} \sum_{j=0}^{n-1} \tilde{H}_0(\gamma_j\gamma)^* \tilde{H}(\gamma_j\gamma)$$

is a.e. an orthogonal projection of $\ell^2 \otimes n$. The definition of the vectors $t_{k,r}(\gamma)$ shows that these $t_{k,r}(\gamma)$ which are non zero form an orthonormal set a.e. Therefore the non zero columns of

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \tilde{H}(\tilde{\rho}(\gamma)) \\ \tilde{H}(\gamma_1\tilde{\rho}(\gamma)) \\ \vdots \\ \tilde{H}(\gamma_{n-1}\tilde{\rho}(\gamma)) \end{pmatrix}$$

form an orthonormal set a.e. in $\widehat{G^*}$. This gives us that the above matrix defines a partial isometry on $\ell^2 \otimes n$ a.e. Therefore, $P_2(\tilde{\rho}(\gamma))$ is an orthogonal projection.

This completes the proof that the operator-valued function $\tilde{H}$ satisfies condition (a) of theorem 7 as well. Therefore, $\tilde{H}$ is a high pass filter associated with the low pass filter M_0 and by theorem 10 defines a frame (in fact a TF) multiwavelet set of vectors associated with the GFMRA $\{V_j\}_j$.

3. Examples of GFMRAs

Throughout this section we exclusively study examples in one dimension defined with respect to the Affine unitary system. These examples are representative but by no means can be considered exhaustive. In fact, after the completion of the present manuscript some very interesting new classes of multidimensional GFMRAs, induced by radial frame scaling functions were discovered ([30]).

In addition, most set equalities and inclusions are valid modulo null sets.

3.1. Singly Generated GFMRAs of $L^2(\mathbb{R})$.

As we mentioned in the introduction singly generated GFMRAs are defined by frame multiscaling sets containing just one element, so we refer to them as FMRAs. We devote this subsection to show how to apply the techniques we developed so far in order to construct frame multiwavelets associated with FMRAs studied in [9, 10, 23].

Let $\{V_j\}_j$ be an FMRA and ϕ be a frame scaling function for $\{V_j\}_j$, i.e. $\{T^n\phi : n \in \mathbb{Z}\}$ is a frame for V_0. In this case G is isomorphic to $\mathbb{Z}$, thus $\widehat{G^*} = \mathbb{T}$. The Fourier transform on $L^1(\mathbb{T})$ is given by the following equation:

$$\hat{f}(n) = \int_0^1 f(\xi)e^{-2\pi i n\xi}d\xi \ .$$

Moreover, we identified $\mathbb{T}$ with the interval $[-\frac{1}{2}, \frac{1}{2})$, where addition is defined by

$$t_1 + t_2 = \begin{cases} t_1 + t_2 & \text{if} \quad -1/2 \leq t_1 + t_2 < 1/2 \\ t_1 + t_2 + 1 & \text{if} \qquad t_1 + t_2 > 1/2 \\ t_1 + t_2 - 1 & \text{if} \qquad t_1 + t_2 < -1/2 \end{cases} \qquad t_1, t_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right) .$$

Since $\{V_j\}_j$ is a singly generated GFMRA, K will be equal to the space $L^2([-\frac{1}{2}, \frac{1}{2}))$. Moreover,

$$\rho(\xi) = \begin{cases} 2\xi & \text{if} \quad -1/2 \leq 2\xi < 1/2 \\ 2\xi - 1 & \text{if} \qquad 2\xi > 1/2 \\ 2\xi + 1 & \text{if} \qquad 2\xi < -1/2 \end{cases} \qquad \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right) .$$

Therefore, $\ker \rho = \{0, 1/2\}$.

Proposition 4 implies

$$A(\xi)^2 = \sum_n |\hat{\phi}(\xi + n)|^2, \qquad \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right)$$

and $P(\xi) = \chi_E(\xi)$, where $E = \{\xi : A(\xi) \neq 0\}$ and χ_E is the characteristic function of E. Moreover, $A(\xi)$, which can be considered as a linear operator defined on $\mathbb{C}$, is invertible a.e. on E. In fact, (ii) of proposition 4 gives that, there exist $B_1, B_2 > 0$ such that

$$B_1 \leq A(\xi) \leq B_2 \quad \text{a.e. on } E. \tag{30}$$

These inequalities imply that if we wish to find a frame scaling function ϕ, which is not a Riesz scaling function (non-exact frame scaling function), then $\hat{\phi}$ cannot be continuous and ϕ simultaneously have even a mild decay, since this would force its autocorrelation function to be continuous ([3]). This explains why we cannot find examples of non-exact frame scaling functions other than MSF (Minimally Supported in the Frequency), i.e. functions ϕ such that $|\hat{\phi}|$ is the characteristic of a measurable subset of $\mathbb{R}$ ([9, 10, 23]). However, as we will show in the next subsection, a way to solve this particular problem is to use non-singly generated GFMRAs.

Another way to circumvent the same problem is to consider multiresolution structures, which satisfy all the properties of the definition of an FMRA, but one; instead of a frame scaling function, there exists a *refinable or pseudo-scaling* function, say ω, generating V_0, i.e. ω satisfies a 2-scale relation (such as eq. (32) below) and $V_0 = [T^n\omega : n \in \mathbb{Z}]^-$. In this case V_0 is shift-invariant ($T(V_0) = V_0$) and therefore is the core subspace of a GFMRA (see example 2 in the next subsection). Several authors have contributed to the study of these multiresolution structures using various techniques and different hypotheses (e.g. [11, 17, 25, 31, 33, 34, 14]). They all construct tight affine frame wavelets from pseudo-scaling functions. Some of these authors refer to these wavelets as framelets. Motivated by the same problem Li ([24]), unlikely with others, constructs ψ and ψ^* so that they form a pair of Affine dual pseudo-frame wavelets of $L^2(\mathbb{R})$, which only in certain cases becomes a pair of dual (not necessarily canonical) frame wavelets.

Since $D^*\phi$ belongs to V_{-1}, we obtain

$$D^*\phi = \sum_n \langle D^*\phi, T^n\phi'\rangle T^n\phi \tag{31}$$

where $\{T^n\phi' : n \in \mathbb{Z}\}$ is the dual frame corresponding to $\{T^n\phi : n \in \mathbb{Z}\}$. In order to be consistent with the notation and terminology used in the classical multiresolution theory, let us denote by m_0 the 1-periodic function defined by $m_0(\xi) := \sum_n \langle D^*\phi, T^n\phi'\rangle e^{-2\pi in\xi}$ and refer to it as the *low pass filter associated with* ϕ. Note that $\mathrm{supp} m_0$ is contained in E, because $\langle D^*\phi, T^n\phi'\rangle = \langle (S^*S)^{-1}D^*\phi, T^n\phi\rangle$, where S is the Analysis operator corresponding to the frame $\{T^n\phi : n \in \mathbb{Z}\}$. Eq. (31) implies

$$\hat{\phi}(2\xi) = \frac{1}{\sqrt{2}} m_0(\xi)\hat{\phi}(\xi) \quad \text{a.e. in } \mathbb{R}. \tag{32}$$

If $\{T^n\phi : n \in \mathbb{Z}\}$ is a Riesz basis for V_0 then, m_0 is uniquely defined by the 2-scale relation (32). So let us assume that $\{T^n\phi : n \in \mathbb{Z}\}$ is *not* a Riesz basis for V_0 and that μ_0 is another 1-periodic measurable function satisfying the 2-scale relation (32). Let $\gamma \in E \cap \mathcal{S}^c$, where $\mathcal{S}$ is the set of all ξ in E such that eq. (32) is not valid. From the definition of E we obtain that there exists $k \in \mathbb{Z}$ such that $\hat{\phi}(\gamma + k) \neq 0$. Then

$$\hat{\phi}(2(\gamma+k)) = \frac{1}{\sqrt{2}} m_0(\gamma)\hat{\phi}(\gamma+k) = \frac{1}{\sqrt{2}}\mu_0(\gamma)\hat{\phi}(\gamma+k) \ ,$$

which implies that $m_0(\gamma) = \mu_0(\gamma)$. Therefore, we have the following result.

Remark 4. *If m_0 is an 1-periodic measurable function supported on E satisfying eq. (32), then m_0 is the low pass filter associated with ϕ.* These issues naturally arise, because if $\{T^n\phi : n \in \mathbb{Z}\}$ is not an exact frame (Riesz basis), then $\{\langle D^*\phi, T^n\phi'\rangle\}_n$ is not the only sequence in $\ell^2(\mathbb{Z})$ satisfying eq. (31). In this case E is a proper subset of $\mathbb{T}$. On the other hand, for almost every $\xi \in \mathbb{R}$ such that $\hat{\phi}(\xi + k) = 0$ for every integer k and eq. (32) is valid (such a ξ does not belong to E) we also have $\hat{\phi}(2(\xi+k)) = 0$, for every $k \in \mathbb{Z}$. Therefore, it makes no sense to try to define outside E an 1-periodic function satisfying eq. (32). We will revisit this particular issue in the next subsection.

Following the notation introduced in Section 2, we have

$$\tilde{m_0} := YD^*\phi = A(\cdot)m_0 \ . \tag{33}$$

Note that $\operatorname{supp}\tilde{m}_0 = \operatorname{supp} m_0$. Now, consider the multiplicative operator acting on K defined by the following equation

$$\tilde{Q}_1 f(\xi) = \left[|\tilde{m}_0(\xi)|^2 + |\tilde{m}_0(\xi+1/2)|^2\right] f(\xi), \qquad f \in K$$

Let $P_1 = \operatorname{supp}\tilde{Q}_1$. From theorem 7 we obtain that there exist constants $A_1, A_2 > 0$ such that

$$A_1 \leq |\tilde{m}_0(\xi)|^2 + |\tilde{m}_0(\xi+1/2)|^2 \leq A_2 \quad \text{a.e. on } P_1. \tag{34}$$

Combing eqs. (30), (33) and (34) we obtain that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq |m_0(\xi)|^2 + |m_0(\xi+1/2)|^2 \leq C_2 \quad \text{a.e. on } P_1. \tag{35}$$

It is not hard to see that $P_1 \subseteq E \cup \tau_{1/2}(E)$, where $\tau_{1/2}$ is the translation by $1/2$ defined on $[-\frac{1}{2}, \frac{1}{2})$.

Let us now find the high pass filter set associated with m_0. We will apply theorem 10.

Case 1: $P_1 = E \cup \tau_{1/2}(E)$. Set $\tilde{h}_1(\xi) = e^{2\pi i \xi}\overline{\tilde{m}_0(\xi+1/2)}\chi_E(\xi)$. We will show that $\tilde{H} := \tilde{h}_1$ satisfies condition (b) of theorem 10. It is trivial to check that condition (a) of the same theorem is satisfied by $\tilde{H}$ as well. More specifically, we must have that $\{(\tilde{m}_0(\xi), \tilde{m}_0(\xi+1/2)), (\tilde{h}_1(\xi), \tilde{h}_1(\xi+1/2))\}$ spans $\tilde{P}(\xi)(\mathbb{C}^2) = P(\xi)(\mathbb{C}) \oplus P(\xi+1/2)(\mathbb{C})$, a.e. in $[-\frac{1}{2}, \frac{1}{2})$. Recall that P is induced by multiplication by χ_E. Let ξ belong to E but not to $\tau_{1/2}(E)$. Since $P_1 = E \cup \tau_{1/2}(E)$, we have

$$\tilde{m}_0(\xi) \neq 0, \quad P(\xi)(\mathbb{C}) = \mathbb{C}, \qquad P(\xi+1/2)(\mathbb{C}) = 0,$$

and

$$\tilde{h}_1(\xi) = \tilde{h}_1\left(\xi + \frac{1}{2}\right) = 0,$$

a.e. in $E \cap (\tau_{1/2}(E))^c$. Therefore, for all such ξ condition (b) of theorem 10 is satisfied. Similarly we prove that the same is true for a.e. $\xi \in E^c \cap (\tau_{1/2}(E))$.

Finally, for a.e. $\xi \in E \cap (\tau_{1/2}(E))$ the subspace $\tilde{P}(\xi)(\mathbb{C}^2)$ is two dimensional, so it is spanned by the pair of the orthogonal, non zero vectors $\{(\tilde{m}_0(\xi), \tilde{m}_0(\xi + 1/2)), (\tilde{h}_1(\xi), \tilde{h}_1(\xi+1/2))\}$. According to proposition 8

$$h_1(\xi) := A(\xi)^{-1}\tilde{h}_1(\xi)$$

is the high pass filter associated to the low pass filter m_0; so, by eqs. (25) and (26) the frame wavelet ψ associated with $\{V_j\}_j$ is given by the following equation:

$$\hat{\psi}(2\xi) = \frac{1}{\sqrt{2}} h_1(\xi)\hat{\phi}(\xi) = \frac{1}{\sqrt{2}} e^{2\pi i \xi} A(\xi)^{-1} A(\xi+1/2)\overline{m_0(\xi+1/2)}\hat{\phi}(\xi) \quad \text{a.e. in } \mathbb{R}. \tag{36}$$

Recall, that due to its definition $A(\cdot)^{-1}$ vanishes outside E.

As mentioned before if $\{T^n\phi : n \in \mathbb{Z}\}$ is a Riesz basis for V_0, then $E = [-\frac{1}{2}, \frac{1}{2})$; so only the last part of the preceding argument is meaningful, yielding a high pass filter corresponding to a semiorthogonal Riesz wavelet associated with the MRA $\{V_j\}_j$.

Now, let ν be the 1-periodic measurable function such that $\nu(\xi) := A(\xi/2)A(\xi/2+1/2)$ a.e. It is not hard to see that

$$\widehat{\tilde{\psi}}(\xi) := \nu(\xi)\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} e^{\pi i \xi} A(\xi/2+1/2)^2 \overline{m_0(\xi/2+1/2)} \chi_E(\xi/2) \quad \text{a.e. in } \mathbb{R}.$$

is another frame wavelet associated with $\{V_j\}_j$.

Case 2: $P_1 \neq E \cup \tau_{1/2}(E)$. Set $F := E \cap \tau_{1/2}(E) \cap P_1^c$, $F_0 := E \cap P_1^c$, $F_1 := F \cap [-1/4, 1/4)$, $F_2 := F \cap ([-1/2,-1/4) \cup [1/4,1/2))$ and

$$\tilde{h}_1(\xi) := e^{2\pi i \xi} \overline{\tilde{m}_0(\xi+1/2)} \chi_E(\xi) + \chi_{F_0}(\xi) \quad .$$

Next, define

$$\tilde{h}_2 := \chi_{F_2} - \chi_{F_1}$$

and

$$\tilde{H}(\xi) := (\tilde{h}_1(\xi) \ , \ \tilde{h}_2(\xi)) \quad .$$

Therefore,

$$\tilde{Q}_2(\xi) := \tilde{H}(\xi)^* \tilde{H}(\xi) + \tilde{H}(\xi+1/2)^* \tilde{H}(\xi+1/2) =$$

$$\begin{pmatrix} \left|\tilde{h}_1(\xi)\right|^2 + \left|\tilde{h}_1(\xi+1/2)\right|^2 & \overline{\tilde{h}_1(\xi)}\tilde{h}_2(\xi) + \overline{\tilde{h}_1(\xi+1/2)}\tilde{h}_2(\xi+1/2) \\ \overline{\tilde{h}_2(\xi)}\tilde{h}_1(\xi) + \overline{\tilde{h}_2(\xi+1/2)}\tilde{h}_1(\xi+1/2) & \left|\tilde{h}_2(\xi)\right|^2 + \left|\tilde{h}_2(\xi+1/2)\right|^2 \end{pmatrix} .$$

First, note that $\tau_{1/2}(P_1^c) = P_1^c$, because $\tau_{1/2}(P_1) = P_1$, and $\tau_{1/2}$ is injective. The latter property implies $\tau_{1/2}(F_1) = F_2$ and $\tau_{1/2}(F_2) = F_1$, as well. Now, it will be easy for the reader to verify that the following are true:

$$\tilde{h}_1(\xi) = \begin{cases} e^{2\pi i \xi} \overline{\tilde{m}_0(\xi+1/2)} & \text{if} \quad \xi \in P_1 \cap E \cap \tau_{1/2}(E) \\ 1 & \text{if} \quad \xi \in F \end{cases} \tag{37}$$

$$\tilde{h}_1\left(\xi + \frac{1}{2}\right) = \begin{cases} -e^{2\pi i \xi} \overline{\tilde{m}_0(\xi)} & \text{if} \quad \xi \in P_1 \cap E \cap \tau_{1/2}(E) \\ 1 & \text{if} \quad \xi \in F \end{cases} \quad . \tag{38}$$

On the other hand,

$$\tilde{h}_2(\xi) = \begin{cases} 0 & \text{if} \quad \xi \in P_1 \cap E \cap \tau_{1/2}(E) \\ -1 & \text{if} \quad \xi \in F_1 \\ 1 & \text{if} \quad \xi \in F_2 \end{cases} \tag{39}$$

In addition for a.e. ξ in $P_1^c \cap \tau_{1/2}(E)^c \cap E$

$$\tilde{h}_1(\xi) = 1, \quad \tilde{h}_1(\xi+1/2) = 0 \tag{40}$$

while, for a.e. ξ in $P_1^c \cap \tau_{1/2}(E) \cap E^c$

$$\tilde{h}_1(\xi) = 0, \quad \tilde{h}_1(\xi+1/2) = 1.$$

In both cases

$$\tilde{h}_2(\xi) = \tilde{h}_2(\xi + 1/2) = 0. \tag{41}$$

The preceding argument combined with eqs. (37), (38) and (39) gives us

$$\tilde{Q}_2(\xi) = \begin{cases} \begin{pmatrix} |\tilde{m}_0(\xi)|^2 + |\tilde{m}_0(\xi+1/2)|^2 & 0 \\ 0 & 0 \end{pmatrix} & \text{if} \quad \xi \in P_1 \cap E \cap \tau_{1/2}(E) \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{l} \text{if} \quad \xi \in P_1^c \cap \tau_{1/2}(E)^c \cap E \\ \text{or if} \quad \xi \in P_1^c \cap \tau_{1/2}(E) \cap E^c \end{array} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \text{if} \quad \xi \in F. \end{cases}$$

If $P_2(\xi)$ is the range projection of $\tilde{Q}_2(\xi)$, it becomes apparent that $\tilde{Q}_2(\xi) \mid_{P_2(\xi)(\mathbb{C}^2)}$ is invertible for a.e. ξ such that $P_2(\xi) \neq 0$. Thus $\tilde{H}$ satisfies condition (a) of theorem 7.

Let us now verify that $\tilde{H}$ satisfies condition (b) of theorem 7. First, it is not hard for someone to verify that

$$\overline{\tilde{m}_0(\xi)}(\tilde{h}_1(\xi), \tilde{h}_2(\xi)) + \overline{\tilde{m}_0(\xi+1/2)}(\tilde{h}_1(\xi+1/2), \tilde{h}_2(\xi+1/2)) = (0,0) \ \text{ a.e.}$$

Finally, *we claim* that

$$\{(\tilde{m}_0(\xi), \tilde{m}_0(\xi+1/2)), (\tilde{h}_1(\xi), \tilde{h}_1(\xi+1/2)), (\tilde{h}_2(\xi), \tilde{h}_2(\xi+1/2))\}$$

spans $\tilde{P}(\xi)(\mathbb{C}^2) = P(\xi)(\mathbb{C}) \oplus P(\xi+1/2)(\mathbb{C})$, a.e. in $[-\frac{1}{2}, \frac{1}{2})$. First, note that

$$\tilde{P}(\xi)(\mathbb{C}^2) = \begin{cases} \mathbb{C} \oplus \mathbb{C} & \text{if } \xi \in E \cap \tau_{1/2}(E) \\ \mathbb{C} \oplus 0 & \text{if } \xi \in E \cap \tau_{1/2}(E)^c \\ 0 \oplus \mathbb{C} & \text{if } \xi \in E^c \cap \tau_{1/2}(E) \\ 0 \oplus 0 & \text{if } \xi \in E^c \cap \tau_{1/2}(E)^c \end{cases}$$

We will prove our claim in the first and second cases. The proof of the claim in the third case is similar to this in the second and, obviously, there is nothing to prove in the last case. Eqs. (37), (38) and (39) imply that for a.e. ξ in $P_1 \cap E \cap \tau_{1/2}(E)$ $(\tilde{m}_0(\xi), \tilde{m}_0(\xi+1/2))$ and $(\tilde{h}_1(\xi), \tilde{h}_1(\xi+1/2))$ are two orthogonal non-zero vectors spanning $\tilde{P}(\xi)(\mathbb{C}^2)$, while for a.e. ξ in $P_1^c \cap E \cap \tau_{1/2}(E)$ $(\tilde{h}_1(\xi), \tilde{h}_1(\xi+1/2))$ and $(\tilde{h}_2(\xi), \tilde{h}_2(\xi+1/2))$ are two orthogonal non-zero vectors spanning $\tilde{P}(\xi)(\mathbb{C}^2)$.

For a.e. ξ in $P_1 \cap E \cap \tau_{1/2}(E)^c$, $\tilde{m}_0(\xi) \neq 0$ and $\tilde{m}_0(\xi+1/2) = 0$, because $\tau_{1/2}(\xi)$ does not belong to E. In this case $(\tilde{m}_0(\xi), \tilde{m}_0(\xi+1/2))$ spans the one-dimensional $\tilde{P}(\xi)(\mathbb{C}^2)$. On the other hand, eqs. (40) and (41) imply that $(\tilde{h}_1(\xi), \tilde{h}_1(\xi+1/2))$ spans $\tilde{P}(\xi)(\mathbb{C}^2)$ for a.e. ξ in $P_1^c \cap E \cap \tau_{1/2}(E)^c$. This argument completes the proof of the claim.

For every MRA of $L^2(\mathbb{R})$ condition $P_1 = E \cup \tau_{1/2}(E)$ is always satisfied. If we take $\hat{\phi} = \chi_{[-1/4,1/4)}$, then ϕ is a frame scaling function. It is not hard to verify $E = [-1/4, 1/4)$. Thus $\tau_{1/2}(E) = [-1/2, -1/4) \cup [1/4, 1/2)$. Note that $m_0 = \tilde{m}_0 =$

$\sqrt{2}\chi_{[-1/8,1/8)}$. Therefore, $P_1 = [-1/2,-3/8) \cup [-1/8,1/8) \cup [3/8,1/2)$, so $P_1 \neq E \cup \tau_{1/2}(E)$. This example has been taken from [10].

Another class of singly generated GFMRAs has been studied by Seleznik and Sendur ([36, 35]). Their result introduces a rich and interesting class of GFMRAs with a significant potential for Signal Processing applications. GFMRAs in this class are generated by a single scaling function, but a frame multiwavelet set with two elements is associated with them. The conditions that lead to the construction of this frame multiwavelet set can be obtained directly from theorem 10. The reader can also find some interesting Image processing applications of these GFMRAs in [36, 35].

3.2. **Non-singly generated GFMRAs of $L^2(\mathbb{R})$ with respect to the Affine system.** Our first example is related with the Journè wavelet (see [21, 26]). This was the first example of an orthonormal wavelet not associated with an MRA that was discovered. Before proceeding we need the following lemma.

Lemma 3.1. ([28]) *Let $I \subseteq \mathbb{N}$ and $\{\phi_k : k \in I\}$ be a subset of $L^2(\mathbb{R})$. Define*

$$a_{l,k}(\xi) := \sum_{m\in\mathbb{Z}} \hat{\phi}_k(\xi+m)\overline{\hat{\phi}_l(\xi+m)} \quad k,l \in I, \ \ \xi \in \left[-\frac{1}{2},\frac{1}{2}\right)$$

and $a_k(\xi) := (a_{1,k}(\xi), a_{2,k}(\xi), \ldots)$, $\xi \in \left[-\frac{1}{2},\frac{1}{2}\right)$.
(a) Assume that for every $k \in I$ the function $\xi \to \|a_k(\xi)\|_{\ell^2}$ is in $L^2([-\frac{1}{2},\frac{1}{2}))$ and that the linear operators $\Phi(\xi)$ defined for a.e. $\xi \in [-\frac{1}{2},\frac{1}{2})$ on $[\delta_k : k \in I]$ by the equation $\Phi(\xi)\delta_k = a_k(\xi)$ satisfy the following properties:

1. *Φ belongs to $L^\infty([-\frac{1}{2},\frac{1}{2}), \mathcal{B}(\ell^2(I)))$.*
2. *Let $P(\xi)$ be the range projection of $\Phi(\xi)$ a.e. There exists $B > 0$ such that for every $x \in P(\xi)(\ell^2(I))$ we have that $B\|x\| \le \|\Phi(\xi)x\|$.*

Then $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ is a frame of its closed linear span with frame bounds B and $\|\Phi\|_\infty$.

Conversely, if $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ is a frame of its closed linear span with frame constants B, C, then there exists $\Phi \in L^\infty([-\frac{1}{2},\frac{1}{2}), \mathcal{B}(\ell^2(I)))$ such that $\|\Phi\|_\infty \le C$ satisfying

$$\Phi(\xi)_{l,k} = \sum_{m\in\mathbb{Z}} \hat{\phi}_k(\xi+m)\overline{\hat{\phi}_l(\xi+m)} \quad k,l \in I, \ \ \xi \in \left[-\frac{1}{2},\frac{1}{2}\right)$$

and property (2). Moreover, if P is the orthogonal projection onto $[T^n\phi_k : k \in I, n \in \mathbb{Z}]^-$, then

$$P\omega(\xi) = P(\xi)\omega(\xi), \quad \omega \in L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right), \ell^2(I)\right).$$

Finally, $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ is a PF of its closed linear span if and only if $\Phi(\xi)$ is an orthogonal projection a.e.

If $\{\phi_k : k \in I\}$ is a frame multiscaling set, then $\Phi = A^2$.

Example 1. Define ϕ_1, ϕ_2 and ϕ_3 such that $\hat{\phi}_1 := \chi_E$, where $E := [-4/7, -1/2) \cup [-2/7, -1/7) \cup [1/7, 2/7) \cup [1/2, 4/7)$, $\hat{\phi}_2 = \chi_{[-8/7,-1)\cup[1,8/7)}$ and $\hat{\phi}_3 = \chi_{[-1/7,1/7)}$. The integral translates of ϕ_1, ϕ_2 and ϕ_3 form a PF for their closed linear span, which we denote by V_0. Indeed, it is not hard to verify that

$$\Phi(\xi) = A(\xi)^2 = \begin{cases} \delta_1 \otimes \delta_1 & \text{if} \quad \xi \in [-\frac{1}{2}, -\frac{3}{7}) \cup [-\frac{2}{7}, -\frac{1}{7}) \\ & \qquad \cup [\frac{1}{7}, \frac{2}{7}) \cup [\frac{3}{7}, \frac{1}{2}) \\ 0 & \text{if} \quad \xi \in [-\frac{3}{7}, -\frac{2}{7}) \cup [\frac{2}{7}, \frac{3}{7}) \\ \delta_1 \otimes \delta_1 \oplus \delta_2 \otimes \delta_2 & \text{if} \quad \xi \in [-\frac{1}{7}, \frac{1}{7}) \end{cases}$$

Note that, for every $\xi \in [-\frac{1}{2}, \frac{1}{2})$, $\Phi(\xi)$ is an orthogonal projection, thus, according to the previous lemma $\{T^n\phi_k : k = 1, 2, 3; n \in \mathbb{Z}\}$ is a PF of its closed linear span, which we will denote by V_0. Moreover,

$$\begin{aligned} \hat{\phi}_1(2\xi) &= \frac{1}{\sqrt{2}}[\sqrt{2}\chi_{[-\frac{2}{7},-\frac{1}{4})\cup[\frac{1}{4},\frac{2}{7})}(\xi)\hat{\phi}_1(\xi) + \sqrt{2}\chi_{[-\frac{1}{7},-\frac{1}{14})\cup[\frac{1}{14},\frac{1}{7})}(\xi)\hat{\phi}_3(\xi)] \quad \text{a.e.} \\ \hat{\phi}_2(2\xi) &= \frac{1}{\sqrt{2}}[\sqrt{2}\chi_{[-\frac{4}{7},-\frac{1}{2})\cup[\frac{1}{2},\frac{4}{7})}(\xi)\hat{\phi}_1(\xi)] \quad \text{a.e.} \\ \hat{\phi}_3(2\xi) &= \frac{1}{\sqrt{2}}[\sqrt{2}\chi_{[-\frac{1}{14},\frac{1}{14})}(\xi)\hat{\phi}_3(\xi)] \quad \text{a.e.} \end{aligned}$$

The previous equations imply that for $k = 1, 2, 3$ each $D^*\phi_k$ belongs to V_0, therefore $V_j := D^j(V_0)$, where $j \in \mathbb{Z}$, is an increasing chain of closed subspaces of $L^2(\mathbb{R})$. Since the $\operatorname{supp}\hat{\phi}_3$ contains a neighborhood of the origin, we have that $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$. On the other hand it is not hard to check that the intersection of all V_j is trivial. Therefore $\{V_j\}_j$ is a GFMRA. In this case $M_0 = \tilde{M}_0$. For notational convenience set $e_n(\xi) := e^{-2\pi i n\xi}$. The following argument generalizes remark 4. Assume that $\mathcal{L}$ is a measurable function defined on $[-\frac{1}{2}, \frac{1}{2})$ whose values are 3×3 matrices such that $\mathcal{L}_{p,q}$ belongs to $L^2([-\frac{1}{2}, \frac{1}{2}))$ for every $p, q = 1, 2, 3$, satisfying

$$\hat{\phi}_p(2\xi) = \frac{1}{\sqrt{2}} \sum_{q=1}^{3} \mathcal{L}_{q,p}(\xi)\hat{\phi}_q(\xi) \tag{42}$$

Let $\mathcal{J} : L^2([-\frac{1}{2}, \frac{1}{2}), \mathbb{C}^3)$ be the preframe operator defined by

$$\mathcal{J}(e_n\delta_k) = T^n\phi_k \qquad n \in \mathbb{Z}, \quad k = 1, 2, 3,$$

where $\{\delta_k : k = 1, 2, 3\}$ is the standard orthonormal basis of $\mathbb{C}^3$. Note that $\mathcal{J}^*$ is the analysis operator corresponding to the PF $\{T^n\phi_p : p = 1, 2, 3; n \in \mathbb{Z}\}$. Set $\mu_p = (\mathcal{L}_{1,p}, \mathcal{L}_{2,p}, \mathcal{L}_{3,p})$, where $p = 1, 2, 3$. Using eq. (42) and the definition of $\mathcal{J}$ it easy to see $\mathcal{J}(m_p) = \mathcal{J}(\mu_p)$, for every $p = 1, 2, 3$. Therefore, $m_p - \mu_p$ belongs to

$\mathrm{Ker}\mathcal{J} = \mathcal{R}(\mathcal{J}^*)^{\perp}$, for every p. Since the projection onto $\mathcal{R}(\mathcal{J}^*)$ is induced by the projection-valued function $P \in L^{\infty}([-\frac{1}{2}, \frac{1}{2}), \mathbb{C}^{3\times 3})$, we obtain

$$m_p(\xi) = P(\xi)\mu_p(\xi) \qquad \text{a.e.} \tag{43}$$

Now, eqs. (42) and (43) imply

$$M_0(\xi) = \sqrt{2}\begin{pmatrix} \chi_{[-\frac{2}{7},-\frac{1}{4})\cup[\frac{1}{4},\frac{2}{7})}(\xi) & \chi_{[-\frac{1}{2},-\frac{3}{7})\cup[\frac{3}{7},\frac{1}{2})}(\xi) & 0 \\ 0 & 0 & 0 \\ \chi_{[-\frac{1}{7},-\frac{1}{14})\cup[\frac{1}{14},\frac{1}{7})}(\xi) & 0 & \chi_{[-\frac{1}{14},\frac{1}{14})}(\xi) \end{pmatrix}, \quad \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Next, we have to compute the projection-valued function $\tilde{P}$ and apply the second algorithm for the construction of a high pass filter corresponding to M_0. This construction gives the following high pass filter

$$\tilde{h}(\xi) := \sqrt{2}\begin{cases} (0,1,0) & \text{if} \quad |\xi| < \frac{1}{7} \\ (1,0,0) & \text{if} \quad \frac{1}{7} \leq |\xi| < \frac{1}{4} \\ (0,0,0) & \text{elsewhere in } [-\frac{1}{2}, \frac{1}{2}). \end{cases}$$

If $\tilde{h}_k$, where $k = 1, 2, 3$ are the coordinate functions of $\tilde{h}$, then we obtain

$$\hat{\psi}(2\xi) = \frac{1}{\sqrt{2}}(\tilde{h}_1(\xi)\hat{\phi}_1(\xi) + \tilde{h}_2(\xi)\hat{\phi}_2(\xi) + \tilde{h}_3(\xi)\hat{\phi}_3(\xi)) \quad \text{a.e.}$$

which implies

$$\hat{\psi} = \chi_{[-\frac{16}{7},-2)\cup[-\frac{1}{2},-\frac{2}{7})\cup[\frac{2}{7},\frac{1}{2})\cup[-2,\frac{16}{7})}.$$

This particular function ψ is known as the Journè wavelet. The preceding argument implies that ψ is a semiorthogonal frame wavelet of $L^2(\mathbb{R})$. We can prove that ψ is an orthonormal wavelet by verifying that $\sum_{\ell\in\mathbb{Z}} \left|\hat{\psi}(\xi+\ell)\right|^2 = 1$ a.e. For more details on certain of the arguments used in this example the reader may refer to [29]. The multiscaling set $\{\phi_i : i = 1, 2, 3\}$ appears first in [26], where it was constructed from the Journè wavelet using the techniques of the proof of Theorem 4 in [26]. A completely different MRA-construction of the Journè wavelet can be found in Section 6 of [15].

Example 2. Let $\phi_0 \in L^2(\mathbb{R})$ be a function with the following properties: $\hat{\phi}_0$ is continuous, $\hat{\phi}_0(\xi) = 0$ for every ξ such that $|\xi| \geq b$, where $0 < b < 1/4$ and $0 < \hat{\phi}_0(\xi) \leq 1$ for every $\xi \in (-b, b)$. Moreover let $c \in (b/2, b)$. We assume that $\hat{\phi}_0(\xi) = 1$ for every $\xi \in [-c, c]$. Set V_0 to be the closed linear span of $\{T^n\phi_0 : n \in \mathbb{Z}\}$ and $V_j := D^j(V_0)$, $j \in \mathbb{Z}$. Then, it is not hard for someone to verify that $\mathcal{F}(V_0) = L^2([-b, b))$, where $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R})$. Therefore, for every $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$, $\cap_j V_j$ is trivial and $\cup_j V_j$ is dense in $L^2(\mathbb{R})$. Corollary 3.1 implies that $\{T^n\phi_0 : n \in \mathbb{Z}\}$ is *not* a frame of V_0. In the example following Theorem 7 of [28] we construct another function, say ω, such that $\{T^n\phi_0, T^n\omega : n \in \mathbb{Z}\}$ is a PF of V_0. Thus

$\{V_j\}_j$ is a GFMRA of $L^2(\mathbb{R})$. However, $\hat{\omega}$ is discontinuous at $\pm b$. In fact even more is true.

Claim: *There is no finite frame multiscaling set* $\{\omega_i : i = 1, 2, \ldots, k\}$ *for* $\{V_j\}_j$, *such that each* $\hat{\omega}_i$ *is continuous.*

Proof of the claim: Let $\{\omega_i : i = 1, 2, \ldots, k\}$ be a finite subset of V_0. Since $\mathcal{F}(V_0) = L^2([-b, b))$, we obtain that for every $i = 1, 2, \ldots, k$ $\operatorname{supp}\hat{\omega}_i$ is contained in $[-b, b)$. Therefore, every entry

$$\Phi(\xi)_{l_1,l_2} := \sum_{m\in\mathbb{Z}} \hat{\omega}_{l_2}(\xi + m)\overline{\hat{\omega}_{l_1}(\xi + m)} \quad 1 \leq l_1, l_2 \leq k$$

of the autocorrelation function Φ corresponding to $\{\omega_i : i = 1, 2, \ldots, k\}$ is a continuous function, thus $\xi \to \|\Phi(\xi)\|$ is a continuous function supported on $[-b, b]$. This, in turn, implies that Φ fails to satisfy condition (2) of (a) of lemma 3.1, equivalently $\{T^n\omega_i : i = 1, 2, \ldots, k; n \in \mathbb{Z}\}$ cannot be a frame for V_0. This completes the proof of the claim.

Assume that $\{a_k\}_{k\in\mathbb{N}}$ is a sequence such that $c < a_k < b$ and $a_k < a_{k+1}$ for every $k \in \mathbb{N}$. Moreover assume that $\lim a_k = b$. Set

$$d_k := a_{k+1} - a_k, \quad b_k := a_k + \frac{d_k}{4} \quad \text{and} \quad c_k := a_k + \frac{3d_k}{4}.$$

Clearly $a_k < b_k < c_k < a_{k+1}$ for every $k \in \mathbb{N}$.

Assume that we have a set of functions $\{\phi_k : k \in \mathbb{N}\}$ satisfying the following properties:

I. $\hat{\phi}_k$ is continuous, for every $k \geq 1$. Moreover, $\hat{\phi}_1$ vanishes outside $[-b_2, -a_1) \cup [a_1, b_2)$. Also every $\hat{\phi}_k$ vanishes outside $[-b_{k+1}, -c_{k-1}) \cup [c_{k-1}, b_{k+1})$ for $k \geq 2$.

II. If $a_1 < |\xi| \leq b_1$ or $a_2 < |\xi| \leq b_2$, then

$$0 < \hat{\phi}_1(\xi) \leq 1.$$

III. For every $k \geq 2$

$$0 < \hat{\phi}_k(\xi) \leq 1$$

if $c_{k-1} < |\xi| \leq a_k$ or $a_{k+1} < |\xi| \leq b_{k+1}$.

IV. $\hat{\phi}_k(\xi) = 1$ if $a_k \leq |\xi| \leq a_{k+1}$, for every $k \geq 2$. Also, $\hat{\phi}_1(\xi) = 1$ if $b_1 \leq |\xi| \leq a_2$.

Obviously, $[T^n\phi_k : k = 0, 1, 2, \ldots;\ n \in \mathbb{N}]^-$ is to equal to V_0. Therefore, if we want to prove that $\{\phi_k : k = 0, 1, 2, \ldots\}$ is a frame multiscaling set of functions for the GFMRA $\{V_j\}_j$, we only need to establish that $\{T^n\phi_k : k = 0, 1, 2, \ldots;\ n \in \mathbb{N}\}$ is a frame of V_0.

According to corollary 3.1 the autocorrelation function Φ corresponding to the set $\{\phi_k : k = 0, 1, 2, \ldots\}$ is given by following equation:

$$\Phi(\xi)_{l,k} = \hat{\phi}_k(\xi)\overline{\hat{\phi}_l(\xi)} \ , \quad k, l \geq 0. \tag{44}$$

Therefore,

$$\Phi(\xi) = v(\xi) \otimes v(\xi), \quad \text{a.e.} \tag{45}$$

where, $v(\xi) := (\hat{\phi}_0(\xi), \hat{\phi}_1(\xi), \hat{\phi}_2(\xi), \dots)$ a.e. in $[-1/2, 1/2)$. Apparently, if $b \le |\xi| < 1/2$, then $\Phi(\xi) = 0$. Note, that for every $\xi \in [-1/2, 1/2)$ at most three of the coordinates of $v(\xi)$ are non zero.

In order to prove that $\{T^n \phi_k : k = 0, 1, 2, \dots;\ n \in \mathbb{N}\}$ is a frame of V_0, first, we must show that a.e. in $[-1/2, 1/2)$ $\Phi(\xi)$ is a well-defined, bounded operator and that $\{\|\Phi(\xi)\| : \xi \in [-1/2, 1/2)\}$ is essentially bounded. From eq. (45) and the fact that all $\hat{\phi}_k$ are all bounded by 1 we can very easily see that $\Phi(\xi)$ is a well-defined, bounded operator, for every $\xi \in [-1/2, 1/2)$ and that $\|\Phi(\xi)\| \le 3$. Second, we will establish that there exists $B > 0$ such that for a.e. $\xi \in [-b, b)$ and for every x in $P(\xi)(\ell^2)$, where $P(\xi)$ is the range projection of $\Phi(\xi)$, we have that $B\,\|x\| \le \|\Phi(\xi)x\|$. For every ξ such that $|\xi| < b$, $P(\xi)$ is the projection onto the one-dimensional subspace $[v(\xi)]$. Thus for every x in $[v(\xi)]$ we have that $\|\Phi(\xi)x\| = \|v(\xi)\|^2 \|x\|$, for every $\xi \in (-b, b)$. But for all such ξ we have that $\|v(\xi)\| \ge \kappa$, where $\kappa := \min\{\left|\hat{\phi}(\xi)\right| : c < |\xi| < b_1\}$. Since $\kappa > 0$, we obtain that $\{\phi_k : k = 0, 1, 2, \dots\}$ is indeed a multiscaling set of functions for $\{V_j\}_j$.

Using the arguments proving eq. (31) we obtain

$$D^*\phi_k = \sum_n \langle D^*\phi_k, T^n \phi_l' \rangle T^n \phi_l \quad k = 0, 1, 2, \dots$$

where $\{T^n \phi_l' : n \in \mathbb{Z}\ ,\ k = 0, 1, 2, \dots\}$ is the dual frame corresponding to $\{T^n \phi_l : n \in \mathbb{Z}\ ,\ k = 0, 1, 2, \dots\}$. Let $m_{l,k}$ be the 1-periodic function defined by

$$m_{l,k}(\xi) := \sum_n \langle D^*\phi_k, T^n \phi_l' \rangle e^{-2\pi i n \xi} \quad l, k \ge 0$$

and $m_k := (m_{0,k}, m_{1,k}, \dots)$. Once again, in order to be consistent with the notation and terminology of classical MRA theory, we refer to the set $\{m_k : k \ge 0\}$ as the *low pass filter associated with* the multiscaling set $\{\phi_k : k \ge 0\}$. Apparently, each m_k is square-integrable, so we can define $\tilde{m}_k := A m_k$ and subsequently derive the normalized low pass filter $\tilde{M}_0$ corresponding to $\{\phi_k : k \ge 0\}$. Let μ_k be the 1-periodic functions satisfying

$$m_k(\xi) := \sqrt{2}(\hat{\phi}_k(2\xi), 0, 0, \dots) \quad \text{a.e. in} \quad \left[-\frac{1}{2}, \frac{1}{2}\right) \quad k \ge 0$$

and $\mu_{l,k}$ $(l \ge 0)$ be the coordinate functions of each μ_k. Then

$$\hat{\phi}_k(2\xi) = \frac{1}{\sqrt{2}} \sum_{l=0}^{\infty} \mu_{l,k}(\xi) \hat{\phi}_l(\xi) \quad \text{a.e. in } \mathbb{R}.$$

Apparently $\text{supp}\mu_k \cap [-1/2, 1/2)$ is contained in $(-b/2, b/2)$ for every $k \geq 0$. Arguments similar to those proving eq. (43) give us

$$m_k(\xi) = P(\xi)\mu_k(\xi) \quad \text{a.e.}$$

On the other hand, for every $\xi \in (-b/2, b/2)$ we have $A(\xi) = \delta_0 \otimes \delta_0$, because $b/2 < c$. Combining, the last two equations we obtain $\tilde{m}_k = \mu_k$, for every $k \geq 0$.

Let us now define the $\infty \times 1$ matrix-valued function $\tilde{H}$:

$$\tilde{H}(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| > b, \quad or \quad |\xi| \leq b/2 \\ \delta_0 \otimes \delta_0 & \text{if} \quad b/2 \leq |\xi| \leq c \\ v(\xi)^T & \text{if} \quad c \leq |\xi| \leq b \end{cases}$$

where T denotes the transpose operation. It is not very hard for the reader to prove that $\tilde{H}$ is in $L^\infty([-1/2, 1/2), \mathcal{B}(\ell^2))$ and the following are true:

(1) For $\xi \in [-1/2, 1/2)$

$$\tilde{Q}_2(\xi) := \begin{cases} \delta_0 \otimes \delta_0 & \text{if} \quad \xi \in \{\frac{b}{2} \leq |\xi| \leq c\} \cup \tau_{1/2}(\{\frac{b}{2} \leq |\xi| \leq c\}) \\ \sum_{k=0}^{\infty} \hat{\phi}_k(\xi)^2 & \text{if} \quad \xi \in \{c < |\xi| < b\} \\ \sum_{k=0}^{\infty} [\hat{\phi}_k(\tau_{1/2}(\xi))]^2 & \text{if} \quad \xi \in \tau_{1/2}(\{c < |\xi| < b\}) \\ 0 & \quad \text{otherwise} \end{cases}$$

where $\tilde{Q}_2(\xi) := \tilde{H}(\xi)^*\tilde{H}(\xi) + \tilde{H}(\xi + 1/2)^*\tilde{H}(\xi + 1/2)$.

(2) $\tilde{H}(\xi)^*\tilde{M}_0(\xi) + \tilde{H}(\xi + 1/2)^*\tilde{M}_0(\xi + 1/2) = 0$ a.e.

(3) The columns of the matrix

$$\begin{pmatrix} \tilde{M}_0(\xi) & \tilde{H}(\xi) \\ \tilde{M}_0(\xi + \frac{1}{2}) & \tilde{H}(\xi + \frac{1}{2}) \end{pmatrix}$$

span $P(\xi)(\ell^2) \oplus P(\tau_{1/2}(\xi))(\ell^2)$ a.e. in $[-1/2, 1/2)$.

Now, according to theorem 10, we obtain the frame wavelet ψ associated with the GFMRA $\{V_j\}_j$ defined by the equation

$$\hat{\psi}(2\xi) = \sum_{l=0}^{\infty} h_l(\xi)\hat{\phi}_l(\xi) \quad \text{a.e. in } \mathbb{R},$$

where h_l are the coordinate functions of h and $h(\xi) = A(\xi)^{-1}\tilde{H}(\xi)^T$. Note that since $A(\xi)^2 = v(\xi) \otimes v(\xi)$ we have that $A(\xi) = \frac{1}{\|v(\xi)\|} v(\xi) \otimes v(\xi)$ for every ξ such that $v(\xi) \neq 0$ and $A(\xi) = 0$ elsewhere. This gives

$$A(\xi)^{-1}(x) = \frac{1}{\|v(\xi)\|} x \quad \text{for every} \;\; x \in [v(\xi)].$$

So

$$h(\xi) = \begin{cases} 0 & \text{if} \quad b < |\xi| < \frac{1}{2}, \quad \text{or} \quad |\xi| \leq b/2 \\ \delta_0 & \text{if} \quad b/2 \leq |\xi| \leq c \\ \frac{1}{\|v(\xi)\|} v(\xi) & \text{if} \quad c \leq |\xi| \leq b \end{cases}$$

Therefore,

$$\hat{\psi}(\xi) = \chi_{[-2c,-b)\cup[b,2c)}(\xi) + \chi_{[-2b,-2c)\cup[2c,2b)}(\xi)\left[\sum_{k=0}^{\infty} \hat{\phi}_k\left(\frac{\xi}{2}\right)^2\right]^{1/2} \quad \text{a.e. in } \mathbb{R}.$$

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References

[1] A. Aldroubi, *Portraits of frames*, Proc. Amer. Math. Soc. **123** (1995), pp. 1661-1668.
[2] A. Aldroubi, *Oblique and hierarchical multiwavelet bases*, Applied Comp. Harmonic Anal. **4** (1997), pp. 231-263.
[3] A. Aldroubi, personal communication.
[4] A. Aldroubi, M. Papadakis, *Characterization and parameterization of multiwavelet bases*, Contemp. Mathem. **216**, (1998), pp. 97-116.
[5] L. Baggett, A. Carey, W. Moran, P. Ohring, *General existence theorems for orthonormal wavelets, an abstract approach*, Publ. Inst. Math. Sci. Kyoto Univ. **31** (1995), pp. 95-111.
[6] L.W. Baggett, H.A. Medina, K.D. Merrill, *Generalized multiresolution analyses and a construction procedure for all wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Applic. **5**, pp. 563-573, 1999.
[7] L.W. Baggett, K.D. Merrill, *Abstract harmonic analysis and wavelets in* $\mathbb{R}^n$, Cont. Math. **247** (2000), pp. 17-28.
[8] J.J. Benedetto, M. Frazier (Eds.) *Wavelets: Mathematics and Applications*, CRC Press, Boca Raton, FL 1994.
[9] J.J. Benedetto, S. Li, *The theory of multiresolution analysis frames and applications to filter banks*, Appl. Comp. Harm. Anal., **5** (1998), pp. 389-427.
[10] J.J. Benedetto, O.M. Treiber, *Wavelet frames: Multiresolution analysis and the unitary extension principle*, in *Wavelet Transforms and Time-Frequency Signal Analysis*, L. Debnath (Ed.), Birkhauser, Boston MA, pp. 3-36, 2001.
[11] C. de Boor, R. DeVore, A. Ron, *Approximation from shift-invariant subspaces of* $L^2(\mathbb{R}^d)$, Trans. Amer. Math. Soc., **341** (1994), pp. 787-806.
[12] M. Bownik, *The structure of shift invariant subspaces of* $L^2(\mathbb{R}^n)$, J. Funct. Anal, **177**, pp. 282-309, 2000.
[13] P. Casazza, *The art of frame theory*, Taiwanesse J. Math. **4** (2000), pp.129-201.
[14] , C. Chui, W. He, J. Stöckler, Q. Sun, *Compactly supported tight affine frames with integer dilations and maximal vanishing moments*, to appear Advances Comput. Mathem. special issue on Frames, 2002.
[15] J. Courter, *Construction of dilation-d wavelets*, Cont. Math. **247** (1999), pp. 183-205.
[16] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs Amer. Math. Soc., Vol. **134**, nr. 640, 1998.
[17] I. Daubechies, *Framelets: MRA-based constructions of wavelet frames*, preprint, 2001.
[18] P.A. Fillmore, *A user's guide to operator algebras*, John Wiley & sons, Inc. 1996.
[19] D. Han, D.R. Larson, *Frames, bases and group representations*, Memoirs Amer. Math. Soc. **147** nr. 697, 2000.

[20] D. Han, D.R. Larson, M. Papadakis, Th. Stavropoulos *Multiresolution analysis of abstract Hilbert spaces and wandering subspaces*, Cont. Math. **247** (1999), pp. 259-284.

[21] E. Hernández, G. Weiss, *A first course on wavelets*, CRC Press, Boca Raton FL, 1996.

[22] R.V. Kadison, J.R. Ringrose, *Fundamentals of the theory of operator algebras*, Vol. II, GSM Vol. **16**, American Mathematical Society, 1997.

[23] H.O. Kim, J.K. Lim, *On frame wavelets associated with frame multiresolution analysis*, Appl. Comp. Harm. Anal. **10** (2001), pp. 61-70.

[24] S. Li, *A theory of generalized multiresolution structure and pseudoframes of translates*, J. Fourier Anal. Appl. **7** (2001), pp. 23-40.

[25] M. Paluszyński, H. Šikić, G. Weiss, S. Xiao, *Generalized low pass filters and MRA frame wavelets*, J. Geom. Anal. **11**, pp. 311-342, 2001.

[26] M. Papadakis, *On the Dimension function of orthonormal wavelets*, Proc. Amer. Math. Soc. **128**, pp. 2043-2046, 2000.

[27] M. Papadakis, *Generalized frame multiresolution analysis of abstract Hilbert spaces and its applications*, SPIE Proc. **4119**, *Wavelet Applications in Signal and Image Processing VIII, A. Aldroubi, A. Laine, M. Unser (Eds.)*, pp. 165-175, 2000.

[28] M. Papadakis, *Frames of translates and the generalized frame multiresolution analysis* , *Trends in Approximation theory*, K. Kopotun, T. Lyche, M. Neamtu (eds.), pp. 353-362, Vanderbilt University Press, Nashville TN, 2001.

[29] M. Papadakis, *Frames of translates and examples generalized frame multiresolution analysis*, SPIE Proc. **4478**, *Wavelet Applications in Signal and Image Processing IX, A. Aldroubi, A. Laine, M. Unser (Eds.)*, pp. 142-150, 2001.

[30] M. Papadakis, G. Gogoshin, I. Kakadiaris, D.J. Kouri, D.K. Hoffman, *Radial frame multiresolution analysis in multidimensions*, preliminary version, 2002.

[31] A. Petukhov, *Explicit construction of framelets*, preprint, 2000.

[32] A. Ron, Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Canad. J. Math. **47** (1995), 1051-1094.

[33] A. Ron, Z. Shen, *Affine systems in* $L^2(\mathbb{R}^d)$*: the analysis of the analysis operator*, J. Funct. Anal. **148** (1997), pp. 408-447.

[34] A. Ron, Z. Shen, *Compactly supported tight affine spline frames in* $L^2(\mathbb{R}^d)$, Math. Comp. **67** (1998), pp. 191-207.

[35] I.W. Seleznik *Smooth wavelet tight frames with zero moments*, Appl. Comp. Harm. Anal. **10** (2001), pp. 163-181.

[36] I.W. Seleznik, L. Sendur, SPIE Proc. **4119** (2000), *Wavelet Applications in Signal and Image Processing, A. Aldroubi, A. Laine, M. Unser (Eds.)*.

[37] T. Stavropoulos, M. Papadakis, *On the multiresolution analyses of abstract Hilbert spaces*, Bull. Greek Math. Soc., **40** (1998), pp. 79-92.

[38] X. Wang, *The study of wavelets from the properties of their Fourier transforms*, Ph.D. Thesis, Washington University in St. Louis, 1995.

[39] E. Weber, *Frames and Single Wavelets for Unitary Groups*, Canadian J. Mathem. **54**, pp. 634-647, 2002.

[40] G. Weiss, E.N. Wilson, *The mathematical theory of wavelets*, Proc. NATO-ASI meeting in *Harmonic Analysis: A Celebration*, Kluwer, 2001.

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