FRAMES OF TRANSLATES IN ABSTRACT HILBERT SPACES AND THE GENERALIZED FRAME MULTIRESOLUTION ANALYSIS

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Abstract. We introduce the concept of frames of translates in abstract Hilbert spaces and characterize the countable families of vectors generating such frames. In this context we generalize the concept of the Grammian introduced by Ron and Shen. We apply this characterization to study Generalized frame MRAs of $L^2(\mathbf{R})$. We also provide the characterization of frame multiwavelet sets associated with GFMRAs of $L^2(\mathbf{R})$.

1. Introduction and preliminaries

Let H be a complex Hilbert space. A *unitary system* is a set of unitary operators acting on H which contains the identity operator I on H. Now, let D be the *(dyadic) Dilation operator*

$$(Df)(t) = \sqrt{2}f(2t) \quad f \in L^2(\mathbf{R}) \tag{1}$$

and T the *Translation operator*

$$(Tf)(t) = f(t-1) \quad f \in L^2(\mathbf{R}). \tag{2}$$

The unitary system $\mathcal{U}_{D,T} := \{D^nT^m : n, m \in \mathbf{Z}\}$ called the *Affine system* has been used extensively in wavelet analysis. Following [4] C is an *orthonormal (Riesz) multiwavelet vector set of H with respect to the unitary system* if $\{U\psi : U \in \mathcal{U},\ \psi \in C\}$ is an orthonormal (Riesz) basis for the Hilbert space H. Riesz wavelet bases are related to exact sampling and have been used for the construction of fast wavelet algorithms. Nevertheless exact sampling is not always the most favorable type of sampling. In fact, in practice we oversample, so for example we are able to deal with noise more effectively (see [2]). Oversampling is related with wavelet frames.

The family $\{x_i : i \in I\}$ is a *frame* for the Hilbert space H if there exist constants $A, B > 0$ such that for every $x \in H$ we have that

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, x_i\rangle|^2 \le B\|x\|^2 .$$

We refer to the positive constants A, B as *frame bounds.* We refer to the frame as *tight* if $A = B$, as *normalized tight* (NTF) if $A = B = 1$ and to the operator $S : H \to \ell^2(I)$ defined by the equation

$$Sx = \sum_i < x, x_i > \delta_i \quad x \in H \tag{3}$$

where $\{\delta_i : i \in I\}$ is the standard orthonormal basis of $\ell^2(I)$, as the *analysis operator* corresponding to the frame $\{x_i : i \in I\}$.

In the abstract Hilbert space setting we can define wavelet frames with respect to a unitary system $\mathcal{U}$ ([6]). Let C be a subset of H and $\mathcal{U}$ a unitary system acting on H. If $\{U\psi : U \in \mathcal{U}, \psi \in C\}$ is a frame (resp. tight, NT) for the Hilbert space H we call the set C a *frame multiwavelet vector set (resp. tight, NT)* and the family $\{U\psi : U \in \mathcal{U},\ \psi \in C\}$ a *multiwavelet vector frame (resp. tight, NT).* If C is a singleton then we drop the prefix multi from all the previous terms.

In the present paper we generalize the theory of multiresolution analysis (MRA) by introducing a very general MRA structure called Generalized Frame MRA (GFMRA). There are numerous contributions in the study of MRAs as well as several generalizations. Most of these generalizations are special cases of our Generalized Frame multiresolution analysis. In [11] we give several details concerning the connection of our work with the work of others.

We focus on unitary systems $\mathcal{U}$ acting on H of the form $\mathcal{U} = \mathcal{U}_0 G$, where $\mathcal{U}_0 = \{D^n : n \in \mathbf{Z}\}$ and G is an abelian unitary group. Obviously unitary systems of this form generalize the affine system.

Definition 1. *A sequence $\{V_j\}_{j\in\mathbb{Z}}$ is a GFMRA of H if every V_j is a closed subspace of H, it is increasing, i.e. $V_j \subseteq V_{j+1}$ for every $j \in \mathbb{Z}$ and satisfies the following properties.*

(a) $V_j = D^j(V_0),\ j \in \mathbb{Z}$

(b) $\bigcap_j V_j = \{0\},\ \overline{\bigcup_j V_j} = H$

(c) There exists a countable subset B of V_0 such that the set $G(B) = \{g\phi : g \in G, \phi \in B\}$ is a frame of V_0.

Every such set B is called a frame multiscaling vector set for $\{V_j\}_j$. Every subset C of V_1 such that $G(C) = \{g\psi : g \in G, \psi \in C\}$ is a frame of $W_0 := V_1 \cap V_0^\perp$ is called a frame multiwavelet vector set associated with $\{V_j\}_j$.

Notice, that if C is a frame multiwavelet vector set associated with the GFMRA $\{V_j\}_j$ then the set C is a frame multiwavelet vector set for H, because $\{D^j g\psi : j \in \mathbb{Z}, g \in G, \psi \in C\}$ is a frame for H with the same frame bounds as the frame $G(C)$. Although we give this very general definition we will characterize only the NTF multiwavelet sets associated with GFMRAs of $L^2(\mathbb{R})$. For the study of GFMRAs of abstract Hilbert spaces the reader may refer to [12].

It becomes apparent from the definiton of GFMRA that most of the MRAs studied in the literature are special cases of GFMRAs. For instance, the MRAs of $L^2(\mathbb{R}^n)$ generated by multiscaling sets are special cases of the more general GFMRA theory.

Singly generated GFMRAs of $L^2(\mathbb{R})$ with respect to the affine system and their associated frame wavelets were studied in [1], [2] and independently in [8] and [14]. We refer to such GFMRAs as FMRAs. Shen. In all these papers frame wavelets are constructed using the frame scaling functions. Their results can be derived from our more general results. An interesting construction of NT wavelet affine frames of $L^2(\mathbf{R})$ based on multiresolution techniques can be found in [9]. The existence of frame multiwavelet sets associated with finitely generated GFMRAs can be deirved by theorem 2 of [15].

At this point we have to introduce a few clarifying remarks on our notation. Let f be a measurable function. We refer to the set $\{t : f(t) \neq 0\}$ as the support of function and we denote this set by $suppf$. In several cases we find convenient and more accurate to denote the function f by $f(\cdot)$, where the $\cdot$ replaces the variable instead of using $f(x)$ which denotes the value of f at x and not the function itself. If S is a bounded linear operator we denote the range of S by $\mathcal{R}(S)$. The range space of S is the closure of $\mathcal{R}(S)$ and the range projection of S is the orthogonal projection onto the range space of S. We use $|A|$ to denote the cardinality of the set A and A^- to denote its closure in the appropriate topology; if A is a subset of a vector space we denote the linear span of A by $[A]$. If x , y belong to a Hilbert space H we define the rank-one operator $x \otimes y$ by the equation: $x \otimes y(z) :=< z, x > y$, for every $z \in H$. We reserve the δ notation for the Kronecker's delta function. The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by the following equation:

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \ .$$

The Fourier transform of a function $f \in L^1([0,1))$ is defined similarly. We use $\mathcal{F}$ to denote Fourier transform on $L^2(\mathbf{R})$. Last but not least comment on our terminology; we reserve the term *subspace* for closed linear manifolds.

Now consider the Hilbert space $\ell^2(G)$. For $g \in G$ let ℓ_g be the unitary operator acting on $\ell^2(G)$ defined by the equation $\ell_g(\delta_k) = \delta_{gh}$. The set $\{\delta_h : h \in G\}$ is an orthonormal basis of $\ell^2(G)$ and the set $G^* := \{\ell_g :$

$g \in G\}$ is an abelian unitary group acting on $\ell^2(G)$. In fact the mapping $g \to \ell_g$, $g \in G$, known as the left regular representation of G is a group isomorphism between G and G^*. In fact it is not hard to verify that if $h, g \in G$, and $a \in \ell^2(G)$ then $\ell_g a(h) = a(g^{-1}h)$. Furthermore G^* is discrete with respect to the SOT (Strong operator topology).

From now on we will work on G^*. By $\widehat{G^*}$ we denote the dual group of G^* and by μ the normalized Haar measure on $\widehat{G^*}$. Let $g \in G$. We define $f_g : \widehat{G^*} \to \widehat{G^*}$ by the formula $f_g(\gamma) = \gamma(l_g)$, $\gamma \in \widehat{G^*}$. It is not hard to verify that $\{f_g : g \in G\}$ is an orthonormal basis of $L^2(\widehat{G^*})$.

2. Frames of translates

In this section we define and characterize the frames of translates in abstract Hilbert spaces (theorem 2). Theorem 2 generalizes theorems 3.4 in [1], 4.2 in [2], 3.3.5 and 3.4.1 in [13] and 2.5 in [3]. The following lemma provides an abstract characterization of frames of abstract Hilbert spaces.

Lemma 2.1. *([12]) Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\{x_r : r \in J\}$ be a complete family of vectors in $\mathcal{H}$, i.e. $[x_r : r \in J]^- = \mathcal{H}$. Assume that $\dim \mathcal{K} = |J|$ and that $\{e_r : r \in J\}$ is an orthonormal basis of $\mathcal{K}$. Then the following are true:*

(a) *If there exists $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $Te_r = x_r$, $r \in J$, then $\{x_r : r \in J\}$ is a Bessel family of $\mathcal{H}$. Conversely, let $T : [e_r : r \in J] \to \mathcal{H}$ be the linear operator such that for every $r \in J$ we have that $Te_r = x_r$. If $\{x_r : r \in J\}$ is a Bessel family of $\mathcal{H}$, then T can be extended to a bounded operator defined on $\mathcal{K}$.*

(b) *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Te_r = x_r$ for every $r \in J$. If V is the range space of T^* and $T^*T|_V : V \to V$ is bounded and invertible (resp. T^*T is an orthogonal projection), then $\{x_r : r \in J\}$ is a frame (resp. a NTF) of $\mathcal{H}$.*

   Conversely, if $\{x_r : r \in J\}$ is a frame (resp. a NTF) of $\mathcal{H}$, then there exists a unique $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying $Te_r = x_r$ for every $r \in J$. Furthermore, $V := \mathcal{R}(T^*)$ is closed and $T^*T|_V : V \to V$ is bounded and invertible (resp. T^*T is an orthogonal projection onto V).

Remark 1. It can be proved that if $\{x_r : r \in J\}$ is a Riesz basis for $\mathcal{H}$, then the subspace V in (b) of the previous lemma, is equal to $\mathcal{K}$ and vice-versa ([12], see also [6]).

Next we will use the previous lemma to characterize all subfamilies of the form $\{\phi_k : k \in \mathbb{N}\}$ of a Hilbert space H such that $\{g\phi_k : k \in \mathbb{N},\ g \in G\}$ is a frame of the Hilbert space H. We call *a frame of* *translates* every frame of the previous form and we refer to G as *a group of translations*.

On more notational comment; the Hilbert space $L^2(\widehat{G^*}, \ell^2)$ consists of all weakly measurable functions $\omega : \widehat{G^*} \to \ell^2$ such that $\int_{\widehat{G^*}} \| \omega(\gamma) \|_{\ell^2}^2 \, d\gamma$ is finite. The norm of ω is defined to be $\| \omega \| := \left(\int_{\widehat{G^*}} \| \omega(\gamma) \|_{\ell^2}^2 \, d\gamma\right)^{1/2}$. It is not hard to see that $\{f_g\delta_k : g \in G, k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\widehat{G^*}, \ell^2)$. We define M_g to be the multiplicative operator acting on $L^2(\widehat{G^*}, \ell^2)$ induced by f_g, i.e., $M_g f = f_g f$, where $f \in L^2(\widehat{G^*}, \ell^2)$. Obviously M_g is unitary for every $g \in G$, and $M_g^* = M_{g^{-1}}$. Finaly, if $\mathcal{A}$ is a subset of $\mathcal{B}(H)$, we will denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$.

Assume that $\{g\phi_k : k \in \mathbb{N},\ g \in G\}$ is a frame of H. By the previous lemma there exists a unique $T \in \mathcal{B}(L^2(\widehat{G^*}, \ell^2), H)$ such that $T(f_g\delta_k) = g\phi_k,\ k \in \mathbb{N},\ g \in G$. Moreover, if $V = \mathcal{R}(T^*)$, then V is closed and $T^*T\mid_V : V \to V$ is bounded and invertible. Now, let P be the orthogonal projection onto V. Using Polar decomposition ([7]) we have that $T^* = Y|T^*|$ and P is the range projection of both T^* and Y. In addition, Y is an isometry, because TT^* is invertible, since $\{g\phi_k : k \in \mathbb{N},\ g \in G\}$ is a frame of H. It will be convenient for the reader if we point that T^* is the analysis operator corresponding to the frame $\{g\phi_k : k \in \mathbb{N},\ g \in G\}$. The following lemma gives two key properties of the projection P and the isometry Y.

Lemma 2.2. *The following are true:*

(a) *For every $g \in G$ we have that $TT^*g = gTT^*$.*
(b) $YgY^* = M_gP, \quad g \in G$.
(c) $P \in \{M_g : g \in G\}'$.

Proof: (a) If g belongs to G we have that

$$\begin{aligned} gTT^*x &= \sum_{h\in G}\sum_{k\in\mathbb{N}} \langle x, h\phi_k\rangle gh\phi_k \\ &= \sum_{h\in G}\sum_{k\in\mathbb{N}} \langle gx, gh\phi_k\rangle gh\phi_k = TT^*gx, \quad x \in H. \end{aligned}$$

(b) Since TT^* is invertible $|T^*|$ is invertible, as well. Define $\tilde{\phi}_k := |T^*|^{-1}\phi_k, \quad k \in \mathbb{N}$. By (a) TT^* belongs to G' and so does $|T^*|$, because G' is a self-adjoint operator algebra closed with respect to the Strong Operator Topology (SOT). Let x be in H. Then we get that

$$Yx = T^* \left|T^*\right|^{-1} x = \sum_g \sum_k \langle x, g\tilde{\phi}_k\rangle f_g\delta_k \ . \tag{4}$$

Since Y is an isometry we obtain that $\|x\|^2 = \sum_g \sum_k \left|\langle x, g\tilde{\phi}_k\rangle\right|^2$, thus $\{\widetilde{g\phi_k} : k \in \mathbb{N},\ g \in G\}$ is a NTF for H.

Let $h \in G$. From eq. (4) we get that

$$\langle Y(g\tilde{\phi}_K),\ Y(h\tilde{\phi}_\ell)\rangle = \langle g\tilde{\phi}_k, h\tilde{\phi}_\ell\rangle = \langle Y(g\tilde{\phi}_k), f_h\delta_\ell\rangle,\ k,\ \ell \in \mathbb{N}.$$

On the other hand we have that

$$\langle Y(g\tilde{\phi}_k), f_h\delta_\ell\rangle = \langle Y(g\tilde{\phi}_k), P(f_h\delta_\ell)\rangle.$$

Thus for every $k, \ell \in \mathbb{N}$ and $g, h \in G$ we have that

$$\langle P(f_h\delta_\ell), Y(g\tilde{\phi}_k)\rangle = \langle Y(h\tilde{\phi}_\ell), Y(g\tilde{\phi}_k)\rangle$$

which implies that $P(f_h\delta_\ell) = Y(h\tilde{\phi}_\ell)$, because $\{Y(g\tilde{\phi}_k) :\ g \in G,\ k \in \mathbb{N}\}$ spans $Y(V_0)$. Equation (4) implies that $Y^*(f_g\delta_k) = g\tilde{\phi}_k$, for every $g \in G$ and $k \in \mathbb{N}$. Therefore, if h is in G and k is in $\mathbb{N}$, we obtain that

$$\begin{aligned} YgY^*(f_h\delta_k) &= Y(gh\tilde{\phi}_k) \\ &= \sum_{h',\ell}\langle gh\tilde{\phi}_k, h'\tilde{\phi}_\ell\rangle f_{h'}\delta_\ell = \sum_{h',\ell}\langle gh\tilde{\phi}_k, gg^{-1}h'\tilde{\phi}_\ell\rangle f_{h'}\delta_\ell \\ &= M_gY(h\tilde{\phi}_k) = M_gP(f_h\delta_k). \end{aligned}$$

This establishes (b). Now for $g, h \in G, k \in \mathbb{N}$ we have that

$$\begin{aligned} M_gP(f_h\delta_k) &= YgY^*(f_h\delta_k) = Y(gh\tilde{\phi}_k) \\ &= P(f_gf_h\delta_k) = PM_g(f_h\delta_k). \end{aligned}$$

This establishes (c).

The space $L^2(\widehat{G^*}, \ell^2)$ is isometrically isomorphic to the space $L^2(\widehat{G^*}) \otimes \ell^2$. Under the same isomorphism, M_g can be identified with $\mu_g \otimes I_{\ell^2}$, where μ_g is the multiplicative operator defined by $\mu_g\omega = f_g\omega$, $\omega \in L^2(\widehat{G^*})$ and I_{ℓ^2} is the identity operator on ℓ^2. Therefore $\{M_g : g \in G\}'$ can be identified with $\mathcal{L}(\widehat{G^*}) \otimes \mathcal{B}(\ell^2)$. The latter Von Neumann algebra is spatially isomorphic (under the inverse of the isomorphism between $L^2(\widehat{G^*}, \ell^2)$ and $L^2(\widehat{G^*}) \otimes \ell^2$) with the Von Neumann $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$. This particular Von Neumann algebra contains exactly all essentially bounded WOT-measurable functions $F : \widehat{G^*} \to \mathcal{B}(\ell^2)$. Therefore, $\{M_g : g \in G\}'$ is the algebra of all bounded operators $\mathbf{F}$, defined in the following way: For every $\mathbf{F}$ there exists $F \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ such that

$$\mathbf{F}\omega(\gamma) = F(\gamma)\omega(\gamma), \qquad \omega \in L^2(\widehat{G^*}, \ell^2)$$

and vice versa. For more details the reader may refer to [5] and in particular to examples 4.3.5 and 4.3.10.

Thus for $P \in \{M_g : g \in G\}'$ there exists a function $P \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ such that

$$(Pf)(\gamma) = P(\gamma)f(\gamma), \quad \gamma \in \widehat{G^*}.$$

Notice that we used the same notation for the projection P and the operator-valued function P in $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$. Throughout the present paper we adopt this convention in order to simplify our notation and we hope that this will not confuse the reader. Since P is a projection it turns out that $P(\gamma)$ is a projection a.e. in $\widehat{G^*}$.

Using (c) of the previous lemma we obtain that

$$M_g Y\,|T^*|\,Y^* = M_g PY\,|T^*|\,Y^* = YgY^*Y\,|T^*|\,Y^* = Yg\,|T^*|\,Y^*$$
$$= Y\,|T^*|\,gY^* = Y\,|T^*|\,Y^*(YgY^*) = Y\,|T^*|\,Y^*M_gP = Y\,|T^*|\,Y^*M_g\,.$$

On the other hand we have that $T^*T = (Y\,|T^*|\,Y^*)^2$, therefore T^*T belongs $\{M_g : g \in G\}'$ as well. This implies that there exists $A \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ such that

$$T^*T\omega(\gamma) = A(\gamma)\omega(\gamma) \ \text{ a.e. in } \widehat{G^*}.$$

Note that the range projection of T^*T is P as well. Since $T^*T(V) = V$ we obtain that $PT^*TP = T^*TP$, thus $T^*TP = PT^*T$. Furthermore, we have that $T^*TP = T^*T$, because $T(I-P) = 0$. Therefore we have that

$$P(\gamma)A(\gamma)P(\gamma) = A(\gamma)P(\gamma) = A(\gamma) \ \text{ a.e.}$$

Since $T^*T\mid_V$ is invertible and its inverse belongs to the Von Neumann algebra $\{M_g : g \in G\}'$, we get that for a.e. $\gamma \in \operatorname{supp}P$ the restriction of $A(\gamma)$ on the range of $P(\gamma)$ is invertible. In fact we have the following: There exist $B_1, B_2 > 0$ such that for a.e. $\gamma \in \operatorname{supp}P$ and $x \in P(\gamma)(l^2)$ we have that

$$B_1\,\|x\| \le \|A(\gamma)x\| \le B_2\,\|x\|\,.$$

Finally, it not hard to verify that for every $k \in \mathbb{N}$ the following equation is satisfied:

$$A(\cdot)\delta_k = \sum_{g\in G}\sum_{l=1}^{\infty}\langle\phi_k, g\phi_l\rangle f_g\delta_l, \ \ \gamma \in \widehat{G^*}. \tag{5}$$

The latter equation shows that A is the *autocorrelation function* corresponding to the frame of translates $\{g\phi_k : k \in \mathbb{N}, g \in G\}$.

Theorem 2. *Let $\{\phi_k : k \in \mathbb{N}\}$ be a subset of the Hilbert space H such that $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is complete in H.*

(a) Define

$$a_k := \sum_g \sum_l < \phi_k, g\phi_l > f_g\delta_l \ \ k \in \mathbb{N}.$$

Assume that for every $k \in \mathbb{N}$ the function a_k is well defined, $\|a_k(\cdot)\|_{\ell^2}$ is in $L^2(\widehat{G^*})$ and the linear operators $A(\gamma)$ defined for a.e. $\gamma \in \widehat{G^*}$ on $[\delta_k : k \in \mathbb{N}]$ by the equation $A(\gamma)\delta_k = a_k(\gamma)$ satisfy the following properties:

1. *A belongs to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$.*
2. *Let $P(\gamma)$ be the range projection of $A(\gamma)$ a.e. There exists $B > 0$ such that for every $x \in P(\gamma)(\ell^2)$ we have that $B\,\|x\| \leq \|A(\gamma)x\|$.*

Then $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a frame of H with frame bounds B and $\|A\|_\infty$. Conversely, if $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a frame of H with frame bounds B, C, then there exists $A \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ such that $\|A\|_\infty \leq C$ satifying equation (5) and property (2). Moreover the function $P(\cdot)$ belongs to $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$.

(b) Assume that $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a frame of H. Then the following are true:

i)$\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a Riesz basis of H if and only if $P(\gamma) = I_{\ell^2}$ a.e.

ii)$\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a NTF of H if and only if $A(\gamma)$ is an orthogonal projection a.e.

iii)$\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is an orthonormal basis of H if and only if $A(\gamma) = I_{\ell^2}$ a.e.

Proof: (a) The neccessity part of (a) was proved in the discussion preceeding the statement of the theorem. Next we prove the sufficiency part of it. Let T be the linear operator defined on $[f_g\delta_k : k \in \mathbb{N}, g \in G]$ satifying $T(f_g\delta_k) = g\phi_k$. We will show that T can be extended to a bounded operator defined on $L^2(\widehat{G^*}, \ell^2)$, thus proving, due to (a) of lemma 2.1, that $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a Bessel family of $\mathcal{H}$. Let $\{b_{k,g}\}$ be a sequence of scalars such that all but a finite number of $b_{k,g}$ are zero. Then we get that $T(\sum_{k,g} b_{k,g} f_g \delta_k) = \sum_{k,g} b_{k,g} g\phi_k$. But

$$< \sum_{k,g} b_{k,g} g\phi_k \ , \ \sum_{k,g} b_{k,g} g\phi_k >= \sum_{k,g}\sum_{k',g'} b_{k,g}\overline{b_{k',g'}} < g\phi_k, g'\phi_{k'} >$$

$$= \sum_{k,g}\sum_{k',g'} b_{k,g}\overline{b_{k',g'}} < \phi_k, g^{-1}g'\phi_{k'} >.$$

On the other hand, the definition of a_k implies that

$$| < \phi_k, g^{-1}g'\phi_{k'} > | = | < a_k, f_{g^{-1}g'}\delta_{k'} > | \ ;$$

now, property (1) gives that

$$| < \phi_k, g^{-1}g'\phi_{k'} > | \leq \|A\|_\infty \ \text{ for all } k \in \mathbb{N}, g, g' \in G.$$

This proves that $\left\|\sum_{k,g} b_{k,g} g\phi_k\right\|^2 \leq \|A\|_\infty^2 \sum_{k,g} |b_{k,g}|^2$, which implies that T can be extended to a bounded operator on the whole $L^2(\widehat{G^*}, \ell^2)$. We denote this extension again by T. This in particular, implies that $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a Bessel family of $\mathcal{H}$.

Now, let V be the range space of T^* and assume that P is the projection onto V. It is not hard to verify that T^*T commutes with $\{M_g : g \in G\}$ and that $A \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ satisfies $T^*T\omega(\gamma) = A(\gamma)\omega(\gamma)$ a.e. As in the proof of the neccessity part of (a) P is induced by a function $P \in L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$, where $P(\gamma)$, a.e. in $\widehat{G^*}$, is the projection we defined in property (2). The same property also implies that $T^*T \mid V$ is invertible, thus proving that $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a frame of H. The conclusion about the frame bounds follows easily.

(b) If $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a Riesz basis of H lemma 2.1 implies that $P = I$, thus $P(\gamma) = I_{\ell^2}$ a.e. On the other hand, if $P(\gamma) = I_{\ell^2}$ a.e. the arguments of the previous paragraph imply that $P = I$, so $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a Riesz basis of H.

If $\{g\phi_k : k \in \mathbb{N}, g \in G\}$ is a NTF of H lemma 2.1 implies that T^*T, where T is the operator we defined in the proof of the neccessity part of (a), is a projection. This implies that $A(\gamma)$ is a projection a.e. Similarly one can prove that the converse statements in (i) and (ii) are true. Statement (iii) follows from the fact that a Riesz basis which is also a NTF is an orthonormal basis and vice-versa.

Remark 2. Property (2) of the previous theorem follows that $A(\gamma)$ is invertible only a.e. in the $\operatorname{supp}P = \operatorname{supp}A$. Although the previous theorem deals only with the case of infinite countable subsets $\{\phi_k : k \in \mathbb{N}\}$ of H the theorem remains true even in the case of finite subsets $\{\phi_k : k = 1, 2, \ldots, n\}$ $(n \in \mathbb{N})$ of H as well, with certain obvious modifications. Specifically, in this case instead of using the spaces $L^2(\widehat{G^*}, \ell^2)$ and $L^\infty(\widehat{G^*}, \mathcal{B}(\ell^2))$ we use the spaces $L^2(\widehat{G^*}, \mathbb{C}^n)$ and $L^\infty(\widehat{G^*}, \mathbb{C}^{n\times n})$.

Let G be the translation group $\{T^n : n \in \mathbb{N}\}$, where T is the translation operator defined by eq. (2). Our final goal in this section is to characterize the subfamilies of functions $\{\phi_k : k \in I\}$ of $L^2(\mathbb{R})$, where I is an arbitrary subset of $\mathbb{N}$ such that $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ is a Riesz basis or a frame, or a NTF, or an orthonormal basis for the closed linear span of $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$. Now, set $e_n(\xi) := e^{2\pi i n\xi}$, $n \in \mathbb{Z}$.

Corollary 3. *Let $I \subseteq \mathbb{N}$ and $\{\phi_k : k \in I\}$ be a subset of $L^2(\mathbb{R})$. Define*

$$a_{l,k}(\xi) := \sum_{m\in\mathbb{Z}} \hat{\phi}_k(\xi + m)\overline{\hat{\phi}_l(\xi + m)} \quad k, l \in I \;\; \xi \in [0, 1)$$

and $a_k(\xi) := (a_{1,k}(\xi), a_{2,k}(\xi), \dots)$, $\xi \in [0,1)$.
(a) Assume that for every $k \in I$ *the function* $\xi \to \|a_k(\xi)\|_{\ell^2}$ *is in* $L^2([0,1))$ *and that the linear operators* $A(\xi)$ *defined for a.e.* $\xi \in [0,1)$ *on* $[\delta_k : k \in I]$ *by the equation* $A(\xi)\delta_k = a_k(\xi)$ *satisfy the following properties:*

(1) *A belongs to* $L^\infty([0,1), \mathcal{B}(\ell^2(I)))$.
(2) *Let* $P(\xi)$ *be the range projection of* $A(\xi)$ *a.e. There exists* $B > 0$ *such that for every* $x \in P(\xi)(\ell^2(I))$ *we have that* $B\|x\| \leq \|A(\xi)x\|$.

Then $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ *is a frame of its closed linear span with frame bounds* B *and* $\|A\|_\infty$. *Conversely, if* $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ *is a frame of its closed linear span with frame constants* B, C, *then there exists* $A \in L^\infty([0,1), \mathcal{B}(\ell^2(I)))$ *such that* $\|A\|_\infty \leq C$ *satifying the following equation*

$$A(\xi)_{l,k} = \sum_{m\in\mathbb{Z}} \hat{\phi}_k(\xi+m)\overline{\hat{\phi}_l(\xi+m)} \quad k,l \in I, \ \ \xi \in [0,1)$$

and property (2). Moreover $P(\cdot)$ *belongs to* $L^\infty([0,1), \mathcal{B}(\ell^2(I)))$.
(b) Assume that $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ *is a frame of its closed linear span. Then the following are true:*
i) $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ *is a Riesz basis of its closed linear span if and only if* $P(\xi) = I_{\ell^2(I)}$ *a.e.*
ii) $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ *is a NTF of its closed linear span if and only if* $A(\xi)$ *is an orthogonal projection a.e.*
iii) $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$ *is orthonormal if and only if* $A(\xi) = I_{\ell^2(I)}$ *a.e.*

Proof. The proof follows directly from theorem 2 and the following two facts. First, the translation group is algebraically isomorphic to $\mathbb{Z}$. By considering the left regular representation of the translation group on $\ell^2(G)$ we get G^* which is discrete and algebraically isomorphic to $\mathbb{Z}$, so its dual $\widehat{G^*}$ is homeomorphicaly isomorphic to the additive group $[0,1)$, where addition is defined modullo 1. Second, a trivial calculation involving Plancherel's theorem shows that for $k,l \in I$ we have that $< \phi_k, T^n\phi_l >= \widehat{a_{l,k}}(n)$, for every $n \in \mathbb{Z}$. □

If I is finite then the function A has a very elegant representation. Set $\Phi := (\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_n)$. Then

$$A(\xi) = \sum_{m\in\mathbb{Z}} \Phi(\xi+m)^*\Phi(\xi+m) \ \text{ a.e.}$$

In the case $|I| = 1$ the previous corollary gives the Benedetto and Li characterization of frames of translates of a single function in $L^2(\mathbb{R})$ (see [1] theorem 3.4, [2] theorem 4.2).

3. Generalized Frame MRA of $L^2(\mathbb{R})$

We begin this section with the characterization of all the NTF multiwavelet sets associated with a GFMRA of $L^2(\mathbb{R})$ with respect to the system $\mathcal{U}_{D,T}$. In order to give a flavor of our more general results we assume throughout this section that $\{V_j\}_j$ is a GFMRA of $L^2(\mathbb{R})$ and $\{\phi_k : k \in \mathbb{N}\}$ is a NTF multiscaling set associated with $\{V_j\}_j$. GFMRAs of $L^2(\mathbb{R})$ where introduced in [10], where it was proved that every orthonormal wavelet ψ is associated with a GFMRA of $L^2(\mathbb{R})$, i.e. $\{T^n\psi : n \in \mathbb{Z}\}$ is an orthonormal basis of $V_1 \cap V_0^{\perp}$.

Define $Y : V_0 \to L^2([0,1), \ell^2)$ by the equation

$$Y(f) = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \langle f, T^n\phi_k\rangle e_n\delta_k\ , \quad f \in V_0.$$

Note that Y is the analysis operator corresponding to the frame $\{T^n\phi_k : k \in I, n \in \mathbb{Z}\}$. Since $\{\phi_k : k \in \mathbb{N}\}$ is a NTF multiscaling set and Y is an isometry, for every $f \in V_0$ we have that $f = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \langle f, T^n\phi_k\rangle T^n\phi_k$. Since $D^*\phi_k$ belong to V_0 for every k, we can define $m_k := YD^*\phi_k$. We call the family $\{m_k : k \in \mathbb{N}\}$ the *low pass filter set corresponding to the multiscaling set* $\{\phi_k : k \in \mathbb{N}\}$. We also have that $D^*\phi_k = \sum_{n\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \langle D^*\phi_k, T^n\phi_l\rangle T^n\phi_l$, for every $k \in \mathbb{N}$. Applying the Fourier transform on both sides of the previous equation we obtain that

$$\hat{\phi}_k(2\xi) = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{N}} m_{l,k}(\xi)\hat{\phi}_l(\xi) \quad \text{a.e., } k \in \mathbb{N}, \tag{6}$$

where $m_{l,k}(\xi) = \sum_n \langle D^*\phi_k, T^n\phi_l\rangle e^{-2\pi in\xi}$. The following proposition generalizes the Quadrature Mirror filter or Smith-Barnwell condition.

Proposition 4. *([12]) There exists a unique $M_0 \in L^\infty([0,1), \mathcal{B}(\ell^2))$ such that $M_0(\xi)\delta_k := m_k(\xi) = (m_{1,k}(\xi), m_{2,k}(\xi), \ldots)$ a.e. satisfying the following property:*

$$P_1(\xi) := \frac{1}{2}[M_0(\xi)^*M_0(\xi) + M_0(\xi + 1/2)^*M_0(\xi + 1/2)]$$

is a.e. an orthogonal projection.

We call the function M_0 the *low pass filter corresponding to* $\{\phi_k : k \in \mathbb{N}\}$.

Let $A \in L^\infty([0,1), \mathcal{B}(\ell^2))$ be the autocorrelation function corresponding to the NTF multiscaling set $\{\phi_k : k \in \mathbb{N}\}$. By corollary 3 $A(\xi)$ is a.e. a projection (this does allow A to vanish on set of positive measure). Now, let ψ_k be a NTF multiwavelet set associated with $\{V_j\}_j$. In the same manner we define $h_k := YD^*\psi_k$, $k \in \mathbb{N}$. We call the family $\{h_k : k \in \mathbb{N}\}$ a *high pass filter set.* We have the following characterization of high pass filter sets.

Theorem 5. *([12]) Let $\{\phi_k : k \in \mathbb{N}\}$ be a set of NTF multiscaling vectors for the GFMRA $\{V_j\}_j$. Assume that $\{\psi_k : k \in \mathbb{N}\}$ is a subset of V_1. Set $h_k := YD^*\psi_k$, $k \in \mathbb{N}$. Then $\{\psi_k : k \in \mathbb{N}\}$ is a set of NTF multiwavelet vectors associated with $\{V_j\}_j$ if and only if the following are true:*

1. *There exists $H \in L^\infty([0,1), \mathcal{B}(\ell^2))$ such that $H(\xi)\delta_k := h_k(\xi)$ a.e.*
2. *$P_2(\xi) := \frac{1}{2}[H(\xi)^*H(\xi) + H(\xi+1/2)^*H(\xi+1/2)]$ is an (orthogonal) projection a.e.*
3. $0 = M_0(\xi)^*H(\xi) + M_0(\xi+1/2)^*H(\xi+1/2)$ *a.e.*
4. *For a.e. ξ in $[0,1)$ we have that the columns of the following matrix*
$$\begin{pmatrix} M_0(\xi) & H(\xi) \\ M_0(\xi+\frac{1}{2}) & H(\xi+\frac{1}{2}) \end{pmatrix}$$
form a complete set for $A(\xi)(\ell^2) \oplus A(\xi+1/2)(\ell^2)$. We call this matrix modulation matrix.

In [12] we prove that we can always construct the function H, which we call *a high pass filter corresponding to the low pass filter* M_0. We also provide two algorithms for the construction of high pass filters. In the example that follows we will apply to some extend the second of the two algorithms. Theorem 5 follows directly from theorems 7 and 10 of [12]. These theorems characterize all frame multiwavelet vector sets associated with a GFMRA of an abstract Hilbert space satisfying two additional very general hypotheses. Similarly with eq. (6), if $\{h_k : k \in \mathbb{N}\}$ is a high pass filter set, then we have that

$$\hat{\psi}_k(2\xi) = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{N}} h_{l,k}(\xi)\hat{\phi}_l(\xi) \quad \text{a.e., } k \in \mathbb{N}, \tag{7}$$

where $h_{l,k}(\xi) = \sum_n \langle D^*\psi_k, T^n\phi_l\rangle e^{-2\pi i n\xi}$.

Example 6. In this example we will use $[-1/2,\ 1/2)$ instead of $[0,1)$. Let $\phi \in L^2(\mathbb{R})$ be a function with the following properties: $\widehat{\phi}$ is continuous, $\widehat{\phi}(\xi) = 0$ for every ξ such that $|\xi| \geq b$, where $0 < b < 1/4$ and $0 < \widehat{\phi}(\xi) \leq 1$ for every $\xi \in (-b, b)$. Moreover let $c \in (b/2, b)$. We assume that $\widehat{\phi}(\xi) = 1$ for every $\xi \in (-c, c)$. Set V_0 to be the closed linear span of $\{T^n\phi : n \in \mathbb{Z}\}$ It is not hard for someone to verify that $\mathcal{F}(V_0) = L^2([-b, b))$. Now, define $\phi_1 = \phi$ and $\phi_2 = \mathcal{F}^{-1}(\widehat{\phi_2})$, where

$$\widehat{\phi}_2(\xi) := \begin{cases} 0 & if \quad |\xi| \geq b \\ \sqrt{1 - \widehat{\phi}(\xi)^2} & if \quad |\xi| < b \end{cases}$$

Apparently $\{T^n\phi_1 : n \in \mathbb{Z}\} \cup \{T^n\phi_2 : n \in \mathbb{Z}\}$ is contained in V_0 and as we will show by calculating the autocorrelation function A corresponding to the set $\{\phi_1,\ \phi_2\}$ the former set is a NTF of V_0. An easy calculation shows that

$$A(\xi) = \begin{cases} 0 & if \quad |\xi| > b \\ v(\xi) \otimes v(\xi) & if \quad c \leq |\xi| \leq b \\ \delta_1 \otimes \delta_1 & if \quad |\xi| < c \end{cases}$$

where $\delta_1 = (1, 0)$, $\delta_2 = (0, 1)$ and $v(\xi) := \widehat{\phi_1}(\xi)\delta_1 + \widehat{\phi_2}(\xi)\delta_2$. Obviously $\|v(\xi)\| = 1$ if $c \leq |\xi| \leq b$. This proves that if $A(\xi) \neq 0$ then it is a rank-one projection, so by corollary 3 $\{T^n\phi_1 : n \in \mathbb{Z}\} \cup \{T^n\phi_2 : n \in \mathbb{Z}\}$ is a NTF of V_0. The intuitive idea behind this particular selection of ϕ_2 is to "compensate" $\hat{\phi}_1$ in the frequency band, specificaly in neighborhoods of $-b$, b, because ϕ_1 cannot guarantee stable reconstruction of functions supported in neighborhoods of $-b$, b in the frequency domain. Now define $V_j = D^j(V_0)$, where D is the dyadic dilation operator and $j \in \mathbb{Z}$. Apparently $\{V_j\}_j$ is a GFMRA of $L^2(\mathbb{R})$.

The hypothesis $c \in (b/2, b)$ yields a fairly simple form for the low pass filter M_0. In fact a proper calculation of the 1-periodic function M_0 has to involve the isometry Y but we will refrain from giving details. We simply give the form of M_0.

$$M_0(\xi) = \begin{pmatrix} m_1(\xi) & m_2(\xi) \\ 0 & 0 \end{pmatrix}$$

where $m_1(\xi) = \sqrt{2}\widehat{\phi_1}(2\xi)$ and $m_2(\xi) = \sqrt{2}\widehat{\phi_2}(2\xi)$, $\xi \in [-1/2, 1/2)$. We hope that the reader will excuse this particular use of the notation $m_{1,}$ m_2 which is inconsistent with the notation we established before theorem 5. Next we have to find a 2×2 matrix valued function H satisfying conditions (1)-(4) of theorem 5. More precisely we will define $H \in L^\infty([0, 1), \mathbb{C}^{2\times 2})$ satisfying the following properties:

- $\frac{1}{2}[H(\xi)^*H(\xi) + H(\xi + \frac{1}{2})^*H(\xi + \frac{1}{2})]$ is an orthogonal projection.
- $M_0(\xi)^*H(\xi) + M_0(\xi + \frac{1}{2})^*H(\xi + \frac{1}{2}) = 0$.
- The columns of the modulation matrix

$$\begin{pmatrix} M_0(\xi) & H(\xi) \\ M_0(\xi + \frac{1}{2}) & H(\xi + \frac{1}{2}) \end{pmatrix}$$

form a frame for $A(\xi)(\mathbb{C}^2) \oplus A(\xi + \frac{1}{2})(\mathbb{C}^2)$ a.e.

It is not hard to see that $A(\xi) \oplus A(\xi + \frac{1}{2})$ vanishes on the intervals $[-1/2 + b, -b) \cup [b, 1/2 - b)$ and on the remaining intervals is a rank-one projection, because the support of A is the interval $[-b, b)$ and $b < 1/4$. Therefore the last of the above properties suggests that at most one of the last two columns of the modulation matrix should be non zero. This suggests to set the fourth column of the modulation matrix identically equal to zero. The second of the above properties implies that the third column of the modulation matrix must be a.e. orthogonal (whenever it is non zero) to the first two columns of the modulation matrix. We will show how this can be accomplished in two cases. The remaining cases follow easily.

If $\frac{c}{2} \leq |\xi| < \frac{b}{2}$ then $m_1(\xi)$ and $m_2(\xi)$ are both non zero a.e. so the first two columns of the modulation matrix are both non zero. This implies that in this case the last two columns of the modulation matrix have to be equal to zero. On the other hand, if $c \leq |\xi| < b$, then $A(\xi) = v(\xi) \otimes v(\xi)$, while $A(\xi + \frac{1}{2}) = 0$. Also $M_0(\xi) = M_0(\xi + \frac{1}{2}) = 0$. In this case, we select the third column of the modulation matrix to be the vector $(v(\xi), 0)$, because $\{v(\xi)\}$ is an orthonornal basis of the range of the rank-one projection $A(\xi)$. Considering all the possible cases for ξ we construct the high pass filter H given below.

$$H(\xi) = \begin{cases} 0 & if \quad |\xi| > b, \quad or \quad |\xi| \leq b/2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & if \quad b/2 \leq |\xi| \leq c \\ \begin{pmatrix} \widehat{\phi}_1(\xi) & 0 \\ \widehat{\phi}_2(\xi) & 0 \end{pmatrix} & if \quad c \leq |\xi| \leq b \end{cases}$$

Therefore we have that the high pass filter set in this case is the singleton $\{h\}$, where

$$h(\xi) = \begin{cases} 0 & if \quad |\xi| > b, \quad or \quad |\xi| \leq b/2 \\ (1, 0) & if \quad b/2 \leq |\xi| \leq c \\ (\widehat{\phi}_1(\xi), \widehat{\phi}_2(\xi)) & if \quad c \leq |\xi| \leq b \end{cases}.$$

Finally, it is not hard to check that the filter h defines the frame wavelet $\psi := \mathcal{F}^{-1}(\chi_{[-2b,-b)\cup[b,2b)})$. We would like to conclude with two remarks focusing on this last result. Since $\mathcal{F}(V_0) = L^2([-b, b))$ we can easily see that the GFMRA we defined above can be generated by the frame scaling function $\mathcal{F}^{-1}(\chi_{[-b,b)})$ as well. Then immediately one realizes that the associated frame wavelet is the particular ψ we found above. We think that the most interesting part of the whole process is not the construction of the frame wavelet but the construction of the filters $(m_1, 0)$, $(m_2, 0)$, h and their particular functional properties.

One of our future goals is to study the process of constructing this type of QMF filters in depth and its connections with the work of others.

4. Acknowledgements

Supported by funds provided by a Ralph E. Powe Junior Faculty Enhancement Award and by a University of Houston New Faculty Research Award.

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