Non-separable Radial Frame Multiresolution Analysis in Multidimensions and Isotropic Fast Wavelet Algorithms

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ABSTRACT

In this paper we present a non-separable multiresolution structure based on frames which is defined by radial scaling functions of the form of the Shannon scaling function. We also construct the resulting frame multiwavelets, which can be isotropic as well. Our construction can be carried out in any number of dimensions and for a great variety of dilation matrices.

Keywords: non-separable multiresolution analysis, wavelets, frames

1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space. A *unitary system* $\mathcal{U}$ is a set of unitary operators acting on H which contains the identity operator I on H. Now, let D be the *(dyadic) Dilation operator*

$$(Df)(\mathbf{t}) = 2^{n/2} f(2\mathbf{t}), \quad f \in L^2(\mathbb{R}^n) \tag{1}$$

and $T_{\mathbf{k}}$ be the Translation operator defined by

$$(T_{\mathbf{k}} f)(\mathbf{t}) = f(\mathbf{t} - \mathbf{k}), \quad f \in L^2(\mathbb{R}^n), \ \ \mathbf{k} \in \mathbb{Z}^n. \tag{2}$$

We refer to the unitary system $\mathcal{U}_{D,\mathbb{Z}^n} := \{D^j T_{\mathbf{k}} : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ as the *n-dimensional separable Affine system.* This system has been extensively used in wavelet analysis for the construction of separable wavelet bases. In fact only a few non-separable wavelet bases have been constructed and all these examples were exclusively given in two dimensions. However, an important drawback of these families of wavelets is the absence of enough symmetry, differentiability and the absolute lack of isotropy. These, examples were also given with respect to a small class of dilation operators and all of them are compactly supported in the time domain. Apparently the whole issue of designing wavelet bases in multidimensions still remains a mostly unexplored area, full of challenges and revealing interesting and surprising results.

The motivation for the present paper stems from the following elementary observation: The low pass filter corresponding to the scaling function of the Shannon MRA is the indicator function of the interval $[-1/2, 1/2)$. This function is even and its Fourier transform is of the form $\hat{\omega} = \chi_A$, where A is a measurable subset of $\mathbb{R}$. Keeping in mind that even functions are also radial (a function is radial if it depends only on the radial variable) one might wonder, what is the multidimensional analogue of even, sinc-like scaling functions. This particular problem motivated us to introduce the radial frame multiresolution analysis. Our construction is

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based on a very general multiresolution scheme of abstract Hilbert spaces developed by Papadakis,[1] namely the Generalized Frame Multiresolution Analysis (GFMRA). The main characteristic of GFMRAs is that they can be generated by redundant sets of frame scaling functions. In fact, GFMRAs encompass all classical MRAs in one and multidimensions as well as the FMRAs of Benedetto and Li.[2]

In this paper we construct non-separable Shannon-like FMRAs of $L^2(\mathbb{R}^n)$ whose scaling functions are radial and are defined with respect to certain unitary systems, which we will later introduce. We also derive certain of their associated frame multiwavelet sets. Our construction is the first of its kind. Scaling functions that are radial have not been constructed in the past. However, certain classes of non separable scaling functions in two dimensions, with some continuity properties with respect to dyadic dilations or dilations induced by the Quincunx matrix only have been constructed in the past.[3–7] All of them have no axial symmetries and are not smooth, except those constructed in,[8] which can be made arbitrarily smooth, but are highly asymmetric. Other constructions in the spirit of digital filter design, but not directly related to wavelets are due to Adelson et al[9] and to Simoncelli et al.[10] These two and the ridgelets and beamlets[11–13] share two properties of our Radial GFMRAs: the separability of the designed filters with respect to polar coordinates and the redundancy of the induced representations. However, our construction in contrast to those due to Simoncelli et. al., Candes, Donoho, Starck et al. are in the spirit of classical multiresolution analysis and can be extended to any number of dimensions and with respect to a great variety of dilation matrices.

The merit of non-separable wavelets and scaling functions is that the resulting processing of images is more compatible with that of human or mammallian vision, because mammals do not process images vertically and horizontally as separable filter banks resulting from separable multiresolution analyses do.[14] As Marr suggests in his book,[15] our visual system critically depends on edge detection. In order to model this detection, Marr and Hildreth used the Laplacian operator, which is the "lowest order isotropic operator",[16] because our visual system is orientation insensitive to edge detection. This suggests that perhaps the most desirable property in filter design for image processing is the isotropy of the filter. Thus radial scaling functions for multiresolutions based on frames are the best types of image processing filters that meet the isotropy requirement.

Before proceeding we need some definitions and certain preliminary results.[1]

The family $\{x_i : i \in I\}$ is a *frame* for the Hilbert space H if there exist constants $A, B > 0$ such that for every $x \in H$ we have

$$A\,\|x\|^2 \le \sum_{i \in I} |\langle x, x_i\rangle|^2 \le B\,\|x\|^2 \ .$$

We refer to the positive constants A, B as *frame bounds.* Apparently for every frame its bounds are not uniquely defined. We refer to the frame as *tight* if $A = B$ and as *Parseval frame* if $A = B = 1$. A frame $\{x_i : i \in I\}$ of H is called *exact* if each one of its proper subsets is not a frame for H. Riesz bases are exact frames and vice-versa. The operator S defined by

$$Sx = \{\langle x, x_i\rangle\}_{i \in I} \quad x \in H$$

is called the *Analysis operator* corresponding to the frame $\{x_i : i \in I\}$.

We are interested in unitary systems $\mathcal{U}$ of the form $\mathcal{U} = \mathcal{U}_0 G$, where $\mathcal{U}_0 = \{U^j : j \in \mathbf{Z}\}$ and G is an abelian unitary group. We will often refer to G as a *translation group.* Obviously unitary systems of this form generalize the affine system.

DEFINITION 1.1. *A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of an abstract Hilbert space H is a Generalized Frame Multiresolution Analysis of H if it is increasing, i.e. $V_j \subseteq V_{j+1}$ for every $j \in \mathbb{Z}$ and satisfies the following properties:*

(a) $V_j = U^j(V_0),\ j \in \mathbb{Z}$

(b) $\bigcap_j V_j = \{0\},\ \overline{\bigcup_j V_j} = H$

(c) There exists a countable subset B of V_0 such that the set $G(B) = \{g\phi : g \in G, \phi \in B\}$ is a frame of V_0. Every such set B is called a frame multiscaling set for $\{V_j\}_j$. Every subset C of V_1 such that $G(C) = \{g\psi : g \in G, \psi \in C\}$ is a frame of $W_0 := V_1 \cap V_0^{\perp}$ is called a semiorthogonal frame multiwavelet vector set associated with $\{V_j\}_j$.

If B is a singleton we refer to its unique element as a *frame scaling vector* and, if $H = L^2(\mathbb{R}^n)$, as a *frame scaling function.* We also let $W_j := U^j(W_0)$, for every $j \in \mathbb{Z}$. Note, that if C is a semiorthogonal frame multiwavelet vector set associated with the GFMRA $\{V_j\}_j$ then the set $\{D^j g\psi : j \in \mathbb{Z}, g \in G, \psi \in C\}$ is a frame for H with the same frame bounds as the frame $G(C)$.

DEFINITION 1.2. *An $n \times n$ invertible matrix A is expanding if all its entries are real and all its eigenvalues have modulus greater than 1. A Dilation matrix is an expanding matrix that leaves $\mathbb{Z}^n$ invariant, i.e. $A(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$.*

The multidimensional affine unitary systems we are interested in are the systems of the form $\mathcal{U}_0 G$, where $\mathcal{U}_0$ is the cyclic torsion free group generated by a dilation operator D defined by

$$Df(\mathbf{t}) = |detA|^{1/2} f(A\mathbf{t}), \quad f \in L^2(\mathbb{R}^n)$$

where A is a dilation matrix and $G = \{T_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^n\}$. Obviously, G is isomorphic with $\mathbb{Z}^n$. Using the definitions of translations and dilations one can easily verify $T_{\mathbf{k}} D = D T_{A\mathbf{k}}$. It is well known that the quotient group $\mathbb{Z}^n / A(\mathbb{Z}^n)$ contains exactly $|detA|$ elements. Now, set $\mathbf{q}_0 = \mathbf{0}$, $p := |detA|$ and fix $\mathbf{q}_r \in \mathbb{Z}^n$, for $r = 1, 2, \ldots, p-1$ so that

$$\mathbb{Z}^n / A(\mathbb{Z}^n) = \{\mathbf{q}_r + A(\mathbb{Z}^n) : r = 0, 1, \ldots, p-1\}.$$

The translation group G is induced by the lattice $\mathbb{Z}^n$. Although our results will be obtained with respect to this particular lattice only, our methods can be easily extended to all regular lattices, i.e. lattices of the form $C(\mathbb{Z}^n)$, where C is an $n \times n$ invertible matrix. Following the tradition of all papers on Harmonic and Fourier analysis, we give the definition of the Fourier transform on $L^1(\mathbb{R}^n)$:

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-2\pi i \mathbf{t} \cdot \xi} d\mathbf{t}, \quad \xi \in \mathbb{R}^n.$$

We reserve $\mathcal{F}$ to denote the Fourier transform on $L^2(\mathbb{R}^n)$. In addition, we adopt the notation $\mathbb{T}^n := [-1/2, 1/2)^n$. Last, but not least, if A is a subset of a topological vector space, then $[A]$ denotes its linear span and A^- denotes the closure of A and $e_{\mathbf{k}}(\xi) := e^{-2\pi i(\xi \cdot \mathbf{k})}$ for every $\xi \in \mathbb{R}^n$.

2. RADIAL FMRAS

In the present section we will develop the theory of singly generated GFMRAs of $L^2(\mathbb{R}^n)$ defined by radial frame scaling functions. We refer to these GFMRAs as *Radial FMRAs.* In this particular paper we will be exclusively using frame scaling functions whose Fourier transform is of the form χ_A, where A is a measurable set. The next lemma, which was first stated by Aldroubi, plays the key-role in our construction of Radial FMRAs, however it has also been instrumental in abstract frame theory.[17]

LEMMA 2.1. [17,18] *Let H be a Hilbert space and P be an orthogonal projection defined on H. If $\{\epsilon_i : i \in I\}$ is an orthonormal subset of H, or a Parseval frame of H, then $\{P\epsilon_i : i \in I\}$ is a Parseval frame of $P(H)$.*

Now, let $\mathbb{D}$ be the sphere with radius $1/2$ centered at the origin, and ϕ be such that $\hat{\phi} := \chi_{\mathbb{D}}$. Since the multiplication with ϕ defines an orthogonal projection on $L^2(\mathbb{R}^n)$, say $\hat{P}$. Using lemma 2.1 we conclude that $\{e_{\mathbf{k}}\chi_{\mathbb{D}} : \mathbf{k} \in \mathbb{Z}^n\}$ Parseval frame for the closed subspace it generates. Applying $\mathcal{F}^{-1}$ we obtain that $\{T_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for the closed subspace it generates. We denote this subspace with V_0. *From now on we will consider dilations induced by dilation matrices A satisfying the following property.*

Property D: There exists $c > 1$ such that for every $x \in \mathbb{R}^n$ we have $c\|x\| \leq \|Ax\|$.

Property D readily implies $\|A^{-1}\| \leq c^{-1} < 1$. However, it is interesting to note that Property D cannot be derived from the definition of dilation matrices as the example of $A = \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$ demonstrates.

Now, define $V_j := D^j(V_0)$, where $j \in \mathbb{Z}$. We will now establish $V_{-1} \subseteq V_0$. First, let $B := A^T$, where the superscript T denotes the transpose operation. Since $(A^T)^{-1} = (A^{-1})^T$ and the operator norm of a matrix is equal to the operator norm of its transpose, we obtain that dilation matrices A satisfying Property D, therefore, satisfies $\|B^{-1}\| < 1$. Thus $B^{-1}(\mathbb{D})$ is contained in $\mathbb{D}$. Next, let μ_0 be the measurable function defined on $\mathbb{R}^n$

such that $\mu_0(\xi) = \chi_{B^{-1}(\mathbb{D})}(\xi)$, for every $\xi \in \mathbb{T}^n$, which is periodically extended on $\mathbb{R}^n$ with respect to the tiling of $\mathbb{R}^n$ induced by the integer translates of $\mathbb{T}^n$. Then μ_0 belongs to $L^2(\mathbb{T}^n)$ and satisfies

$$\hat{\phi}(B\xi) = \mu_0(\xi)\hat{\phi}(\xi) \quad \text{a.e.},$$

because $\hat{\phi}(B\xi) = \chi_{B^{-1}(\mathbb{D})}(\xi)$, for every $\xi \in \mathbb{R}^n$. This implies that $D^*\phi$, belongs to V_0, which in turn establishes $V_{-1} \subseteq V_0$, and thus $V_j \subseteq V_{j+1}$, for every integer j. Since $\mathcal{F}(V_j) = L^2(B^j(\mathbb{D}))$, for all $j \in \mathbb{Z}$, we finally obtain that both properties in (b) of the definition of a GFMRA are satisfied. From the preceding argument we conclude that $\{V_j\}_j$ is a GFMRA of $L^2(\mathbb{R}^n)$, singly generated by the radial scaling function ϕ. So $\{V_j\}_j$ is a Radial FMRA of $L^2(\mathbb{R}^n)$. We may also occasionally refer to ϕ as a *Parseval frame scaling function* in order to indicate that $\{T_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for V_0. Following the terminology and the notation introduced in,[1] the analysis operator S induced by the frame scaling set $\{\phi\}$ maps V_0 into $L^2(\mathbb{T}^n)$ and is defined by

$$Sf = \sum_{\mathbf{k}\in\mathbb{Z}} \langle f,\, T_{\mathbf{k}}\phi\rangle e_{\mathbf{k}}.$$

Since ϕ is a Parseval frame scaling function we obtain that S is an isometry. Moreover it is not hard to verify that the range of S is the subspace $L^2(\mathbb{D})$.

According to Definition 3,[1] we define the low pass filter m_0 corresponding to ϕ is given by $m_0 := SD^*\phi$. Using the definition of S and taking the Fourier transforms of both sides of

$$D^*\phi = \sum_{\mathbf{k}\in\mathbb{Z}^n} \langle D^*\phi, T_{\mathbf{k}}\phi\rangle T_{\mathbf{k}}\phi$$

we obtain

$$\hat{\phi}(B\xi) = |detA|^{-1/2}\, m_0(\xi)\hat{\phi}(\xi) \qquad \text{a.e.} \tag{3}$$

Now, recall

$$\hat{\phi}(B\xi) = \mu_0(\xi)\hat{\phi}(\xi) \qquad \text{a.e.} \tag{4}$$

Unfortunately, the fact that the set of the integer translates of ϕ is not a basis for V_0, but an overcomplete frame, does not automatically imply $|detA|^{1/2}\,\mu_0 = m_0$. However, both m_0 and μ_0 vanish outside $\mathbb{D}$, so eqs. (3) and (4) imply

$$m_0(\xi) = |detA|^{1/2}\,\chi_{B^{-1}(\mathbb{D})}(\xi), \quad \xi \in \mathbb{T}^n. \tag{5}$$

Obviously, all radial functions of the form $\chi_{\mathbb{D}}$, where $\mathbb{D}$ is a sphere centered at the origin with radius $r < 1/2$, are radial Parseval frame scaling functions. We will not distinguish this particular case from the case $r = 1/2$, because the latter case is generic and also optimizes the frequency spectrum subject to subband filtering, induced by this particular selection of the scaling function ϕ. This frequency spectrum is equal to the support of the autocorrelation function of ϕ, because every signal in V_0 will be encoded by the Analysis operator with an $\ell^2(\mathbb{Z})$-sequence, whose Fourier transform has support contained in $\mathbb{D}$. Therefore, the frequency spectrum subject to subband filtering induced by $\{V_j\}_j$ equals $\mathbb{D}$. This suggests that a prefiltering step transforming a random digital signal into another signal whose frequency spectrum is contained in $\mathbb{D}$ is necessary prior to the application of the decomposition algorithm induced by $\{V_j\}_j$. This prefiltering step is called initialization of the input signal. In the light of these remarks one might wonder whether we may be able to increase the frequency spectrum that these FMRAs can filter by allowing $r > 1/2$. We proved (see Proposition 5[19]) that the selection $r = 1/2$ is optimum.

The frame scaling function can be determined in terms of Bessel functions, because it is a radial function.

$$\phi(R) = \frac{J_{\frac{n}{2}}(\pi R)}{(2R)^{\frac{n}{2}}}, \quad R > 0. \tag{6}$$

The proof of eq. (6) is contained in the proof of Lemma 2.5.1.[20]

We will not give any details regarding Bessel functions.[20–22] We only include the following formula:

$$J_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+a}}{k!\Gamma(k+a+1)}, \quad a > -1, \ x > 0.$$

The function J_a given by the above equation is called the *Bessel function of the first kind of order a.*

Apparently every function in V_0 is bandlimited, because its Fourier transform is supported on $\mathbb{D}$. Since $\mathbb{D}$ is contained in $\mathbb{T}^n$, we can readily infer from the classical sampling theorem that if f is in V_0, then

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}) T_{\mathbf{k}} \omega , \tag{7}$$

where the RHS of the previous equation converges in the L^2-norm and $\omega(x_1, x_2, \ldots, x_n) = \prod_{q=1}^{n} \frac{\sin(\pi x_q)}{\pi x_q}$. If P_0 is the projection onto V_0, then applying P_0 on both sides of eq. (7) gives

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}) P_0(T_{\mathbf{k}}\omega) = \sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}) T_{\mathbf{k}} P_0(\omega) ,$$

because P_0 commutes with the translation operator $T_{\mathbf{k}}$, for every $\mathbf{k} \in \mathbb{Z}^n$. Since $P_0(\omega) = \phi$, we conclude the following sampling theorem:

THEOREM 2.2. *Let f be in V_0. Then,*

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}) T_{\mathbf{k}} \phi , \tag{8}$$

where the RHS of equation (8) converges in the L^2 norm. Moreover, the same series converges uniformly to f, if we assume that f is continuous.

REMARK 1. Although ϕ is a radial function, its dilations $D^j\phi$, for $j \neq 0$ may cease to be radial, for if $j = -1$, then $\mathcal{F}(D^*\phi) = |\det A|^{1/2} \chi_{B^{-1}(\mathbb{D})}$ and $B^{-1}(\mathbb{D})$ may not be an isotropic domain. However, in several interesting cases of dilation matrices A all the dilations of ϕ are radial. The preceding remark motivates the following definition:

DEFINITION 2.3. *An expansive matrix A is called radially expansive if $A = aU$, where $a > 0$ and U is a unitary matrix.* Expansive matrices obviously satisfy $a^n = |detA|$ and $||A|| = a$. Apparently radially expansive dilation matrices satisfy Property D. When this is the case, we immediately obtain that all $D^j\phi$ are radial functions as well, and, in particular,

$$(D^{-1}\phi)(R) = \frac{J_{\frac{n}{2}}(\pi a^{-1} R)}{(2R)^{\frac{n}{2}}}, \quad R > 0. \tag{9}$$

Combining eqs. (3), (5) and (9) we conclude

$$\widehat{m_0}(\mathbf{k}) = \frac{J_{\frac{n}{2}}(\pi a^{-1} ||\mathbf{k}||)}{(2\,||\mathbf{k}||)^{\frac{n}{2}}}, \ \ \mathbf{k} \in \mathbb{Z}^n.$$

In the discussion that follows we present one construction of frame multiwavelet sets associated with $\{V_j\}_j$. The merit of this construction is that it does not depend on the dimension of the underlying Euclidean space $\mathbb{R}^n$. We will then discuss Decomposition and Reconstruction algorithms induced by $\{V_j\}_j$.

Generic algorithm for the construction of frame multiwavelet sets: We adopt the proof of Theorem 13 of[1] to the radial FMRA $\{V_j\}_j$. First, set $\hat{V}_j := \mathcal{F}(V_j)$ and $\hat{W}_j := \mathcal{F}(W_j)$, where $j \in \mathbb{Z}$. Recall that $\hat{V}_0 = \mathcal{F}(V_0) = L^2(\mathbb{D})$, and that the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$. Combining these facts with $\hat{V}_{-1} = L^2(B^{-1}(\mathbb{D}))$, we conclude

$$\hat{W}_{-1} = \hat{V}_0 \cap \hat{V}_{-1}^{\perp} = L^2(\mathcal{Q}),$$

where $\mathcal{Q}$ is the annulus $\mathbb{D} \cap (B^{-1}(\mathbb{D}))^c$, and the superscript c denotes the set-theoretic complement. Now Lemma 2.1 implies that the orthogonal projection defined on $L^2(\mathbb{T}^n)$ by multiplication with the indicator function of $\mathcal{Q}$ gives a Parseval frame for $L^2(\mathcal{Q})$, namely the set $\{e_{\mathbf{k}}\chi_{\mathcal{Q}} : \mathbf{k} \in \mathbb{Z}^n\}$.

Next, observe that each $\mathbf{k} \in \mathbb{Z}^n$ belongs to exactly one of the elements of the quotient group $\mathbb{Z}^n/A(\mathbb{Z}^n)$; thus there exist $\mathbf{q}$ and $r \in \{0, 1, \ldots, p-1\}$ such that $\mathbf{k} = \mathbf{q}_r + A(\mathbf{q})$. Therefore, $e_{\mathbf{k}} = e_{\mathbf{q}_r} e_{A(\mathbf{q})}$. We now define the following functions:

$$h_r := e_{\mathbf{q}_r}\chi_{\mathcal{Q}} \quad r \in \{0, 1, \ldots, p-1\}. \tag{10}$$

Apparently $\{e_{A(\mathbf{k})}h_r : \mathbf{k} \in \mathbb{Z}^n,\ r = 0, 1, \ldots, p-1\}$ is a Parseval frame for $L^2(\mathcal{Q})$, thus for $\hat{W}_{-1}$ as well. Therefore, $\{T_{A(\mathbf{k})}\mathcal{F}^{-1}h_r : \mathbf{k} \in \mathbb{Z}^n,\ r = 0, 1, \ldots, p-1\}$ is a Parseval frame for W_{-1}, because the Fourier transform is unitary. Setting $\psi_r := D\mathcal{F}^{-1}h_r$ $(r = 0, 1, \ldots, p-1)$ we finally have that $\{T_{\mathbf{k}}\psi_r : \mathbf{k} \in \mathbb{Z}^n,\ r = 0, 1, \ldots, p-1\}$ is a Parseval frame for W_0, therefore $\{\psi_r : r = 0, 1, \ldots, p-1\}$ is a Parseval frame multiwavelet set associated with the FMRA $\{V_j\}_j$. This concludes the construction of a frame multiwavelet set associated with $\{V_j\}_j$.

The reader might wonder whether it is possible to give a more explicit formula for the frame wavelets ψ_r. In the light of remark 1, ψ_0 may not be radial as well. This may yield a rather unattractive time domain formula for all these wavelets. It worths mentioning that ψ_r, where $r > 0$, are never radial if ψ_0 is radial. However, if A is a radially expansive dilation matrix and $a = \|A\|$, then

$$(\mathcal{F}^{-1}h_0)(R) = \frac{J_{\frac{n}{2}}(\pi R)}{(2R)^{\frac{n}{2}}} - \frac{J_{\frac{n}{2}}(\pi \frac{R}{a})}{(2aR)^{\frac{n}{2}}}, \quad R > 0.$$

Therefore, under this assumption, ψ_0 is radial and

$$\psi_0(R) = \frac{a^{\frac{n}{2}} J_{\frac{n}{2}}(\pi a R) - J_{\frac{n}{2}}(\pi R)}{(2aR)^{\frac{n}{2}}}, \quad R > 0;$$

and for $r = 1, 2, \ldots, p-1$.

$$\begin{aligned}\psi_r(t) &= DT_{\mathbf{q}_r}D^*\psi_0(t) = \psi_0(t - A^{-1}\mathbf{q}_r) \\ &= \frac{a^{\frac{n}{2}} J_{\frac{n}{2}}(\pi a \left\|t - A^{-1}\mathbf{q}_r\right\|) - J_{\frac{n}{2}}(\pi \left\|t - A^{-1}\mathbf{q}_r\right\|)}{(2a\left\|t - A^{-1}\mathbf{q}_r\right\|)^{\frac{n}{2}}}, \quad t \in \mathbb{R}^n.\end{aligned}$$

Notice that in this case $p = |detA| = a^n$.

This is not the only construction of frame multiwavelet sets associated with Radial FMRAs, but it is a very general one. Its main advantage is that it generates frame wavelets that are very symmetric. In fact, they may even be isotrpic if A is strictly expansive and in this case one of them is radial. This construction leads to a very interesting hybrid Fast Wavelet algorithm (see Proposition 2.4). Before discussing this algorithm we would like to conclude the discussion of the frame multiwavelet sets. If $n = 2$ and $A = 2I_2$, then a very interesting frame multiwavelet set associated with $\{V_j\}_j$ is $\{\psi_r : r = 0, 1, 2, 3\}$, where for every such r the Fourier transform of ψ_r, is the characteristic function of each of the four sets depicted in Fig. 2. The elements of the latter set are not as symmetric as those presented in the generic construction of the frame multiwavelet set associated with Radial FMRAs. If we allow frame multiwavelet sets with cardinality greater than four, then we can obtain more symmetric frame wavelets. If $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then a frame multiwavelet set associated with $\{V_j\}_j$ is $\{\psi_r : r = 0, 1\}$, where the Fourier transform of ψ_r $(r = 0, 1)$ is the characteristic function of each one of the two sets depicted in Fig. 2.

Let us now present the Isotropic Fast Wavelet algorithm induced by this multiresolution structure. Reflecting upon the generic construction of frame multiwavelet sets associated with a Radial FMRA, we infer the following:

PROPOSITION 2.4. *The set* $\{e_{A(\mathbf{k})}m_0 : \mathbf{k} \in \mathbb{Z}\} \cup \{e_{\mathbf{k}}\chi_{\mathcal{Q}} : \mathbf{k} \in \mathbb{Z}\}$ *is a Parseval frame for the space of all functions belonging to* $L^2(\mathbb{T}^n)$ *(so they are* $\mathbb{Z}^n$*-periodic) which vanish almost everywhere outside of* $\mathbb{D}$.

Proof of Proposition 2.4: Recall that S is an isometry and that its range is the space of all square-integrable $\mathbb{Z}^n$-periodic functions vanishing outside of $\mathbb{D}$. On the other hand, using the definition of S, it is not hard to see

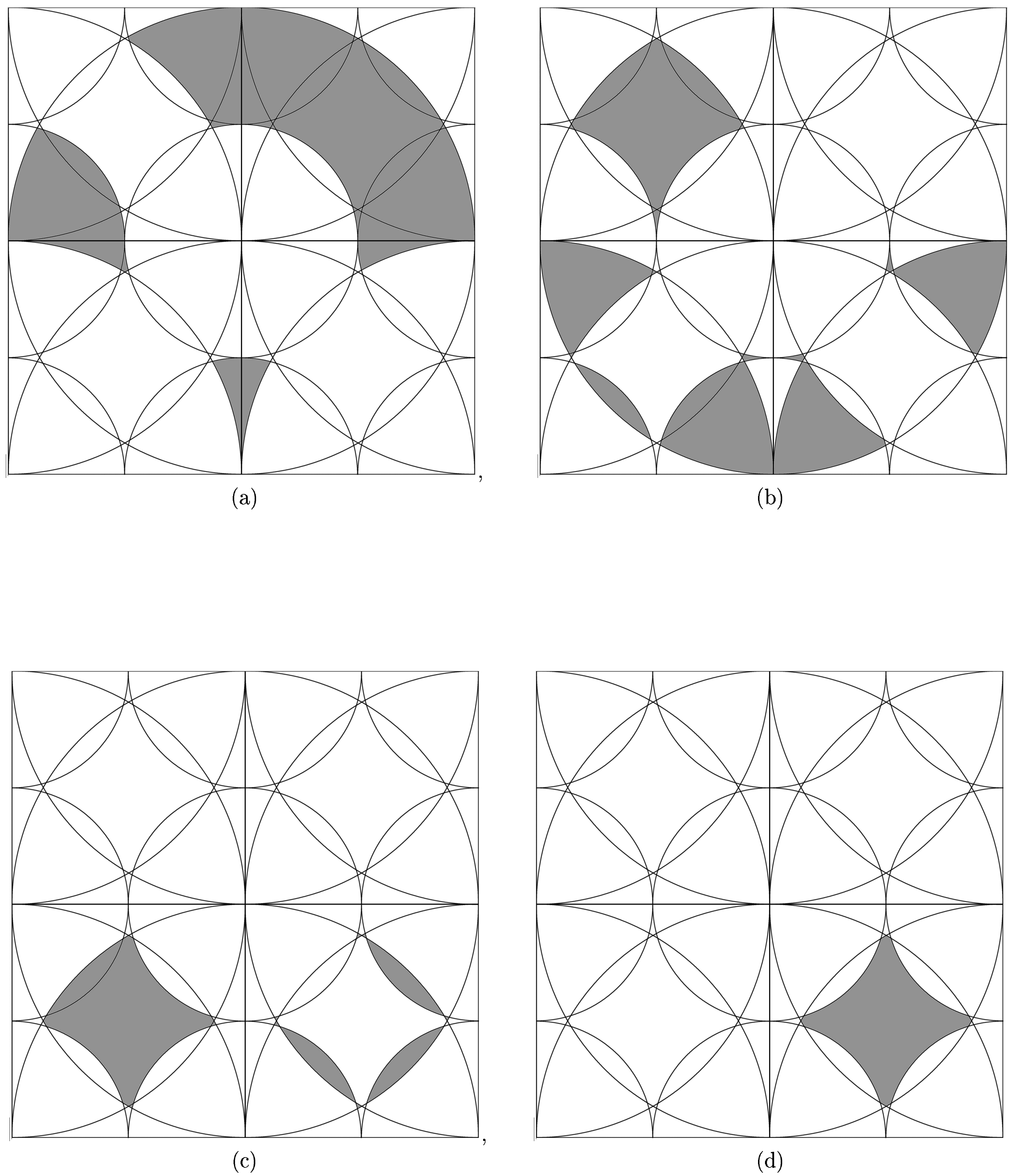
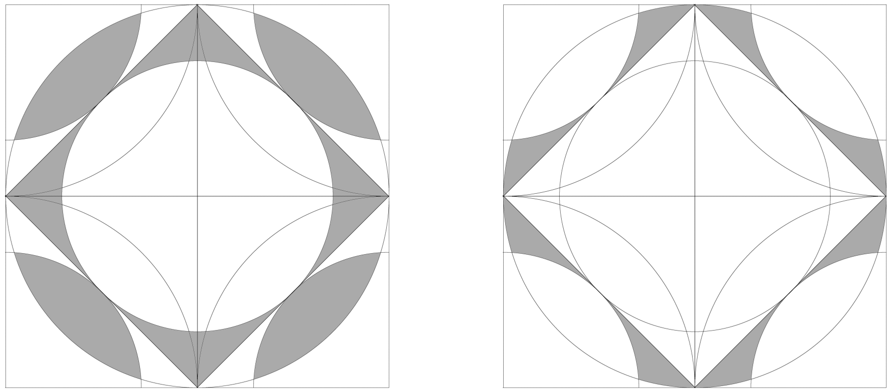


Figure 1. (a), (b), (c) and (d): Support of $\hat{\psi}_r$.



(a) (b)

Figure 2. (a), (b)

that $ST_{\mathbf{k}}f = e_{\mathbf{k}}Sf$, for every $f \in V_0$. Combining all these with the facts that $\{T_{A(\mathbf{k})}D^*\phi : \mathbf{k} \in \mathbb{Z}\}$ is a Parseval frame of V_{-1}, and $S(D^*\phi) = m_0$, we conclude that $\{e_{A(\mathbf{k})}m_0 : \mathbf{k} \in \mathbb{Z}\}$ is a Parseval frame of $S(V_{-1})$. The latter space contains all square-integrable $\mathbb{Z}^n$-periodic functions vanishing outside of $B^{-1}(\mathbb{D})$.

Now, recall that $\{e_{\mathbf{k}}\chi_{\mathcal{Q}} : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for the space of $\mathbb{Z}^n$-periodic functions vanishing outside of $\mathcal{Q}$. If we define m_1 to be the $\mathbb{Z}^n$-periodic function whose restriction on $\mathbb{T}^n$ is $\chi_{\mathcal{Q}}$, we readily infer that $\{e_{\mathbf{k}}m_1 : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for the space of all square-integrable $\mathbb{Z}^n$-periodic functions vanishing outside of $\mathcal{Q}$. The conclusion of Proposition 2.4 now follows from the fact that each square-integrable $\mathbb{Z}^n$-periodic function vanishing outside of $\mathbb{D}$ is the sum of two $\mathbb{Z}^n$-periodic functions, one vanishing outside of $B^{-1}(\mathbb{D})$ and another one vanishing outside of $\mathcal{Q}$.

The conclusion of Proposition 2.4 implies

$$f = \sum_{\mathbf{k}\in\mathbb{Z}} \langle f, e_{A(\mathbf{k})}m_0\rangle e_{A(\mathbf{k})}m_0 + \sum_{\mathbf{k}\in\mathbb{Z}} \langle f, e_{\mathbf{k}}m_1\rangle e_{\mathbf{k}}m_1 \ ,$$

for every square-integrable $\mathbb{Z}^n$-periodic function f vanishing outside $\mathbb{D}$. The previous equation is actually an exact reconstruction formula and it gives rise to decomposition and exact reconstruction algorithms. We refer to these algorithms as *Isotropic Fast Wavelet Algorithms*. Notice that since both, low and high pass filters have infinite length in the time domain we choose to implement the filtering processes in both the decomposition and reconstruction algorithms in the frequency domain. Another important feature of our decomposition algorithm is that low pass outputs are followed by downsampling, while the high pass outputs remains undecimated. The corresponding statement is true for the reconstruction algorithm as well. The reader can inspect the results of the application of the Isotropic Fast Wavelet Algorithms on two stil images, Barbara and King Phillip's of Macedonia royal emblem which was found in his burrial site in Vergina, Macedonia, Greece (see Figs. 3 (a), 5 (a)). We applied the Decomposition algorithm twice on each image. The reader may notice how edges are detected by the Isotropic Fast Wavelet Algorithm regardless of their orientation (see Figs. 4 (b,c); Figs. 6 (b)). Since the support of the Fourier transform of the scaling function ϕ is the disk $\mathbb{D}$, part of the power spectrum of an image may not vanish outside of $\mathbb{D}$ (the power spectrum of an image will always be supported on $\mathbb{T}^n$). This results in some loss of very high frequency content during the application of the algorithm. The observed loss of energy due to this reason is 0.2463% for Barbara and 0.0031% for King Phillip's emblem.

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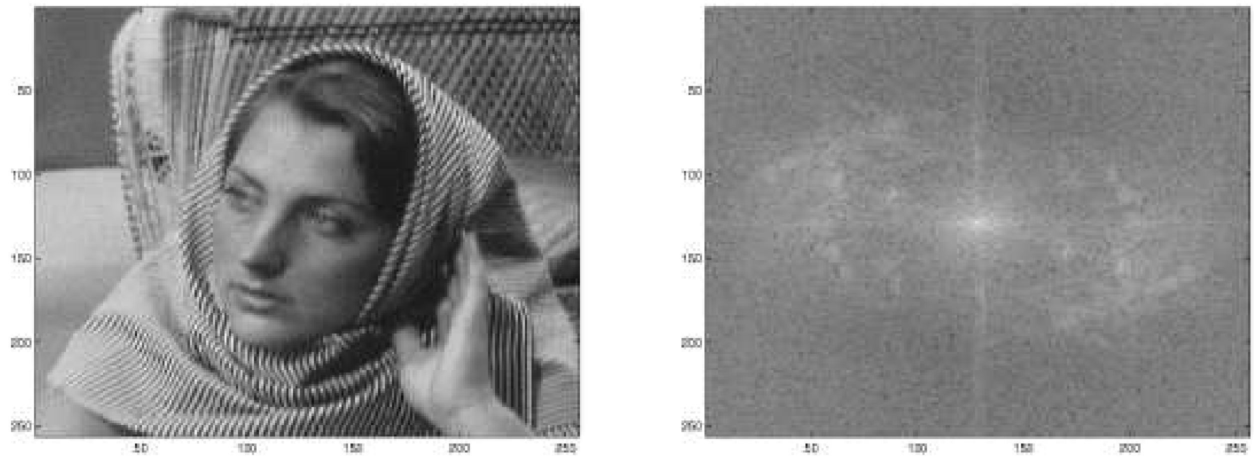

(a) (b)

Figure 3. (a) Barbara; original 256x256, (b) The power spectrum of (a)

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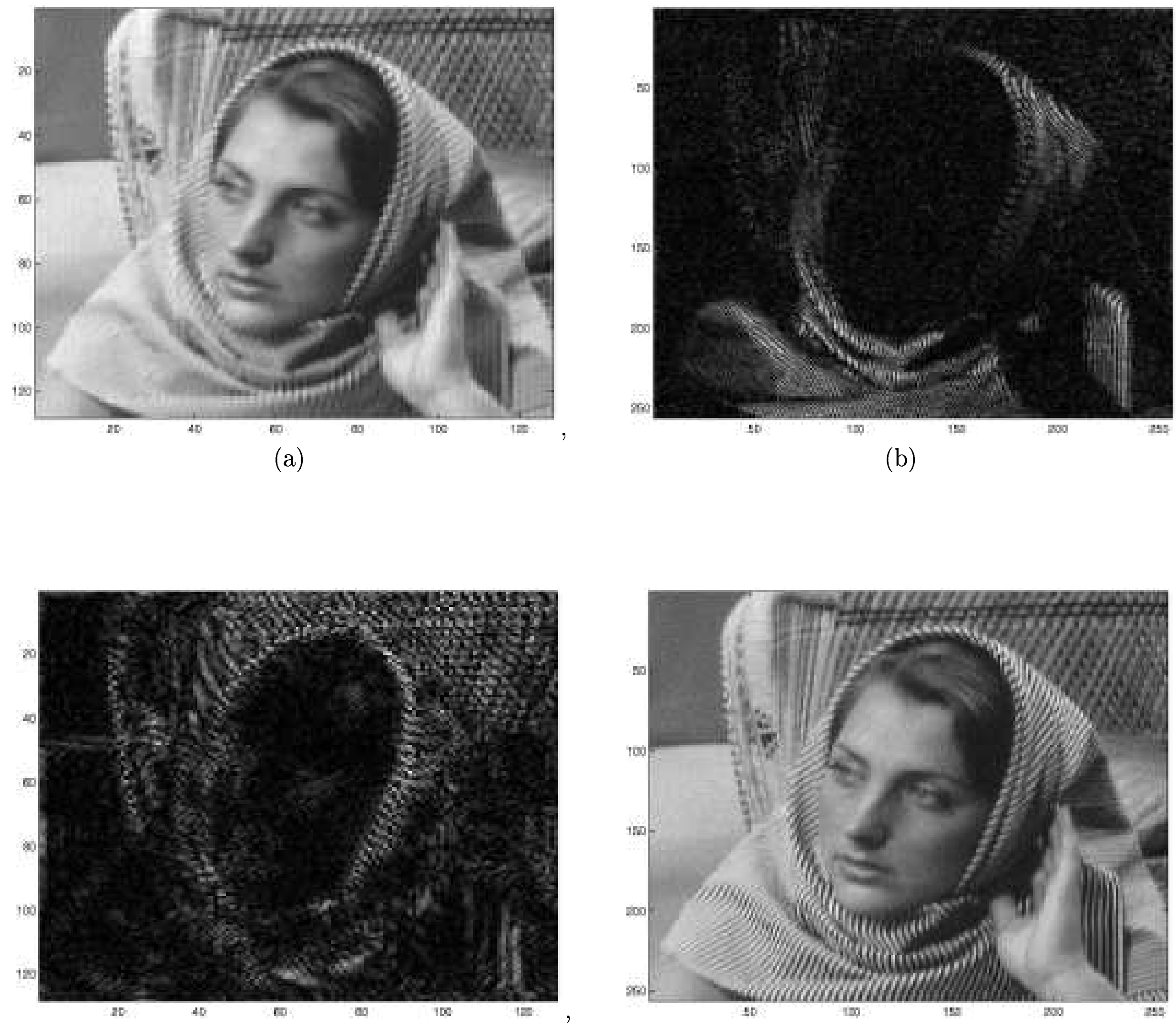
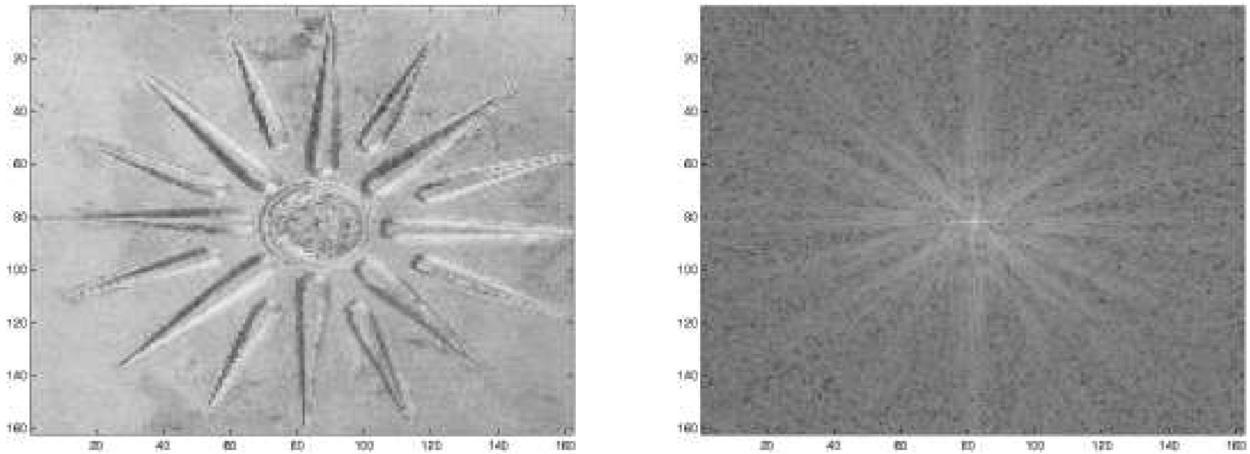


Figure 4. (a) Low pass, first iteration, (b) high pass, first iteration, (c) high pass, second iteration, (d) reconstructed image.



(a) (b)

Figure 5. (a) King Phillip's emblem ; original, (b) The power spectrum of (a)

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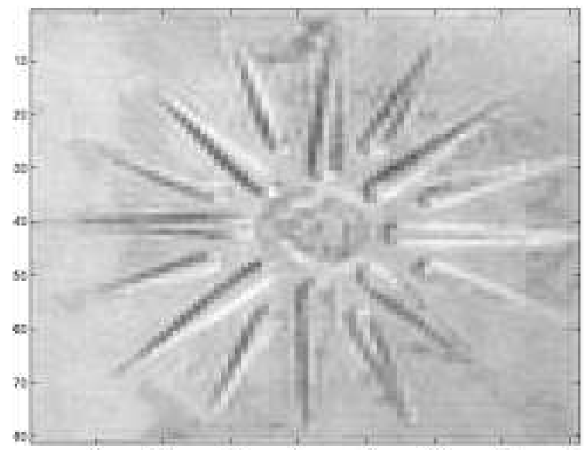
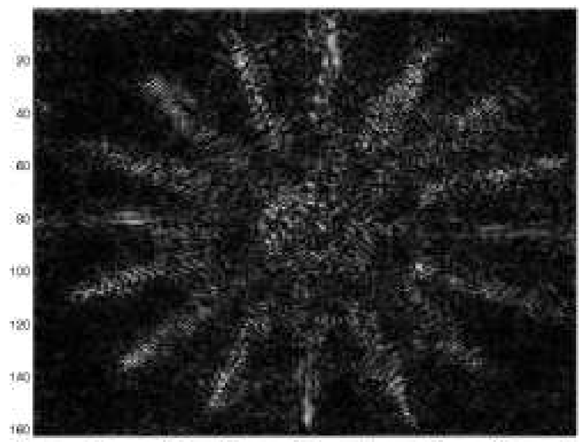
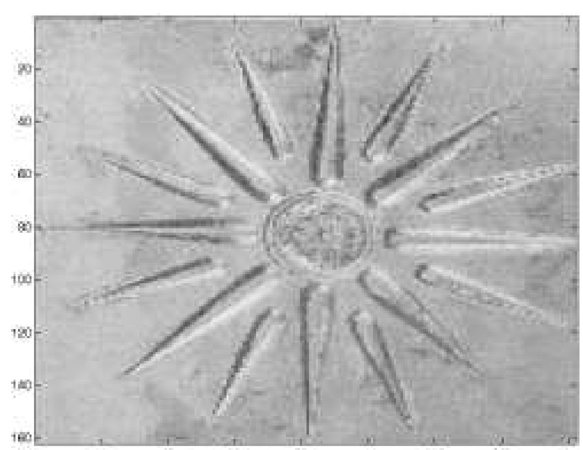


,

(a)



(b)



(c)

Figure 6. (a) Low pass, first iteration, (b) high pass, first iteration, (c) high pass, second iteration, (d) reconstructed image.