LARGE DEVIATIONS AND CENTRAL LIMIT THEOREMS FOR SEQUENTIAL AND RANDOM SYSTEMS OF INTERMITTENT MAPS

MATTHEW NICOL, FELIPE PEREZ PEREIRA, AND ANDREW TÖRÖK

Abstract. We obtain large and moderate deviations estimates for both sequential and random compositions of intermittent maps. We also address the question of whether or not centering is necessary for the quenched central limit theorems (CLT) obtained by Nicol, Török and Vaienti for random dynamical systems comprised of intermittent maps. Using recent work of Abdelkader and Aimino, Hella and Stenlund we extend the results of Nicol, Török and Vaienti on quenched central limit theorems (CLT) for centered observables over random compositions of intermittent maps: first by enlarging the parameter range over which the quenched CLT holds; and second by showing that the variance in the quenched CLT is almost surely constant (and the same as the variance of the annealed CLT) and that centering is needed to obtain this quenched CLT.

1. Introduction

The theory of limit laws and rates of decay of correlations for uniformly hyperbolic and some non-uniformly hyperbolic sequential and random dynamical systems has recently seen major progress. Results in this area include: in [CR07] strong laws of large numbers and centered central limit theorems for sequential expanding maps; in [AHN+15], polynomial decay of correlations for sequential intermittent systems; in [NTV18], sequential and quenched (self-centering) central limit theorems for intermittent systems; in [ANV15], annealed versions of a central limit theorem, large deviations principle, local limit theorem and almost sure invariance principle are proven for random expanding dynamical systems, as well as quenched versions of a central limit theorem, dynamical Borel-Cantelli lemmas, Erdős-Rényi laws and concentration inequalities; in [AA16], necessary and sufficient conditions are given for a central limit theorem without random centering for uniformly expanding maps; and in [BB16b] mixing rates and central limit theorems are given for random intermittent maps using a Tower construction. Recently the preprint [BBR17] considered quenched decay of correlation for slowly mixing systems and the preprint [AM18] used martingale techniques to obtain large deviations for systems with stretched exponential decay rates.

In this article we obtain large deviations estimates for both sequential and random compositions of intermittent maps. We also address the question of whether or not centering is necessary for the quenched central limit theorems (CLT) obtained in [NTV18] for random dynamical systems comprised of intermittent maps. More precisely, we consider in the first

---

Date: January 19, 2021.

2010 *Mathematics Subject Classification.* 37H99, 37A99, 60F10, 60F05, 60F99.

Key words and phrases. Large Deviations, Central Limit Theorems, Stationary Stochastic Processes, Random Dynamical Systems.

MN was supported in part by NSF Grant DMS 1600780. FPP thanks the University of Houston for hospitality while this work was completed. FPP was partially supported by the Becas Chile scholarship scheme from CONICYT and thanks the University of Houston for hospitality. AT was supported in part by NSF Grant DMS 1816315.

instance a fixed deterministically chosen sequence of maps $\ldots T_{\alpha_n}, \ldots, T_{\alpha_1}$ in the sequential case, or a randomly drawn sequence $\ldots T_{\omega_n}, \ldots, T_{\omega_1}$ with respect to a Bernoulli measure ν on $\Sigma := \{T_1, \ldots, T_k\}^{\mathbb{N}}$, where each of the maps T_j is a Liverani-Saussol-Vaienti [LSV99] intermittent map of form

$$T_{\alpha_j}(x) = \begin{cases} x + 2^{\alpha_j} x^{1+\alpha_j}, & 0 \le x \le 1/2, \\ 2x - 1, & 1/2 \le x \le 1 \end{cases},$$

for numbers $0 < \alpha_j \le \alpha < 1$. We consider the asymptotic behavior of the centered (that is, after substracting their expectation) sums

$$S_n := \sum_{k=1}^{n} \varphi \circ (T_{\alpha_k} \circ \ldots \circ T_{\alpha_1})$$

for sufficiently regular observables φ.

Denote by m Lebesgue measure on $X := [0,1]$, and by $m(\varphi)$ the integral of φ with respect to m. We will also consider the measure $\widetilde{m}$ given by $d\widetilde{m}(x) = x^{-\alpha} dm$, where $0 < \alpha_j \le \alpha < 1$. The motivation for introduction of this measure is that in the case of a stationary system, if $\alpha_k = \alpha$ for each k, then a natural and convenient measure to use is the invariant measure μ_α for T_α, which behaves near 0 as $x^{-\alpha}$. In the stationary case large deviation estimates are given with respect to μ_α and m in [MN08] for $\alpha < \frac{1}{2}$ and for all $0 \le \alpha < 1$ in [Mel09].

In the sequential case of a fixed realization we are interested in the large deviations of the self-centered sums:

$$m\left\{x : \frac{1}{n}\left|\sum_{k=1}^{n} \varphi \circ (T_{\alpha_k} \circ \ldots \circ T_{\alpha_1}) - \sum_{k=1}^{n} m(\varphi \circ T_{\alpha_k} \circ \ldots \circ T_{\alpha_1})\right| > \epsilon\right\}$$

for $\epsilon > 0$. We also obtain large deviations with respect to $\widetilde{m}$, which are in a sense sharper. In the sequential case centering is clearly necessary.

In the annealed case we consider the random dynamical system (RDS) $F \colon \Sigma \times [0,1] \to \Sigma \times [0,1]$ given by $F(\omega, x) = (\tau\omega, T_{\alpha_1} x)$ for $\omega = (\alpha_1, \alpha_2, \ldots) \in \Sigma$, where τ is the left-shift operator on Σ. For ν a Bernoulli measure on Σ, we suppose μ is a stationary measure for the stochastic process on $[0,1]$, that is, a measure such that $\nu \otimes \mu$ is F invariant. This assumption is valid in the setting we consider. If φ is an observable such that $\mu(\varphi) = 0$, we estimate

$$\nu \otimes \mu\left\{(\omega, x) : \frac{1}{n}\left|\sum_{k=1}^{n} \varphi \circ (T_{\alpha_k} \circ \ldots \circ T_{\alpha_1})\right| > \epsilon\right\}.$$

In the quenched case, once again assuming $\mu(\varphi) = 0$, we give bounds for

$$m\left\{x : \frac{1}{n}\left|\sum_{k=1}^{n} \varphi \circ (T_{\alpha_k} \circ \ldots \circ T_{\alpha_1})\right| > \epsilon\right\}$$

for ν-almost every realization $\omega \in \Sigma$.

Since the maps we are considering are not uniformly hyperbolic, spectral methods used to obtain limits laws are not immediately available. Our techniques to establish large and moderate deviations estimates are based on those developed for stationary systems, in particular the martingale methods of [MN08, Mel09].

Using recent work of [AA16] and [HS20] we extend the results of [NTV18] on quenched central limit theorems (CLT) for centered observables over random compositions of intermittent maps in two ways, first by enlarging the parameter range over which the quenched CLT holds and second by showing as a consequence of results in [HS20] that the variance in the quenched CLT is almost surely constant and equal to the variance of the annealed CLT.

We also study the necessity of centering to achieve a quenched CLT using ideas of [AA16] and [ANV15]. The work of [ANV15] together with our observations show that centering is necessary 'generically' (in a sense made precise later) to obtain the quenched CLT in fairly general hyperbolic situations.

Improvements of earlier results. With this paper we improve some results of [NTV18]:

- we show that the sequential CLT in [NTV18, Theorem 3.1], [HL19], holds for the sharp $\alpha < 1/2$ (from $\alpha < 1/9$) if the variance grows at the rate specified.
- we show that the CLT holds not only with respect to Lebesgue measure m but also for $d\widetilde{m} = x^{-\alpha} dm$, which scales at the origin as the invariant measure of T_α.
- in the case of quenched CLT's of [NTV18, Theorem 3.1], using results of Hella and Stenlund [HS20] we show that the variance σ^2_ω is almost-surely the same for any sequence of maps and equal to the annealed variance σ^2.

Remark 1.1. *After this work was finished we learned about a preprint by Korepanov and Leppänen* [KL20], *in which interesting related results are obtained.*

2. Notation and assumptions

Throughout this article, m denotes the Lebesgue measure on $X := [0, 1]$ and $\mathcal{B}$ the Borel σ-algebra on $[0, 1]$. We consider the family of intermittent maps given by

$$T_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha}, & 0 \le x \le 1/2, \\ 2x - 1, & 1/2 \le x \le 1 \end{cases}, \tag{2.1}$$

for $\alpha \in (0, 1)$.

For $\beta_k \in (0, 1)$ denote by $P_{\beta_k} = P_k \colon L^1(m) \to L^1(m)$ the transfer operator (or Ruelle-Perron-Frobenius operator) with respect to m associated to the map $T_{\beta_k} = T_k$, defined as the "pre-dual" of the Koopman operator $f \mapsto f \circ T_k$, acting on $L^\infty(m)$. The duality relation is given by

$$\int_X P_k f \; g \; dm = \int_X f \; g \circ T_k \; dm$$

for all $f \in L^1(m)$ and $g \in L^\infty(m)$ [BG97, Proposition 4.2.6]. For a fixed sequence $\{\beta_k\}$ such that $0 < \beta_k \le \alpha$ for all k, define

$$\begin{aligned} \mathcal{T}^\infty &:= \dots, T_{\beta_n}, \dots, T_{\beta_1} \\ \mathcal{T}^n_m &:= T_{\beta_n} \circ \dots \circ T_{\beta_m}, \qquad \mathcal{T}^n := \mathcal{T}^n_1 \\ \mathcal{P}^n_m &:= P_{\beta_n} \circ \dots \circ P_{\beta_m}, \qquad \mathcal{P}^n := \mathcal{P}^n_1 \end{aligned}$$

We will often write, for ease of exposition when there is no ambiguity, $T_{\beta_n} \circ \dots \circ T_{\beta_m}$ as $T_n \circ \dots \circ T_m$ and $P_{\beta_n} \circ \dots \circ P_{\beta_m}$ as $P_n \circ \dots \circ P_m$.

Since $L^1(m)$ is invariant under the action of the transfer operators, the duality relation extends to compositions

$$\int_X \mathcal{P}_k^n f \ g \ dm = \int_X f \ g \circ \mathcal{T}_k^n \ dm.$$

We will write $\mathbb{E}_m[\varphi|\mathcal{F}]$ for the conditional expectation of φ on a sub-σ-algebra $\mathcal{F}$ with respect to the measure m. To simplify notation we might write $\mathbb{E}$ for $\mathbb{E}_m$.

Remark 2.1. *In* [CR07, NTV18] *it is shown that*

$$\mathbb{E}_m[\varphi \circ \mathcal{T}^\ell | \mathcal{T}^{-k}\mathcal{B}] = \frac{P_k \circ \ldots \circ P_{\ell+1}(\varphi \cdot \mathcal{P}^\ell(\mathbf{1}))}{\mathcal{P}^k(\mathbf{1})} \circ \mathcal{T}^k \tag{2.2}$$

for $0 \leq \ell \leq k$.

One of the main tools to study sequential and random systems of intermittent maps is the use of cones (see [LSV99], [AHN$^+$15], [NTV18]). Define the cone $\mathcal{C}_2$ by

$$\mathcal{C}_2 := \{f \in \mathcal{C}^0((0,1]) \cap L^1(m) \mid f \geq 0, f \text{ non-increasing }, X^{\alpha+1}f \text{ increasing }, f(x) \leq ax^{-\alpha}m(f)\},$$

where $X(x) = x$ is the identity function and $m(f)$ is the integral of f with respect to m. In [AHN$^+$15] it is proven that for a fixed value of $\alpha \in (0,1)$, provided that the constant a is big enough, the cone $\mathcal{C}_2$ is invariant under the action of all transfer operators P_β with $0 < \beta \leq \alpha$.

Notation 2.2. *In general we will denote the transfer operator with respect to a non-singular*[1] *measure* μ *(not necessarily Lebesgue measure) by* P_μ*. Similarly, the (conditional) expectation will be denoted by* $\mathbb{E}_\mu$*.*

Denote the centering with respect to μ *of a function* $\varphi \in L^1(X,\mu)$ *by*

$$[\varphi]^\mu := \varphi - \frac{1}{\mu(X)} \int_X \varphi \, d\mu \tag{2.3}$$

In particular, for $g(x) := x^{-\alpha}$*, denote the measure* gm *by* $\widetilde{m}$*, the corresponding transfer operator by* $\widetilde{P} := P_{gm}$*, and the (conditional) expectation by* $\mathbb{E}_{\widetilde{m}} := \mathbb{E}_{gm}$*.*

Random dynamical systems. Now we introduce a randomized choice of maps: consider a finite family of intermittent maps of the form (2.1), indexed by a set $\Omega = \{\beta_1, \ldots, \beta_m\} \subset (0,\alpha)$. Given a probability distribution $\mathbb{P} = (p_1, \ldots, p_m)$ on Ω, define a Bernoulli measure $\mathbb{P}^{\otimes\mathbb{N}}$ on $\Sigma := \Omega^{\mathbb{N}}$ by $\mathbb{P}^{\otimes\mathbb{N}}\{\omega : \omega_{j_1} = \beta_{j_1}, \ldots, \omega_{j_k} = \beta_{j_k}\} = \prod_{i=1}^k p_{j_i}$ for every finite cylinder and extend to the sigma-algebra generated by the cylinders of Σ by Kolmogorov's extension theorem. This measure is invariant and ergodic with respect to the shift operator τ on Σ, $\tau \colon \Sigma \to \Sigma$ acting on sequences by $(\tau(\omega))_k = \omega_{k+1}$. We will denote $\mathbb{P}^{\otimes\mathbb{N}}$ by ν from now on.

For $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$ define $\mathcal{T}_\omega^n := T_{(\tau^n\omega)_1} \circ \ldots \circ T_{\omega_1} = T_{\omega_n} \circ \ldots \circ T_{\omega_1}$. The random dynamical system is defined as

$$\begin{aligned} F : \Sigma \times X &\to \Sigma \times X \\ (\omega, x) &\mapsto (\tau\omega, T_{\omega_1}x) \,. \end{aligned}$$

The iterates of F are given by $F^n(\omega, x) = (\tau^n(\omega), \mathcal{T}_\omega^n(x))$.

[1]The measure μ is non-singular for the transformation T if $\mu(A) > 0 \implies \mu(T(A)) > 0$.

We will also use Ω-indexed subscripts for random transfer operators associated to the maps T_{ω_i}, so that $P_{\omega_i} := P_{T_{\omega_i}}$. We will also abuse notation and write P_ω for P_{ω_1} if $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots)$.

A probability measure μ on X is said to be stationary with respect to the RDS F if

$$\mu(A) = \int_\Sigma \mu\left(T_{\omega_1}^{-1}(A)\right) d\nu(\omega) = \sum_{\beta\in\Omega} p_\beta \mu\left(T_\beta^{-1}(A)\right)$$

for every measurable set A, where p_β is the $\mathbb{P}$-probability of the symbol β. This is equivalent to the measure $\nu \otimes \mu$ being invariant under the transformation $F\colon \Sigma \times X \to \Sigma \times X$.

See Remark 4.5 about the existence and ergodicity of such a stationary measure in our setting.

The annealed transfer operator $P\colon L^1(m) \to L^1(m)$ is defined by averaging over all the transformations:

$$P = \sum_{\beta\in\Omega} p_\beta P_\beta = \int_\Sigma P_\omega \; d\nu(\omega).$$

This operator is "pre-dual" to the annealed Koopman operator $U\colon L^\infty(m) \to L^\infty(m)$ defined by

$$(U\varphi)(x) := \sum_{\beta\in\Omega} p_\beta \varphi(T_\beta x) = \int_\Sigma \varphi(T_\omega x) d\nu(\omega) = \int_\Sigma F(\tilde{\varphi})(\omega, x) d\nu(\omega)$$

where $\tilde{\varphi}(\omega, x) := \varphi(x)$. The annealed operators satisfy the duality relationship

$$\int_X (U\varphi) \cdot \psi \; dm = \int_X \varphi \cdot P\psi \; dm$$

for all observables $\varphi \in L^\infty(m)$ and $\psi \in L^1(m)$.

3. Background results and the Martingale approximation

In this section we describe the main technique used to prove some of the limit law results: the martingale approximation, introduced by Gordin [Gor69]. Since there is no common invariant measure for the set of maps $\{T_k\}$, for a given $\mathcal{C}^1$ observable φ we center along the orbit by

$$[\varphi]_k (\omega, x) := \varphi(x) - \int_X \varphi \circ \mathcal{T}_\omega^k \; dm,$$

with $\mathcal{T}_\omega^k = Id$ for $k = 0$.

This implies that $\mathbb{E}_m([\varphi]_k \circ \mathcal{T}^k) = 0$ and consequently the centered Birkhoff sums

$$\widehat{S}_n := \sum_{k=1}^{n} [\varphi]_k \circ \mathcal{T}^k,$$

have zero mean with respect to m. Following [NTV18], define

$$H_1 := 0 \text{ and } H_n \circ \mathcal{T}^n := \mathbb{E}_m(\widehat{S}_{n-1} | \mathcal{B}_n) \text{ for } n \geq 2 \tag{3.1}$$

and the (reverse) martingale sequence $\{M_n\}$ by

$$M_0 := 0 \text{ and } \widehat{S}_n = M_n + H_{n+1} \circ \mathcal{T}^{n+1},$$

where the filtration here is $\mathcal{B}_n = \mathcal{T}^{-n}\mathcal{B}$. Define $\psi_n \in L^1(m)$ by setting

$$\psi_n = [\varphi]_n + H_n - H_{n+1} \circ T_{n+1},$$

then $M_n - M_{n-1} = \psi_n \circ \mathcal{T}^n$ and we have that $\mathbb{E}(M_n|\mathcal{B}_{n+1}) = 0$. Thus $\{\psi_n \circ \mathcal{T}^n\}$ is a reverse martingale difference scheme. An explicit expression for H_n is given by

(3.2)

$$H_n = \frac{1}{\mathcal{P}^n\mathbf{1}} \left[P_n([\varphi]_{n-1} \mathcal{P}_{n-1}\mathbf{1}) + P_nP_{n-1}([\varphi]_{n-2} \mathcal{P}_{n-2}\mathbf{1}) + \ldots + P_nP_{n-1}\ldots P_1([\varphi]_0 \mathcal{P}_0\mathbf{1}) \right].$$

Remark 3.1. *The formulas derived so far with m being the Lebesgue measure actually hold for any measure μ that is non-singular for the transformations T_β considered. The conditional expectations $\mathbb{E}_\mu$ will be with respect to μ and the transfer operator P_μ will be with respect to the measure space (X, μ). In particular the centering will have the form*

$$[\varphi]_k(\omega, x) := \varphi(x) - \frac{1}{\mu(X)} \int_X \varphi \circ \mathcal{T}_\omega^k \, d\mu,$$

but all other equations are the same, with the notational changes just described.

We collect and extend some results from [NTV18] concerning the properties of H_n, as well as the non-stationary decay of correlations for the sequential system.

We state first a few formulas for changing from a measure m to the measure $g(x)\,dm(x)$ with $g \in L^1(m)$; for simplicity, we denote this new measure as gm when there is no possibility of confusion.

Lemma 3.2 (Change of measure)**.** *We state this result only for the situation we need, but it holds also for any measure μ non-singular with respect to T in place of m the Lebesgue measure, and instead of $g(x) = x^{-\alpha}$ for any $g \in L^1(\mu)$, $g > 0$.*

Note that $L^1(gm) = g^{-1}L^1(m)$, so all formulas below make sense for φ in the appropriate L^1-space.

We have:

$$\begin{aligned} m(\varphi) &= m(P_m\varphi) \\ P_{gm}(\varphi) &= g^{-1}P_m(g\varphi) \\ g\,[\varphi]^{gm} &= [g\varphi]^m - \frac{m(g\varphi)}{m(g)}\,[g]^m \\ \mathbb{E}_{gm}(\varphi|\mathcal{B}) &= \mathbb{E}_m(g\varphi|\mathcal{B})/\mathbb{E}_m(g|\mathcal{B}) \end{aligned} \tag{3.3}$$

Therefore

$$(\mathcal{P}_{gm})_\ell^k([\varphi]^{gm}) = g^{-1}(\mathcal{P}_m)_\ell^k\left([g\varphi]^m - \frac{m(g\varphi)}{m(g)}\,[g]^m\right) \tag{3.4}$$

Proof. The first two properties are standard and follow from the definition of the transfer operator. The third is a direct computation using the notation (2.3).

For the fourth, $\mathbb{E}_{gm}(\varphi|\mathcal{B})$ is the function Φ that is $\mathcal{B}$-measurable and $\int \Phi\psi \, d(gm) = \int \varphi\psi \, d(gm)$ for each $\psi \in L^\infty(\mathcal{B})$. Expanding the LHS,

$$\int \Phi\psi \, d(gm) = \int \Phi\psi g \, dm = \int \Phi\psi\mathbb{E}_m(g|\mathcal{B}) \, dm$$

whereas the RHS becomes

$$\int \varphi\psi \, d(gm) = \int \varphi\psi g \, dm = \int \mathbb{E}_m(g\varphi|\mathcal{B})\psi \, dm$$

Thus $\Phi\mathbb{E}_m(g|\mathcal{B}) = \mathbb{E}_m(g\varphi|\mathcal{B})$, as claimed. $\square$

Proposition 3.3 ([NTV18])**.** *If φ, ψ are both in the cone $\mathcal{C}_2$ and have the same mean, $\int_X \varphi dm = \int_X \psi dm$, then by* [NTV18, Theorem 1.2]

$$\|\mathcal{P}^n(\varphi) - \mathcal{P}^n(\psi)\|_{L^1(m)} \leq C_\alpha(\|\varphi\|_{L^1(m)} + \|\psi\|_{L^1(m)})n^{-\frac{1}{\alpha}+1}(\log n)^{\frac{1}{\alpha}}$$

Moreover [NTV18, Remark 2.5 and Corollary 2.6]*, for $\varphi \in \mathcal{C}^1$, $h \in \mathcal{C}_2$ and any sequence of maps $\mathcal{T}^\infty$:*

$$\|\mathcal{P}^n([h\varphi]^m)\|_{L^1(m)} \leq C_\alpha \mathcal{F}\left(\|\varphi\|_{\mathcal{C}^1} + m(h)\right) n^{-\frac{1}{\alpha}+1}(\log n)^{\frac{1}{\alpha}}$$

where C_α depends only on the map T_α, and $\mathcal{F}\colon \mathbb{R} \to \mathbb{R}$ is an affine function.

The decay result of Proposition 3.3 for products of elements in the cone with C^1 observables (see also [LSV99, Theorem 4.1]), follows from Lemma 3.4, which was stated in [LSV99, proof of Theorem 4.1]. The proof of Lemma 3.4 is given in the Appendix; a different – less transparent – proof is given in [NTV18, Lemma 2.4].

Lemma 3.4. *Suppose $\varphi \in \mathcal{C}^1$ and $h \in \mathcal{C}_2$. Then there exist constants $\lambda, A, B \in \mathbb{R}$ such that $(\varphi + A + \lambda x)h + B$ and $(A + \lambda x)h + B$ are both in $\mathcal{C}_2$ and hence if $\int \varphi h dm = 0$ then $\|\mathcal{P}^j(\varphi h)\|_{L^1(m)} \leq C\rho(j)\|\varphi h\|_{L^1(m)}$ where $\rho(j)$ is the $L^1(m)$-decay for centered functions from the cone $\mathcal{C}_2$.*

Note that in our setting $\rho(j) = j^{-\frac{1}{\alpha}+1}(\log j)^{\frac{1}{\alpha}}$.

A consequence of Proposition 3.3 is the non-stationary decay of correlations ([NTV18, Page 1130])

$$\left|\int_X \varphi \cdot \psi \circ T_{\omega_n} \circ \ldots \circ T_{\omega_1} dm - m(\varphi) \cdot m(\psi \circ T_{\omega_n} \circ \ldots \circ T_{\omega_1})\right|$$
$$\leq \|\psi\|_\infty \|\mathcal{P}^n_\omega(\varphi) - \mathcal{P}^n_\omega(\mathbf{1}\int_X \varphi dm)\|_{L^1(m)}$$

We derive next decay estimates with respect to the measure $\widetilde{m}$, which are better in L^p, $p > 1$, than those for m.

Proposition 3.5. *For $\varphi : [0,1] \to \mathbb{R}$ bounded, $h \in \mathcal{C}_2$ and $1 \leq p \leq \infty$:*

$$\|\widetilde{\mathcal{P}}^n\,(\varphi)\,\|_{L^\infty(\widetilde{m})} \leq m(g)\|\varphi\|_{L^\infty(\widetilde{m})} \tag{3.5}$$

For $\varphi \in C^1([0,1])$, $h \in \mathcal{C}_2$

$$\|\widetilde{\mathcal{P}}^n\left(\left[(g^{-1}h)\varphi\right]^{\widetilde{m}}\right)\|_{L^1(\widetilde{m})} \leq C_\alpha \mathcal{F}\left(\|\varphi\|_{\mathcal{C}^1} + m(h)\right) n^{-\frac{1}{\alpha}+1}(\log n)^{\frac{1}{\alpha}} \tag{3.6}$$

and therefore, if $1 \leq p \leq \infty$,

(3.7)
$$\|\widetilde{\mathcal{P}}^n\left(\left[(g^{-1}h)\varphi\right]^{\widetilde{m}}\right)\|_{L^p(\widetilde{m})} \leq C_\alpha^{\frac{1}{p}}\left(m(g)\|\varphi\|_{L^\infty(\widetilde{m})}\right)^{1-\frac{1}{p}} \mathcal{F}^{\frac{1}{p}}\left(\|\varphi\|_{\mathcal{C}^1} + m(h)\right) n^{\frac{1}{p}\left(-\frac{1}{\alpha}+1\right)}(\log n)^{\frac{1}{p\alpha}}$$

where C_α depends only on T_α and $\mathcal{F}$ is an affine function.

Note that the L^1 and L^p bounds are relevant only for $\varphi \in C^1$.

Proof. The L^1 and L^∞ bounds give (3.7), since

$$\|f\|_{L^p} \le \|f\|_{L^\infty}^{1-\frac{1}{p}} \|f\|_{L^1}^{\frac{1}{p}} \tag{3.8}$$

because

$$\int |f|^p \le \int \|f\|_{L^\infty}^{p-1} |f| = \|f\|_{L^\infty}^{p-1} \|f\|_{L^1}.$$

To prove the L^∞ estimate (3.5) note that by the invariance of the cone $\mathcal{C}_2$, $\mathcal{P}^n(g) \in \mathcal{C}_2$, so $\mathcal{P}^n(g) \le x^{-\alpha} m(\mathcal{P}^n(g)) = x^{-\alpha} m(g)$. That is, using (3.3),

$$\widetilde{\mathcal{P}}^n(\mathbf{1}) = g^{-1}\mathcal{P}^n(g) \le m(g)$$

Since $-\|\varphi\|_{L^\infty}\mathbf{1} \le \varphi \le \|\varphi\|_{L^\infty}\mathbf{1}$ and $\widetilde{\mathcal{P}}^n$ are positive operators, we obtain (3.5).

For (3.6) assume that $\varphi \in C^1$ (otherwise it is clearly satisfied). In view of (3.4):

$$\begin{aligned}
\|\widetilde{\mathcal{P}}^n\left(\left[(g^{-1}h)\varphi\right]^{\widetilde{m}}\right)\|_{L^1(\widetilde{m})} &= \|g^{-1}\mathcal{P}^n([h\varphi]^m) - \frac{m(g\varphi)}{m(g)} g^{-1}\mathcal{P}^n([g]^m)\|_{L^1(\widetilde{m})} \\
&= \|\mathcal{P}^n([h\varphi]^m) - \frac{m(g\varphi)}{m(g)} \mathcal{P}^n([g]^m)\|_{L^1(m)} \qquad (3.9)\\
&\le \|\mathcal{P}^n([h\varphi]^m)\|_{L^1(m)} + \left|\frac{m(g\varphi)}{m(g)}\right| \|\mathcal{P}^n([g]^m)\|_{L^1(m)}
\end{aligned}$$

By [NTV18, Lemma 2.3], there is an affine function $\mathcal{F}: \mathbb{R} \to \mathbb{R}$ such that for $\varphi \in C^1([0,1])$ and $h \in \mathcal{C}_2$ can write $[\varphi h]^m = \Psi_1 - \Psi_2$ with $\Psi_1, \Psi_2 \in \mathcal{C}_2$ and $\|\Psi_{1,2}\|_{L^1(m)} \le \mathcal{F}(\|\varphi\|_{C^1} + m(h))$. By [NTV18, Theorem 1.2], for an observable ψ in the cone $\mathcal{C}_2$ and for any sequence of maps $\mathcal{T}^\infty$, we have

$$\int_X |\mathcal{P}^n([\psi]^m)| dm \le C_\alpha \|\psi\|_{L^1(m)} n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}$$

where C_α depends only on T_α. Applying these to (3.9), we obtain (3.6). □

Lemma 3.6. *Let $\varphi \in C^1$ and $0 < \alpha < 1$. Then*

$$\|H_n \circ \mathcal{T}^n\|_{L^p(m)} \le \begin{cases} C_{\alpha, \|\varphi\|_{C^1} + m(g)} (\log n)^{1+\frac{1}{1-\alpha}} & \text{if } 1 \le p = \frac{1}{\alpha} - 1 \\ \frac{1}{1-\frac{1}{p}\left(\frac{1}{\alpha}-1\right)} C_{\alpha, \|\varphi\|_{C^1} + m(g)} n^{1+\frac{1}{p}(1-\frac{1}{\alpha})} (\log n)^{\frac{1}{p\alpha}} & \text{if } p > \max\{1, \frac{1}{\alpha} - 1\} \end{cases}$$

(the first case is valid for $0 < \alpha \le \frac{1}{2}$) and the same bounds hold for $\|\widetilde{H}_n \circ \mathcal{T}^n\|_{L^p(\widetilde{m})}$, where

$$H_n \circ \mathcal{T}^n := \mathbb{E}_m([S_{n-1}]^m | \mathcal{B}_n),\ \widetilde{H}_n \circ \mathcal{T}^n := \mathbb{E}_{\widetilde{m}}([S_{n-1}]^{\widetilde{m}} | \mathcal{B}_n),\ \mathcal{B}_n := \mathcal{T}^{-n}\mathcal{B}.$$

Note that if $1 \le p < \frac{1}{\alpha} - 1$, then $\|H_n \circ \mathcal{T}^n\|_{L^p(m)} \le C_{p,\alpha,\|\varphi\|_{C^1}+m(g)}$, though this observation does not play a role in our subsequent analysis.

Proof. We prove the statement for $\widetilde{H}_n$. The one for H_n is obtained the same way, using Proposition 3.3 instead of (3.6).

Using the definition of $\widetilde{H}_n$:

$$\|\widetilde{H}_n \circ \mathcal{T}^n\|_{L^p(\widetilde{m})} = \|\sum_{k=1}^{n-1} \mathbb{E}_{\widetilde{m}}(\left[\varphi \circ \mathcal{T}^k\right]^{\widetilde{m}} | \mathcal{B}_n)\|_{L^p(\widetilde{m})} \le \sum_{k=1}^{n-1} \|\mathbb{E}_{\widetilde{m}}(\left[\varphi \circ \mathcal{T}^k\right]^{\widetilde{m}} | \mathcal{B}_n)\|_{L^p(\widetilde{m})} \tag{3.10}$$

We will bound each term of the above sum in both L^1 and L^∞, and then use (3.8) to obtain an L^p-bound.

In L^∞ we have

$$\|\mathbb{E}_{\widetilde{m}}(\left[\varphi \circ \mathcal{T}^k\right]^{\widetilde{m}} |\mathcal{B}_n)\|_{L^\infty(\widetilde{m})} \leq \| \left[\varphi \circ \mathcal{T}^k\right]^{\widetilde{m}} \|_{L^\infty(\widetilde{m})} \leq 2\|\varphi\|_{L^\infty(\widetilde{m})}.$$

In L^1 we use (2.2) to compute the conditional expectation. Since the conditional expectation preserves the expected value, one can check that the centering holds as written below[2]. We can then use (3.6) for the decay, with $h = \mathcal{P}^k(g)$, because $\widetilde{\mathcal{P}}^k(\mathbf{1}) = g^{-1}\mathcal{P}^k(g)$.

$$\begin{aligned}
&\|\mathbb{E}_{\widetilde{m}}(\left[\varphi \circ \mathcal{T}^k\right]^{\widetilde{m}} |\mathcal{B}_n)\|_{L^1(\widetilde{m})} = \|\frac{\widetilde{P}_n \circ \ldots \circ \widetilde{P}_{k+1}(\left[\varphi \cdot \widetilde{\mathcal{P}}^k(\mathbf{1})\right]^{\widetilde{m}})}{\widetilde{\mathcal{P}}^n(\mathbf{1})} \circ \mathcal{T}^n\|_{L^1(\widetilde{m})} \\
&= \|\widetilde{P}_n \circ \ldots \circ \widetilde{P}_{k+1}(\left[\varphi \cdot \widetilde{\mathcal{P}}^k(\mathbf{1})\right]^{\widetilde{m}})\|_{L^1(\widetilde{m})} = \|\widetilde{P}_n \circ \ldots \circ \widetilde{P}_{k+1}(\left[\varphi \cdot g^{-1}\mathcal{P}^k(g)\right]^{\widetilde{m}})\|_{L^1(\widetilde{m})} \\
&\leq C_\alpha \mathcal{F}_1(\|\varphi\|_{C^1} + m(\mathcal{P}^k(g)))(n-k)^{-\frac{1}{\alpha}+1}(\log(n-k))^{\frac{1}{\alpha}}.
\end{aligned}$$

Note that $m(\mathcal{P}^k(g)) = m(g)$, so the coefficient above does not depend on k.

Apply now (3.8), noting that $\|f\|_\infty^{1-\frac{1}{p}} \leq \max\{1, \|f\|_\infty\}$, to obtain for $1 \leq p \leq \infty$ that

$$\|\mathbb{E}_{\widetilde{m}}(\left[\varphi \circ \mathcal{T}^k\right]^{\widetilde{m}} |\mathcal{B}_n)\|_{L^p(\widetilde{m})} \leq C_{\alpha, \|\varphi\|_{C^1}+m(g)} \left[(n-k)^{-\frac{1}{\alpha}+1}(\log(n-k))^{\frac{1}{\alpha}}\right]^{\frac{1}{p}}$$

If $p = \frac{1}{\alpha} - 1 \geq 1$ we bound the last sum in (3.10) by $\sum_{k=1}^{n-1} C_{\alpha, \|\varphi\|_{C^1}+m(g)} \left[k^{-1}(\log(n))^{\frac{1}{p\alpha}}\right]$ to obtain

$$\|\widetilde{H}_n \circ \mathcal{T}^n\|_{L^p(\widetilde{m})} \leq C_{\alpha, \|\varphi\|_{C^1}+m(g)}(\log n)^{1+\frac{1}{1-\alpha}}.$$

If $p > \max\{1, \frac{1}{\alpha} - 1\}$ we bound the sum in (3.10) by $\sum_{k=1}^{n-1} C_{\alpha, \|\varphi\|_{C^1}+m(g)} \left[k^{-\frac{1}{\alpha}+1}(\log(n))^{\frac{1}{\alpha}}\right]^{\frac{1}{p}}$ to obtain the bound

$$\|\widetilde{H}_n \circ \mathcal{T}^n\|_{L^p(\widetilde{m})} \leq \frac{1}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)} C_{\alpha, \|\varphi\|_{C^1}+m(g)} n^{1+\frac{1}{p}(1-\frac{1}{\alpha})}(\log n)^{\frac{1}{p\alpha}}.$$

Note that if $1 \leq p < \frac{1}{\alpha} - 1$ the series converges to a constant $C_{p,\alpha,\|\varphi\|_{C^1}+m(g)}$. $\square$

A useful remark is the following lower bound for functions in the cone $\mathcal{C}_2$:

Proposition 3.7 ([LSV99, Lemma 2.4]). *For every function $f \in \mathcal{C}_2$ one has*

$$\inf_{x \in [0,1]} f(x) = f(1) \geq \min\left\{a, \left[\frac{\alpha(1+\alpha)}{a^\alpha}\right]^{\frac{1}{1-\alpha}}\right\} m(f).$$

Denote the constant in the above expression by D_α. Then $\mathcal{P}^n\mathbf{1} \geq D_\alpha > 0$ for all $n \geq 1$.

We will also use Rio's inequality, taken from [MPU06]. This is a concentration inequality that allows us to bound the moments of Birkhoff sums.

[2] $\widetilde{m}(\varphi \cdot \widetilde{\mathcal{P}}^k(\mathbf{1})) = \widetilde{m}(\varphi \circ \mathcal{T}^k)$ because, by the definition of the transfer operator, $\int \varphi \cdot \widetilde{\mathcal{P}}^k(\mathbf{1}) d\widetilde{m} = \int \varphi \circ \mathcal{T}^k \cdot \mathbf{1} d\widetilde{m}$

Proposition 3.8 ([MPU06, Rio17]). *Let $\{X_i\}$ be a sequence of L^2 centered random variables with filtration $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$. Let $p \geq 1$ and define*

$$b_{i,n} = \max_{i \leq u \leq n} \|X_i \sum_{k=i}^{u} \mathbb{E}(X_k | \mathcal{F}_i)\|_{L^p},$$

then

$$\mathbb{E}|X_1 + \ldots + X_n|^{2p} \leq \left(4p \sum_{i=1}^{n} b_{i,n}\right)^p.$$

4. Polynomial large and moderate deviations estimates

4.1. **Sequential dynamical systems.** Recall we fixed a sequence $\mathcal{T}^\infty = \ldots T_{\alpha_n}, \ldots, T_{\alpha_1}$ where each of the maps is of the form

$$T_{\alpha_j}(x) = \begin{cases} x + 2^{\alpha_j} x^{1+\alpha_j}, & 0 \leq x \leq 1/2, \\ 2x - 1, & 1/2 \leq x \leq 1 \end{cases},$$

for $0 < \alpha_j \leq \alpha < 1$. In the first part of this section we prove that for such a fixed sequence of maps $\mathcal{T}^\infty$, a polynomial large deviations bound holds for the centered sums.

Theorem 4.1 (Sequential LD). *Let $0 < \alpha < 1$ and $\varphi \in \mathcal{C}^1([0,1])$. Then the centered sums satisfy the following large deviations upper bound: for any $\epsilon > 0$ and $p > \max\{1, \frac{1}{\alpha} - 1\}$,*

$$m\left\{x : \left|\sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right| > n\epsilon\right\} \leq \left(\frac{4p}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)}\right)^p C^p_{\alpha,\|\varphi\|_{C^1}} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2p}$$

where $C = C_{\alpha,\|\varphi\|_{C^1}}$ is a constant depending on α and the C^1 norm of φ, but not *on the sequence $\mathcal{T}^\infty$.*

In particular, for $p > \max\{1, \frac{1}{\alpha} - 1\}$ we obtain the following moment estimate:

$$\mathbb{E}_m \left|\sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right|^{2p} \leq \left(\frac{4p}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)}\right)^p C^p_{\alpha,\|\varphi\|_{C^1}} n^{2p+(1-\frac{1}{\alpha})} (\log n)^{\frac{1}{\alpha}} \tag{4.1}$$

The same estimates (by the same proof) hold for the measure $\widetilde{m}$. More precisely,

$$\widetilde{m}\left\{x : \left|\sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - \widetilde{m}(\varphi(\mathcal{T}^j))]\right| > n\epsilon\right\} \leq \left(\frac{4p}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)}\right)^p \widetilde{C}^p_{\alpha,\|\varphi\|_{C^1}} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2p}$$

Remark 4.2. *Our result gives that the dependence on ϵ is better in the case $\alpha > \frac{1}{2}$, where we may take $p \to 1$ to obtain*

$$m\left\{x : \left|\sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right| > n\epsilon\right\} \leq \tilde{C}_{\alpha,\|\varphi\|_{C^1}} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2}$$

where $\tilde{C}_{\alpha,\|\varphi\|_{C^1}} = \frac{4\alpha}{2\alpha-1} C_{\alpha,\|\varphi\|_{C^1}}$. The worse bound for $\alpha < \frac{1}{2}$ is probably an artefact of our proof, and not an optimal result.

Remark 4.3. *In* [Mel09, Corollary A.2]*, improving* [MN08]*, these bounds are shown to be basically optimal if a single map T_α, $0<\alpha<1$, is being iterated, with respect to its absolutely continuous invariant measure μ: there exists an open and dense set of Hölder observables φ such that*

$$\mu\left\{x : \sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - \mu(\varphi(\mathcal{T}^j))] > n\epsilon\right\} \geq C_\epsilon n^{1-\frac{1}{\alpha}} \qquad \textit{infinitely often.}$$

As a corollary of Theorem 4.1 we obtain moderate deviations estimates.

Theorem 4.4 (Sequential Moderate Deviations)**.** *Let $0<\alpha<1$, $\beta := \frac{1}{\alpha}-1$, $\varphi \in \mathcal{C}^1([0,1])$ and $\tau \in (\frac{1}{2}, 1]$. Then the centered sums satisfy the following moderate deviations upper bounds, where $C_{\alpha,\|\varphi\|_{C^1}}$ is a constant depending on α and the C^1 norm of φ, but* not *on the sequence $\mathcal{T}^\infty$:*

(a) If $\alpha > \frac{1}{2}$ then for any $t>0$

$$m\left\{x : \left|\sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right| > n^\tau t\right\} \leq \frac{4}{2-\frac{1}{\alpha}} C_{\alpha,\|\varphi\|_{C^1}} n^{-\beta+2(1-\tau)}(\log n)^{\frac{1}{\alpha}} t^{-2}$$

(b) If $\alpha \leq \frac{1}{2}$ then

$$m\left\{x : \left|\sum_{j=1}^{n}[\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right| > n^\tau t\right\} \leq (4\beta)^\beta C^\beta_{\alpha,\|\varphi\|_{C^1}+m(g)}(\log n)^{\frac{2}{\alpha}-1} n^{-\beta(2\tau-1)} t^{-2\beta}$$

The same estimates (by the same proof) hold for the measure $\widetilde{m}$.

Proof of Theorem 4.1. We prove the estimate for m, the one for $\widetilde{m}$ is obtained the same way.

Fix n and for $i \in \{1,\dots,n\}$, define the sequence of $\sigma-$algebras $\mathcal{F}_{i,n} = \mathcal{F}_i = \mathcal{T}^{-(n-i)}(\mathcal{B})$. Note that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ hence $\{\mathcal{F}_i\}_{i=1}^n$ is an increasing sequence of $\sigma-$algebras. Take $X_i = [\varphi]_{n-i} \circ \mathcal{T}^{n-i}$, so that X_i is $\mathcal{F}_i$ measurable. Recall that $\psi_j = [\varphi]_j + H_j - H_{j+1} \circ T_{j+1}$ for all $j \geq 0$. We define $Y_i = \psi_{n-i} \circ \mathcal{T}^{n-i}$, $h_i = H_{n-i} \circ \mathcal{T}^{n-i}$ for $i \in \{1,\dots,n\}$. Hence $Y_i = X_i + h_i - h_{i-1}$.

Note also that $\mathcal{G}_i := \sigma(X_1,\dots,X_i) \subset \sigma(\mathcal{F}_1,\dots,\mathcal{F}_i) = \mathcal{F}_i$, as $\sigma(X_i) \subset \mathcal{F}_i$ for all i. Since $\mathbb{E}(\psi_i \circ \mathcal{T}^i | \mathcal{T}^{-i-1}\mathcal{B}) = 0$, $\mathbb{E}(Y_i|\mathcal{F}_j) = 0$ for all $i > j$. Hence $\mathbb{E}(Y_i|\mathcal{G}_j)) = \mathbb{E}(\mathbb{E}(Y_i|\mathcal{F}_j)|\mathcal{G}_j) = 0$ for $i > j$.

For $p \geq 1$ define $b_{i,n}$ as in Rio's inequality, with $\mathcal{G}_i$, X_i as described above so that

$$b_{i,n} = \max_{i \leq u \leq n} \left\| X_i \sum_{k=i}^{u} \mathbb{E}(X_k|\mathcal{G}_i) \right\|_{L^p(m)}.$$

Here all the expectations are taken with respect to m.

Recalling the expression we have for the martingale difference, we can write the sum inside the p-norm as

$$\begin{aligned}\sum_{k=i}^{u} \mathbb{E}(X_k|\mathcal{G}_i) &= \sum_{k=i}^{u} \left[\mathbb{E}(Y_k|\mathcal{G}_i) - \mathbb{E}(h_k|\mathcal{G}_i) + \mathbb{E}(h_{k-1}|\mathcal{G}_i)\right] \\ &= [\sum_{k=i}^{u} \mathbb{E}(Y_k|\mathcal{G}_i)] + \mathbb{E}(h_{i-1}|\mathcal{G}_i) - \mathbb{E}(h_u|\mathcal{G}_i).\end{aligned}$$

If $k > i$, then $\mathbb{E}(Y_k|\mathcal{G}_i) = 0$. This reduces the expression above to

$$\mathbb{E}(Y_i|\mathcal{G}_i) + \mathbb{E}(h_{i-1}|\mathcal{G}_i) - \mathbb{E}(h_u|\mathcal{G}_i).$$

We note that $\|E[f|\mathcal{G}]\|_p \le \|f\|_p$ for any $f \in L^p(m)$, $p \ge 1$. Therefore, we may bound $b_{i,n}$ by $\max_{i\le u\le n} \|X_i\|_\infty(\|Y_i\|_p + \|h_{i-1}\|_p + \|h_u\|_p)$.

We now pick $p > \max\{1, \frac{1}{\alpha} - 1\}$. Since $\|X_i\|_\infty$ is uniformly bounded by $2\|\varphi\|_\infty$ and $Y_i = X_i + h_i - h_{i-1}$, we may bound $\max_{i\le u\le n} \|X_i\|_\infty(\|Y_i\|_p + \|h_{i-1}\|_p + \|h_u\|_p)$ by

$$\frac{1}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)} C_{\alpha,\|\varphi\|_{C^1}} n^{1+\frac{1}{p}(1-\frac{1}{\alpha})}(\log n)^{\frac{1}{p\alpha}}$$

where $C_{\alpha,\|\varphi\|_{C^1}}$ is independent of n. This is a consequence of Proposition 3.6.

Therefore $(4p\sum_{i=1}^n b_{i,n})^p \le \left(\frac{4p}{1-\frac{1}{p}\left(\frac{1}{\alpha}-1\right)}\right)^p C^p_{\alpha,\|\varphi\|_{C^1}} n^{2p+(1-\frac{1}{\alpha})}(\log n)^{\frac{1}{\alpha}}$, which, by Rio's inequality (see Proposition 3.8), is an upper bound for $\mathbb{E}_m|X_1 + X_2 + \cdots + X_n|^{2p}$; this proves (4.1). Thus, by Markov's inequality,

$$\begin{aligned} m(|X_1 + \ldots + X_n|^{2p} > n^{2p}\epsilon^{2p}) &\le \left(\frac{4p}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)}\right)^p C^p_{\alpha,\|\varphi\|_{C^1}} (n^{-2p}\epsilon^{-2p}) n^{2p+(1-\frac{1}{\alpha})}(\log n)^{\frac{1}{\alpha}} \\ &= \left(\frac{4p}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)}\right)^p C^p_{\alpha,\|\varphi\|_{C^1}} n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}}\epsilon^{-2p} \end{aligned}$$

which is the claimed Large Deviation bound. □

Proof of Theorem 4.4. Assume the hypotheses of Theorem 4.4 and let $\tau \in (\frac{1}{2}, 1]$.

(a) Let $\alpha > \frac{1}{2}$ so that $\frac{1}{\alpha} - 1 < 1$. For $\tau \in (\frac{1}{2}, 1]$ define $tn^\tau = n\epsilon$ so that $\epsilon = tn^{\tau-1}$. Then by Theorem 4.1 for any $t > 0$ and $p > 1$,

$$m\left\{x : \left|\sum_{j=1}^n [\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right| > n^\tau t\right\} \le \left(\frac{4p}{1 - \frac{1}{p}\left(\frac{1}{\alpha} - 1\right)}\right)^p C^p_{\alpha,\|\varphi\|_{C^1}} n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}} n^{2p(1-\tau)} t^{-2p}$$

where $C_{\alpha,\|\varphi\|_{C^1}}$ is a constant depending on α and the C^1 norm of φ, but *not* on the sequence $\mathcal{T}^\infty$ or p. Fix $t > 0$ and let $p \to 1$ to obtain, where $\beta := \frac{1}{\alpha} - 1$,

$$m\left\{x : \left|\sum_{j=1}^n [\varphi(\mathcal{T}^j)(x) - m(\varphi(\mathcal{T}^j))]\right| > n^\tau t\right\} \le \frac{4}{2 - \frac{1}{\alpha}} C_{\alpha,\|\varphi\|_{C^1}} n^{-\beta+2(1-\tau)}(\log n)^{\frac{1}{\alpha}} t^{-2}$$

(b) If $\alpha \le \frac{1}{2}$ we take $p = \frac{1}{\alpha} - 1 \ge 1$ and have from Lemma 3.6 that $\|H_n \circ \mathcal{T}^n\|_{L^p(m)} \le C_{\alpha,\|\varphi\|_{C^1}+m(g)}(\log n)^{1+\frac{1}{1-\alpha}}$. In the proof of Theorem 4.1 we can then bound $(4p\sum_{i=1}^n b_{i,n})^p \le (4p)^p C^p_{\alpha,\|\varphi\|_{C^1}+m(g)} n^p (\log n)^{p+\frac{p}{1-\alpha}}$ and hence by Rio's inequality

$$\mathbb{E}_m|X_1 + \ldots + X_n|^{2p} \le (4p)^p C^p_{\alpha,\|\varphi\|_{C^1}+m(g)} n^p (\log n)^{p+\frac{p}{1-\alpha}}.$$

Markov's inequality gives

$$\begin{aligned} m(|X_1 + \ldots + X_n| > n\epsilon) &\le n^{-2p}\epsilon^{-2p}(4p)^p C^p_{\alpha,\|\varphi\|_{C^1}+m(g)} n^p (\log n)^{p+\frac{p}{1-\alpha}} \\ &= n^{-p}\epsilon^{-2p}(4p)^p C^p_{\alpha,\|\varphi\|_{C^1}+m(g)} (\log n)^{p+\frac{p}{1-\alpha}}. \end{aligned}$$

Taking $n\epsilon = n^\tau t$ for $\tau \in (\frac{1}{2}, 1]$ and the choice $p = \beta = \frac{1}{\alpha} - 1$ we obtain

$$m(|X_1 + \ldots + X_n| > n^\tau t) \leq (4\beta)^\beta C^\beta_{\alpha, \|\varphi\|_{C^1} + m(g)} (\log n)^{\frac{2}{\alpha} - 1} n^{-\beta(2\tau - 1)} t^{-2\beta}$$

as claimed. $\square$

4.2. **Random dynamical systems.** Now we prove large deviations estimates for the randomized systems. First we recall some notation. The annealed transfer operator $P\colon L^1(m) \to L^1(m)$ is defined by averaging over all the transformations:

$$P = \sum_{\beta \in \Omega} p_\beta P_\beta = \int_\Sigma P_\omega \, d\nu(\omega).$$

This operator is dual to the annealed Koopman operator $U\colon L^\infty(m) \to L^\infty(m)$ defined by

$$(U\varphi)(x) = \sum_{\beta \in \Omega} p_\beta \varphi(T_\beta x) = \int_\Sigma \varphi(T_\omega x) d\nu(\omega) = \int_\Sigma \tilde{\varphi}(F(\omega, x)) d\nu(\omega)$$

where $\tilde{\varphi}(\omega, x) := \varphi(x)$. The annealed operators satisfy the duality relationship

$$\int_X (U\varphi) \cdot \psi \, dm = \int_X \varphi \cdot P\psi \, dm$$

for all observables $\varphi \in L^\infty(m)$ and $\psi \in L^1(m)$.

Remark 4.5. *It is easy to see that the averaged transfer operator P has no worse rate of decay in L^1 than the slowest of the maps (so better than $n^{-\frac{1}{\alpha}+1}(\log n)^{\frac{1}{\alpha}}$, by Proposition 3.3). By taking a limit point of $\frac{1}{n}\sum_{k=1}^n P^k(\mathbf{1})$, there is an invariant vector h for P in the cone $\mathcal{C}_2$, see* [LSV99]. *The measure $\mu = hm$ is stationary for the RDS; by Proposition 3.7, $h \geq D_\alpha > 0$.*

Moreover, Bahsoun and Bose [BB16b, BB16a] *have shown that there exists a unique absolutely continuous (with respect to the Lebesgue measure) stationary measure μ, and $\nu \otimes \mu$ is mixing — so also ergodic.*

Using the same idea as in the proof of Theorem 4.1, we can obtain an annealed result for the random dynamical system. Note that P_μ, the transfer operator with respect to the stationary measure μ, satisfies $P_\mu \mathbf{1} = \mathbf{1}$ and so $\|P_\mu \varphi\|_\infty \leq P_\mu(\|\varphi\|_\infty) = \|\varphi\|_\infty \|P_\mu \mathbf{1}\|_\infty = \|\varphi\|_\infty$. An easy calculation shows that $P_\mu(\varphi) = \frac{1}{h} P(h\varphi)$ where $h \in \mathcal{C}_2$ is the density of the invariant measure μ and hence $h \geq D_\alpha m(h)$ is bounded below. As before this observation allows us to bootstrap in some sense the $L^1(\mu)$ decay rate to $L^p(\mu)$ for $p \geq 1$, a technique used in [MN08, Mel09].

Theorem 4.6 (Annealed LD)**.** *Let $\varphi \in \mathcal{C}^1([0,1])$ with $\mu(\varphi) = 0$ and let $0 < \alpha < 1$. Then the Birkhoff averages have annealed large deviations with respect to the measure $\nu \otimes \mu$ with rate*

$$(\nu \otimes \mu)\{(\omega, x) : \left| \sum_{j=1}^n \varphi \circ \mathcal{T}^j_\omega(x) \right| \geq n\epsilon\} \leq C_{\alpha, p, \|\varphi\|_{C^1}} n^{1 - \frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2p}$$

for any $p > \max\{1, \frac{1}{\alpha} - 1\}$.

Note that the Birkhoff sums above are not centered for a given realization ω, *only on average over Σ.*

Proof. To prove this result we will use the construction used to prove the annealed CLT in [ANV15]: let $\Sigma_X := X^{\mathbb{N}_0}$, endowed with the σ-algebra $\mathcal{G}$ generated by the cylinders, and the left shift operator $\tau\colon \Sigma_X \to \Sigma_X$.

Denote by π the projection from Σ_X onto the 0-th coordinate, that is, $\pi(x) = x_0$ for $x = (x_0, x_1, \ldots)$. We can lift any observable $\varphi\colon X \to \mathbb{R}$ to an observable on Σ_X by setting $\varphi_\pi := \varphi \circ \pi\colon \Sigma_X \to \mathbb{R}$.

Following [ANV15, §4], one can introduce a τ-invariant probability measure μ_c on Σ_X such that

$\mathbb{E}_\mu(\varphi) = \mathbb{E}_{\mu_c}(\varphi_\pi)$, and the law of $S_n(\varphi)$ on $\Sigma \times X$ under $\nu \otimes \mu$ is the same as the law of the n-th Birkhoff sum of φ_π on Σ_X under μ_c and τ; thus it suffices to establish large deviations for the latter.

Define now

$$H_n := \sum_{k=1}^{n} P_\mu^k(\varphi) : X \to \mathbb{R}$$

From the relation $P_\mu(.) = \frac{1}{h}P(.h)$, we have that $\|P_\mu^n(\varphi)\|_{L^1(\mu)} \le C_{\alpha,\varphi} n^{1-\frac{1}{\alpha}}(\log n)^{1/\alpha}$ because $\mu(\varphi) = 0$. We calculate $E_\mu|P_\mu^i(\varphi)|^p = E_\mu[|P_\mu^i(\varphi)|^{p-1}|P_\mu^i(\varphi)|] \le \|P_\mu^i(\varphi)\|_\infty^{p-1}\|P_\mu^i(\varphi)\|_{L^1(\mu)}$. Hence $\|P_\mu^k(\varphi)\|_{L^p(\mu)} \le Ck^{(1-1/\alpha)/p}(\log k)^{1/(p\alpha)}$ and thus $\|H_n\|_{L^p(\mu)}$ satisfies the bounds of Lemma 3.6.

We lift φ and H_n to Σ_X and denote them by φ_π and $H_{n,\pi}$ respectively, and define

$$\chi_n := \varphi_\pi + H_{n,\pi} - H_{n,\pi} \circ \tau : \Sigma_X \to \mathbb{R}.$$

We now continue as in the proof of Theorem 4.1, applying Rio's inequality. For $i = 1, \ldots, n$ take the sequences $\{X_i = \varphi_\pi \circ \tau^{n-i}\}$, $\{Y_i = \chi_{n-i} \circ \tau^{n-i}\}$ and $\mathcal{G}_i = \tau^{-(n-i)}\mathcal{G}$. We have $\mathbb{E}_{\mu_c}[Y_i|\mathcal{G}_k] = 0$ for $i > k$ and so, for $p > \max\{1, \frac{1}{\alpha} - 1\}$,

$$b_{i,n} = \max_{i \le u \le n} \left\| X_i \sum_{k=i}^{u} \mathbb{E}_{\mu_c}(X_k|\mathcal{G}_i) \right\|_{L^p(\mu_c)} \le Cn^{1+\frac{1}{p}(1-\frac{1}{\alpha})}(\log n)^{\frac{1}{p\alpha}}$$

which gives, as in Theorem 4.1,

$$\mu_c(|X_1 + \ldots + X_n|^{2p} > n^{2p}\epsilon^{2p}) \le C_{\alpha,\varphi,p} n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}}\epsilon^{-2p}$$

□

Using similar ideas, it is possible to obtain an annealed central limit theorem. This has been established already by Young Tower techniques in [BB16a, Theorem 3.2]. We include the statement of the annealed central limit and an alternative proof for completeness and to give an expression for the annealed variance.

Proposition 4.7 (Annealed CLT)**.** *If $\alpha < \frac{1}{2}$ and $\varphi \in C^1$ with $\mu(\varphi) = 0$ then a central limit theorem holds for $S_n\varphi$ on $\Sigma \times X$ with respect to the measure $\nu \otimes \mu$, that is, $\frac{1}{\sqrt{n}}S_n\varphi$ converges in distribution to $\mathcal{N}(0, \sigma^2)$, with variance σ^2 given by*

$$\sigma^2 = -\mu(\varphi^2) + 2\sum_{k=0}^{\infty} \mu(\varphi U^k \varphi)$$

Proof. We will use the results of [ANV15, Section 4] and [Liv96, Theorem 1.1] (see Theorem 6.3 in the Appendix). We proceed as in Theorem 4.6, using the averaged operators U

and P. As in [ANV15, Section 4], to U corresponds a transition probability on X given by $U(x,A) = \sum_\beta \{p_\beta : T_\beta x \in A\}$. The stationary measure μ is invariant under U. Extend μ to the unique probability measure μ_c on $\Sigma_X := X^{N_0} = \{\underline{x} = (x_0, x_1, x_2, \dots, x_n, \dots)\}$, endowed with the σ-algebra $\mathcal{G}$ given by cylinder sets, such corresponding to μ such that $\{x_n\}_{n\geq 0}$ is a Markov chain on $(\Sigma_X, \mathcal{G}, \mu_c)$ (where x_n is the n-th coordinate of $\underline{x}$) induced by the random dynamical system. The left shift τ on Σ_X preserves μ_c. Given $\varphi : X \to \mathbb{R}$, $\mu(\varphi) = 0$, we define φ_π on Σ_X by $\varphi_\pi(x_0, x_1, x_2, \dots, x_n, \dots) := \varphi(x_0)$. As in [ANV15, Section 4], to prove the CLT for $S_n(\varphi)$ with respect to $\nu \otimes \mu$ on $\Sigma \times X$ it suffices to prove the CLT for the Birkhoff sum $\sum_{j=0}^n \varphi_\pi \circ \tau^k$ with respect to μ_c on Σ_X.

We introduce the Koopman operator $\widetilde{U}$ and transfer operator $\widetilde{P}$ for the map τ on the probability space $(\Sigma_X, \mathcal{G}, \mu_c)$. We define the decreasing sequence of σ-algebras $\mathcal{G}_k = \tau^{-k}\mathcal{G}$, and note that $\widetilde{P}$, $\widetilde{U}$ satisfy $\widetilde{P}^k \widetilde{U}^k f = f$ and $\widetilde{U}^k \widetilde{P}^k f = \mathbb{E}_{\mu_c}(f|\mathcal{G}_k)$ for every μ_c-integrable f. We note that $\varphi_\pi \in L^\infty(\mu_c)$. As in [ANV15, Lemma 4.2] we have $\widetilde{P}^n(\varphi_\pi) = (P^n\varphi)_\pi$. Thus $\sum_{k=0}^\infty \widetilde{P}^k \varphi_\pi$ converges in $L^1(\mu_c)$ if $\alpha < \frac{1}{2}$ and therefore $\sum_{k=0}^\infty |\int \varphi_\pi \widetilde{U}^k \varphi_\pi d\mu_c| < \infty$. Thus the result for $\sum_{j=0}^n \varphi_\pi \circ \tau^k$ follows from [Liv96, Theorem 1.1]. The stated formula for σ^2 is also given in [Liv96, Theorem 1.1]. □

We will use the annealed and sequential results to obtain quenched large deviations for random systems of intermittent maps. We denote the Birkhoff sums by $S_{n,\omega}(x)$ to stress the dependence on the realization ω.

Theorem 4.8 (Quenched LD)**.** *Suppose $\varphi \in \mathcal{C}^1$ and $\mu(\varphi) = 0$. Fix $0 < \alpha < 1$. Then, given $p > \max\{1, \frac{1}{\alpha} - 1\}$ and $\kappa := \lceil \frac{4p}{1-\alpha} \rceil$ (rounded up), for ν-almost every realization $\omega \in \Sigma$ the Birkhoff averages have large deviations with polynomial rate,* even without centering*: there is an $N(\omega)$ such that for each $\epsilon > 0$*

$$m\{x : |S_{n,\omega}\varphi| > 4n\epsilon\} \leq C_{\alpha,p,\varphi} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-\kappa} \text{ for } n \geq N(\omega).$$

Note that the Birkhoff sums $S_{n,\omega}\varphi$ above are not centered with respect to the realization *ω, only on average over Σ.*

Remark 4.9. *The point of the above Theorem, compared to the sequential Theorem 4.1, is that for almost each realization the large deviation estimates hold* even without centering*. That is, the contribution of the means (with respect to the measure m on X) can be ignored for almost each realization ω.*

Proof of Theorem 4.8. Choose $p > \max\{1, \frac{1}{\alpha} - 1\}$ and $\epsilon > 0$. By Theorem 4.1, for all $\omega \in \Sigma$,

$$m\left\{x : \left|\frac{1}{n} S_{n,\omega}\varphi(x) - \frac{1}{n}\sum_{j=1}^n m(\varphi \circ T_\omega^j)\right| \geq \epsilon\right\} \leq C_{\alpha,p,\varphi} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2p}$$

with $C_{\alpha,\varphi,\delta}$ independent of ω. Integrating over Σ with respect to ν we obtain

$$\nu \otimes m\left\{(\omega, x) : \left|\frac{1}{n} S_{n,\omega}\varphi(x) - \frac{1}{n}\sum_{j=1}^n m(\varphi \circ T_\omega^j)\right| \geq \epsilon\right\} \leq C_{\alpha,p,\varphi} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2p}$$

By Theorem 4.6, we also have the annealed estimate for the non-centered sums:

$$\nu \otimes m\left\{(\omega, x) : \left|\frac{1}{n} S_{n,\omega}\varphi(x)\right| \geq \epsilon\right\} \leq C_{\alpha,p,\varphi} n^{1-\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \epsilon^{-2p}$$

Theorem 4.6 refers to the measure $\nu\otimes\mu$ but since $\frac{dm}{d\mu}=\frac{1}{h}\leq\frac{1}{D_\alpha}$, the large deviations estimate applies also to $\nu\otimes m$. Observe now that

$$\left\{(\omega,x):\left|\frac{1}{n}\sum_{j=1}^{n}m(\varphi\circ T_\omega^j)\right|>2\epsilon\right\}$$

$$\subset\left\{(\omega,x):\left|\frac{1}{n}S_{n,\omega}\varphi(x)\right|<\epsilon,\left|\frac{1}{n}S_{n,\omega}\varphi(x)-\frac{1}{n}\sum_{j=1}^{n}m(\varphi\circ T_\omega^j)\right|\geq\epsilon\right\}$$

$$\bigcup\left\{(\omega,x):\left|\frac{1}{n}S_{n,\omega}\varphi(x)\right|>\epsilon\right\}.$$

Thus

$$\nu\otimes m\left\{(\omega,x):\left|\frac{1}{n}\sum_{j=1}^{n}m(\varphi\circ T_\omega^j)\right|>2\epsilon\right\}\leq K_{\alpha,p,\varphi}n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}}\epsilon^{-2p}$$

and, as there is no dependence on $x\in X$, this means

$$\nu\left\{\omega:\left|\frac{1}{n}\sum_{j=1}^{n}m(\varphi\circ T_\omega^j)\right|>2\epsilon\right\}\leq K_{\alpha,p,\varphi}n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}}\epsilon^{-2p}\tag{4.2}$$

Denote $\beta:=\frac{1}{\alpha}-1>0$.

The proof we give does not give an optimal value of κ. In the case $\beta>1$ a simpler proof may be given but the resulting exponent κ is also not optimal and no better than the estimate we give.

Let $\tau=\frac{2}{\beta}$ and $\delta>0$ small. Choose $\gamma=\frac{1}{2p}(\beta-\frac{1}{\tau})-\delta=\frac{\beta}{4p}-\delta$ and $\kappa=\lceil(1+\beta^{-1})(4p)\rceil=\lceil\frac{4p}{1-\alpha}\rceil$. The notation $\lceil x\rceil$ indicates the smallest integer greater than or equal to x. Then $(2p\gamma-\beta)\tau<-1$ and $\gamma\kappa>\beta$ for $\delta>0$ small enough.

For $\epsilon=n^{-\gamma}$ the bound (4.2) becomes

$$\nu\left\{\omega:\left|\frac{1}{n}\sum_{j=1}^{n}m(\varphi\circ T_\omega^j)\right|>2n^{-\gamma}\right\}\leq K_{\alpha,p,\varphi}n^{2p\gamma}n^{-\beta}(\log n)^{\frac{1}{\alpha}}$$

Consider the subsequence $n_k:=k^\tau$. As $(2p\gamma-\beta)\tau<-1$, for ν almost every ω there exists an $N(\omega)$ such that for all $n_k>N(\omega)$,

$$\left|\frac{1}{n_k}\sum_{j=1}^{n_k}m(\varphi\circ T_\omega^j)\right|\leq 2n_k^{-\gamma}$$

If $n_k\leq n<n_{k+1}$ then

$$\left|\frac{1}{n}\sum_{j=1}^{n}m(\varphi\circ T_\omega^j)\right|\leq\frac{1}{n_k}\left|\sum_{j=1}^{n_k}m(\varphi\circ T_\omega^j)+\sum_{j=n_k+1}^{n}m(\varphi\circ T_\omega^j)\right|$$

$$\leq 2n_k^{-\gamma}+\frac{\|\varphi\|_\infty}{n_k}|n_{k+1}-n_k|$$

There is $K>0$, independent of ω, depending only on τ, γ and $\|\varphi\|_\infty$, such that

$$2n_k^{-\gamma}+\frac{\|\varphi\|_\infty}{n_k}|n_{k+1}-n_k|<3n^{-\gamma}\qquad\text{if } k\geq K.$$

Indeed, $\lim_{k\to\infty}\frac{n_{k+1}}{n_k}=1$, $\frac{1}{n_k}|n_{k+1}-n_k|=O(\frac{1}{k})$, $\frac{1}{k}=O(\frac{1}{n^{1/\tau}})$ and $n^{-1/\tau}<n^{-\gamma}$ because $1/\tau>\gamma$.

Increase $N(\omega)$ such that $n>N(\omega)$ implies $n\ge K^\tau$ and $C_{\alpha,p,\varphi}n^{\gamma\kappa-\beta}(\log n)^{1/\alpha}>1$.

We will show that for $n>N(\omega)$

$$m(x:|\frac{1}{n}S_{n,\omega}\varphi(x)|\ge 4\epsilon)\le C_{\alpha,p,\varphi}\epsilon^{-\kappa}n^{-\beta}(\log n)^{1/\alpha}.$$

Suppose $\epsilon<n^{-\gamma}$. Then $C_{\alpha,p,\varphi}\epsilon^{-\kappa}n^{-\beta}(\log n)^{1/\alpha}\ge C_{\alpha,p,\varphi}n^{\gamma\kappa-\beta}(\log n)^{1/\alpha}>1$ and there is nothing to prove.

If $\epsilon\ge n^{-\gamma}$ and $n>N(\omega)$ then, as $|\frac{1}{n}\sum_{j=1}^n m(\varphi\circ T_\omega^j)|<3\epsilon$,

$$\left\{x:|\frac{1}{n}S_{n,\omega}\varphi(x)|\ge 4\epsilon\right\}\subset\left\{x:|\frac{1}{n}S_{n,\omega}\varphi(x)-\frac{1}{n}\sum_{j=1}^n m(\varphi\circ T_\omega^j)|\ge\epsilon\right\}$$

Hence the result holds by Theorem 4.1, as

$$m(x:|\frac{1}{n}S_{n,\omega}\varphi(x)-\frac{1}{n}\sum_{j=1}^n m(\varphi\circ T_\omega^j)|\ge\epsilon)\le C_{\alpha,p,\varphi}\epsilon^{-2p}n^{-\beta}(\log n)^{1/\alpha}$$

and $2p<\kappa$. □

We remark that the methods used to prove these results in the uniformly expanding case are not applicable here, as they rely on the quasi-compactness of the transfer operator. In the uniformly expanding case, which has exponential large deviations for Hölder observables, it is possible to obtain a rate function.

5. The Role of Centering in the Quenched CLT for RDS

In this section we discuss two results: Proposition 5.1, that the quenched variance is the same for almost all realizations $\omega\in\Sigma$, and Theorem 5.3, that generically one must center the observations in order to obtain a CLT (as opposed to LD Theorem 4.8, where centering did not affect the quenched LD). Note that these hinge on the rate of growth of the mean of the Birkhoff sums; we see that it is $o(n)$ but not $o(\sqrt{n})$. We use the recent paper by Hella and Stenlund [HS20] to extend and clarify results of [NTV18].

In [NTV18, Theorem 3.1] a self-norming quenched CLT is obtained for ν-a.e. realization ω of the random dynamical system of Theorem 4.6. More precisely, recalling the definition of the centered observables $[\varphi]_k(\omega,x)=\varphi(x)-m(\varphi\circ\mathcal{T}_\omega^k)$ and $\sigma_n^2(\omega):=\int\left[\sum_{k=1}^n[\varphi]_k(\omega,\mathcal{T}_\omega^k x)\right]^2dx$ it is shown that $\frac{1}{\sigma_n(\omega)}\sum_{k=1}^n[\varphi]_k(\omega,\cdot)\circ\mathcal{T}_\omega^k\to N(0,1)$ provided $\sigma_n^2\approx n^\beta$, with $\alpha<\frac{1}{9}$ and $\beta>\frac{1}{2(1-2\alpha)}$. Various scenarios under which $\sigma_n^2(\omega)>n^\beta$ are given in [NTV18]. See also [HL19].

If the maps T_{ω_i} preserved the same invariant measure then it suffices to consider observables with mean zero, since the mean would be the same along each realization. In the setting of [ALS09] this is the case, namely all realizations preserve Haar measure, and the authors address the issue of whether the variance $\sigma_n^2(\omega)$ can be taken to be the "same" for almost every quenched realization in the setting of random toral automorphisms. They show that for almost every quenched realization the variance in the quenched CLT may be taken as a uniform constant. The technique they use is adapted from random walks in random environments and consists in analyzing a random dynamical system on a product space.

A natural question is whether in our setup of random intermittent maps, after centering, $\sigma_n(\omega)$ can be taken to be "uniform" over ν-a.e. realization. Recent results of Hella and Stenlund [HS20] give conditions under which $\frac{1}{n}\sigma_n^2(\omega) \to \sigma^2$ for ν-a.e. ω, as well as information about rates of convergence. Note that this is also true in the context of uniformly expanding maps considered by [AA16] using the same method used in [HS20].

A related question is whether we need to center at all. For example, if $\mu(\varphi) = 0$, where μ is the stationary measure on X, then for ν-a.e ω

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}[\varphi(\mathcal{T}_\omega^j x) - m(\varphi(\mathcal{T}_\omega^j))] \to 0 \qquad \text{for } \mu\text{-a.e. } x$$

by the ergodicity of $\nu \otimes \mu$, but also

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} m(\varphi(\mathcal{T}_\omega^j)) \to 0 \qquad \text{for } \nu\text{-a.e. } \omega,$$

by the proof of Theorem 4.8. So for the strong law of large numbers centering is not necessary. Using ideas of [AA16] we consider the related question of whether centering is necessary to obtain a quenched CLT with almost surely constant variance. We show the answer to this is positive: to obtain an almost surely constant variance in the quenched CLT we need to center.

5.1. **Non-random quenched variance.** For Proposition 5.1, we verify that our system satisfies the conditions SA1, SA2, SA3 and SA4 of [HS20]; then, by [HS20, Theorem 4.1], the quenched variance is almost surely the same, equal to the annealed variance.

Proposition 5.1. *Let $\alpha < \frac{1}{2}$, $\varphi \in C^1$ and define the annealed variance*

$$\sigma^2 := \lim_{n\to\infty} \frac{1}{n} \| \left[S_n\right]^{\nu\otimes m} \|^2_{L^2(\nu\otimes m)} = \lim_{n\to\infty}\frac{1}{n}\|S_n - \int_{\Sigma\times X} S_n d\, \nu\otimes m\|^2_{L^2(\nu\otimes m)}$$

$$= \sum_{k=0}^{\infty}(2-\delta_{0k}) \lim_{i\to\infty}\int_{\Sigma}[m(\varphi_i\varphi_{i+k}) - m(\varphi_i)m(\varphi_{i+k})]d\nu$$

If $\sigma^2 > 0$ then for ν-a.e. ω

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left[\varphi(\mathcal{T}_\omega^j \cdot)\right]^m \to^d N(0,\sigma^2)$$

in distribution with respect to m.

Remark 5.2. *Proposition 4.7 shows that the annealed CLT holds for $\alpha < \frac{1}{2}$ and under the usual genericity conditions the annealed variance satisfies $\sigma^2 > 0$. Thus Proposition 5.1 extends* [NTV18, Theorem 5.3] *from the parameter range $\alpha < \frac{1}{9}$ to $\alpha < \frac{1}{2}$. Note that* [HL19], *proved the CLT for $\alpha < \frac{1}{3}$.*

Proof of Proposition 5.1. We will verify conditions SA1, SA2, SA3 and SA4 of [HS20, Theorem 4.1] in our setting, with $\eta(k) = Ck^{-\frac{1}{\alpha}+1}(\log k)^{\frac{1}{\alpha}}$ in the notation of [HS20].
SA1: If $j > i$ then

$$\left|\int \varphi\circ\mathcal{T}_\omega^i(x)\varphi\circ\mathcal{T}_\omega^j(x)dm - \int\varphi\circ\mathcal{T}_\omega^i(x)dm\int\varphi\circ\mathcal{T}_\omega^j(x)dm\right|$$

$$= \left| \int \varphi \circ \mathcal{T}_\omega^{j-i+1}(\mathcal{T}_\omega^i x)\varphi(x) P_\omega^i \mathbf{1} dm - \int \varphi \mathcal{P}_\omega^i \mathbf{1} dm \int \varphi(x) \mathcal{P}_\omega^j \mathbf{1} dm \right| \leq C(j-i)^{-\frac{1}{\alpha}+1}(\log(j-i))^{\frac{1}{\alpha}}$$

by the same argument as in the proof of [NTV18, Proposition 1.3].
SA2: Our underlying shift $\sigma : \Sigma \to \Sigma$ is Bernoulli hence α-mixing.
SA3: We need to check [HS20, equation (4)] that

$$\left| \int \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_1} x) dm - \int \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_{r+1}} x) dm \right| \leq C\eta(k-r)$$

and

$$\left| \int \varphi \cdot \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_1} x) dm - \int \varphi \cdot \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_{r+1}} x) dm \right| \leq C\eta(k-r).$$

Using the transfer operators, rewrite

$$\left| \int \psi \cdot \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_1} x) dm - \int \psi \cdot \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_{r+1}} x) dm \right|$$
$$= \left| \int \varphi \cdot P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_1}(\psi) dm - \int \varphi \cdot P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_{r+1}}(\psi) dm \right|$$
$$\leq \|\varphi\|_\infty \| P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_{r+1}} [\psi - P_{\omega_r} \cdots P_{\omega_1}(\psi)] \|_{L^1}$$

We have to bound this for ψ either $\mathbf{1}$ or φ. If $\psi = \mathbf{1}$ then

$$\| P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_{r+1}} [\mathbf{1} - P_{\omega_r} \cdots P_{\omega_1} \mathbf{1}] \|_{L^1} \leq C(k-r)^{-\frac{1}{\alpha}+1}(\log(k-r))^{\frac{1}{\alpha}}$$

with C independent of ω and r by [NTV18, Theorem 1.2] (see Proposition 3.3) because $\mathbf{1}$ and $P_{\omega_r} \cdots P_{\omega_1} \mathbf{1}$ both lie in the cone and have the same m-mean. If $\psi = \varphi$, using Lemma 3.4, can write $\varphi - (\int \varphi dm)\mathbf{1}$ as a difference of two functions in the cone, and then the same decay estimate holds.
SA4: (σ, Σ, ν) is stationary so SA4 is automatic. □

5.2. **Centering is generically needed in the CLT.** Now we address the question of the necessity of centering in the quenched central limit theorem. We show that if $\int \varphi d\mu_{\beta_i} \neq \int \varphi d\mu_{\beta_j}$ for two maps T_{β_i}, T_{β_j}, where μ_{β_i} is the invariant measure of T_{β_i}, then centering is needed: although

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \left[\varphi(\mathcal{T}_\omega^j) - m(\varphi(\mathcal{T}_\omega^j)) \right] \to^d N(0, \sigma^2)$$

for ν-a.e. ω, it is not the case that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \varphi(\mathcal{T}_\omega^j) \to^d N(0, \sigma^2)$$

for ν-a.e. ω.

Our proof has the same outline as that of [AA16], adapted to our setting of polynomial decay of correlations. First we suppose that the maps T_{β_i} do not preserve the same measure. After reindexing we can suppose that T_{β_1} and T_{β_2} have different invariant measures and that $\int \varphi d\mu_{\beta_1} \neq \int \varphi d\mu_{\beta_2}$, a condition satisfied by an open and dense set of observables. Recall that the RDS has the stationary measure $d\mu = h dm$, $h \geq D_\alpha > 0$ and we have assumed $\mu(\varphi) = 0$, $\varphi \in \mathcal{C}^1$.

Here are the steps:

- construct a product random dynamical system on $X \times X$ and prove that it satisfies an annealed CLT for $\tilde{\varphi}(x, y) = \varphi(x) - \varphi(y)$ with distribution $N(0, \tilde{\sigma}^2)$;
- observe that almost every uncentered quenched CLT has the same variance only if $2\sigma^2 = \tilde{\sigma}^2$, where the original RDS with stationary measure $d\mu = hdm$ satisfies an annealed CLT for φ with distribution $N(0, \sigma^2)$;
- observe that the conclusions of [AA16, Theorem 9] hold in our setting and $\tilde{\sigma}^2 = 2\sigma^2$ if and only if $\lim_{n\to\infty} \frac{1}{n} \int_\Sigma \left(\sum_{k=1}^{n-1} \int_X \varphi \circ \mathcal{T}_\omega^k hdm \right)^2 d\nu = 0$;
- use ideas of [AA16] to show the limit above is zero only if a certain function G on Σ is a Hölder coboundary, which in turn implies $\int \varphi d\mu_{\beta_1} = \int \varphi d\mu_{\beta_2}$, a contradiction.

Let $\varphi \colon X \to \mathbb{R}$ be $\mathcal{C}^1$, with $\int_X \varphi d\mu = 0$, and define $S_n(\varphi) = \sum_{k=0}^{n-1} \varphi(\mathcal{T}_\omega^k x)$ on $\Sigma \times X$. Recall the standard expression (e.g. see [AA16]) for the annealed variance,

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \int_X [S_n(\varphi)]^2 \, d\mu d\nu.$$

We also consider the product random dynamical system $(\tilde{\Sigma} := \Sigma \times X \times X, \tilde{\nu} := \nu \otimes \mu \otimes \mu, \tilde{T})$ defined on X^2 by $\tilde{T}_\omega(x, y) = (T_\omega x, T_\omega y)$. For an observable φ, define $\tilde{\varphi} \colon X^2 \to \mathbb{R}$ by $\tilde{\varphi}(x, y) = \varphi(x) - \varphi(y)$, and its Birkhoff sums $S_n(\tilde{\varphi})$. In Theorem 6.1 and Corollary 6.2 of the Appendix we show $\frac{1}{\sqrt{n}} \sum_{j=1}^n \tilde{\varphi} \circ \tilde{T}^j \to^d N(0, \tilde{\sigma}^2)$ with respect to $\nu \otimes \mu \otimes \mu$ for some $\tilde{\sigma}^2 \geq 0$.

The following lemma from [ANV15] is general and does not depend upon the underlying dynamics. It is a consequence of Levy's continuity theorem (Theorem 6.5 in [Kar93]).

Lemma ([ANV15, Lemma 7.2])**.** *Assume that $\sigma^2 > 0$ and $\tilde{\sigma}^2 > 0$ are such that*

(1) $\frac{S_n(\varphi)}{\sqrt{n}}$ *converges in distribution to $\mathcal{N}(0, \sigma^2)$ under the probability $\nu \otimes \mu$,*
(2) $\frac{S_n(\tilde{\varphi})}{\sqrt{n}}$ *converges in distribution to $\mathcal{N}(0, \tilde{\sigma}^2)$ under the probability $\nu \otimes \mu \otimes \mu$,*
(3) $\frac{S_{n,\omega}(\varphi)}{\sqrt{n}}$ *converges in distribution to $\mathcal{N}(0, \sigma^2)$ under the probability μ, for ν almost every ω.*

Then $2\sigma^2 = \tilde{\sigma}^2$.

Suppose two of the maps T_{β_1} and T_{β_2} have different invariant measures. It is possible to find a $\mathcal{C}^1$ φ such that $\int \varphi d\mu_{\beta_1} \neq \int \varphi d\mu_{\beta_2}$. In fact, $\int \varphi d\mu_{\beta_1} \neq \int \varphi d\mu_{\beta_2}$ for a $\mathcal{C}^2$ open and dense set of φ.

Theorem 5.3. *Let $\varphi \in C^1$ with $\mu(\varphi) = 0$ and suppose that $\int \varphi \, d\mu_{\beta_1} \neq \int \varphi \, d\mu_{\beta_2}$. Then it is not the case that*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \varphi(\mathcal{T}_\omega^j .) \to N(0, \sigma^2)$$

for almost every $\omega \in \Sigma$. Hence, the Birkhoff sums need to be centered along each realization.

Proof. We follow the counterexample method of [AA16, Section 4.3]. We show that in the uncentered case $2\sigma^2 \neq \tilde{\sigma}^2$. To do this we use [AA16, Theorem 9] which holds in our setting, namely $\tilde{\sigma}^2 = 2\sigma^2$ if and only if

$$\lim_{n\to\infty} \int_\Sigma \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} \int_X \varphi P_{\omega_k} \dots P_{\omega_n}(h) dm \right)^2 d\nu = 0 \tag{5.1}$$

(as in [AA16, Section 4.3] we change the time direction and replace $(\omega_1, \omega_2, \dots, \omega_n)$ by $(\omega_n, \omega_2, \dots, \omega_1)$; this does not affect integrals with respect to ν over finitely many symbols).

Note that the sequence $P_{\omega_1}P_{\omega_2}\dots P_{\omega_n}h$ is Cauchy in L^1, as $\alpha < \frac{1}{2}$ and

$$\|P_{\omega_1}P_{\omega_2}\dots P_{\omega_n}(h) - P_{\omega_1}P_{\omega_2}\dots P_{\omega_n}\dots P_{\omega_{n+k}}(h)\|_1 \leq Cn^{-\frac{1}{\alpha}+1}(\log n)^{\frac{1}{\alpha}}$$

by Proposition 3.3. Thus $P_{\omega_1}P_{\omega_2}\dots P_{\omega_n}h \to h_\omega$ in L^1 for some $h_\omega \in \mathcal{C}_2$. This limit defines h_ω, in terms of $\bar{\omega} := (\dots, \omega_n, \omega_2, \dots, \omega_1)$, i.e. ω reversed in time. We define $G(\omega) := \int_X \varphi h_\omega dm$. Note also that $\|P_{\omega_1}P_{\omega_2}\dots P_{\omega_n}h - h_\omega\|_1 \leq Cn^{-1-\delta}$ for some $\delta > 0$, uniformly for $\omega \in \Sigma$. Hence

$$\int_\Sigma \left(\sum_{k=1}^{n-1} \frac{1}{\sqrt{n}} \int_X \varphi P_{\omega_k}\dots P_{\omega_n}hdm\right)^2 d\nu$$
$$= \int_\Sigma \left(\sum_{k=1}^{n-1} \frac{1}{\sqrt{n}} \left(\int_X \varphi h_{\tau^k\omega}\, dm + O\left(\sum_{k=1}^{n-1} \frac{1}{(n-k)^{1+\delta}}\right)\right)\right)^2 d\nu$$

which gives, using (5.1), that

$$\lim_{n\to\infty} \int_\Sigma \left(\frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n-1} G(\tau^k\omega)\right)\right)^2 d\nu = 0. \tag{5.2}$$

We put a metric on Σ by defining $d(\omega, \omega') = s(\omega, \omega')^{-1-\frac{\epsilon}{2}}$ where $s(\omega, \omega') = \inf\{n : \omega_n \neq \omega'_n\}$. With this metric Σ is a compact and complete metric space. Note that $\|h_\omega - h_{\omega'}\|_{L^1} \leq Cs(\omega, \omega')^{-\frac{\epsilon}{2}}$ hence $G(\omega)$ is Hölder with respect to our metric.

As in the Abdulkader-Aimino counterexample, (5.2) implies that $G = H - H \circ \tau$ for a Hölder function H on the Bernoulli shift (τ, Σ, ν): by [Liv96, Theorem 1.1] (see Theorem 6.3 in the Appendix) G is a measurable coboundary, and therefore a Hölder coboundary, by the standard Livšic regularity theorem (see for instance [VO16, Section 12.2]). Now consider the points $\beta_1^* := (\beta_1, \beta_1, \cdots)$ and $\beta_2^* := (\beta_2, \beta_2, \cdots)$ in Σ; they are fixed points for τ, and correspond to choosing only the map T_{β_1}, respectively only the map T_{β_2}. This implies $G(\beta_1^*) = G(\beta_2^*) = 0$ which in turn implies $\int \varphi d\mu_{\beta_1} = \int \varphi d\mu_{\beta_2}$, a contradiction. $\square$

6. Appendix

We will show that the system $\widetilde{F}(\omega, x, y) = (\tau\omega, T_{\omega_1}x, T_{\omega_1}y)$ with respect to the measure $\nu \otimes \mu^2$ on $\Sigma \times [0,1]^2$ (recall that $\nu := \mathbb{P}^{\otimes\mathbb{N}}$ and μ is a stationary measure of the RDS) has summable decay of correlations in L^2 for $\alpha < \frac{1}{2}$, and as a corollary it satisfies the CLT.

Theorem 6.1. *Suppose that for $\omega \in \Sigma$, $h = \frac{d\mu}{dm} \in \mathcal{C}_2$ and each $\varphi \in \mathcal{C}^1$ with $m(\varphi h) = 0$*

$$\|P_{\omega_n}\dots P_{\omega_1}(\varphi h)\|_{L^1(m)} \leq C\rho(n)(\|\varphi\|_{\mathcal{C}^1} + m(h))$$

(that is, the setting of Proposition 3.3).

Then there is a constant $\widetilde{C}$, independent of ω, such for each $\psi \in \mathcal{C}^1(X \times X)$ and $\varphi \in L^\infty(X \times X)$ with $(\mu \otimes \mu)(\psi) = 0$, one has

$$\left|\int \varphi(\mathcal{T}_\omega^n x, \mathcal{T}_\omega^n x)\psi(x, y)d\mu(x)d\mu(y)\right| \leq \widetilde{C}\rho(n)\|\varphi\|_{L^\infty}(\|\psi\|_{\mathcal{C}^1} + 1)$$

Proof. Since $X \times X$ is compact, ψ is uniformly $\mathcal{C}^1$ in both variables in the sense that $\psi(x_0, y)$ is uniformly $\mathcal{C}^1$ for each x_0 and similarly for $\psi(x, y_0)$. We want to estimate

$$I := \int \varphi(\mathcal{T}_\omega^n x, \mathcal{T}_\omega^n y)\psi(x,y)d\mu(x)d\mu(y).$$

Define

$$\overline{\psi}(x) := \int \psi(x,y)d\mu(y), \qquad h_x(y) := \psi(x,y) - \overline{\psi}(x).$$

Then $\overline{\psi}, h_x \in \mathcal{C}^1(X)$, with $\mathcal{C}^1$-norms bounded by $2\|\psi\|_{\mathcal{C}^1}$, uniformly with respect to x.

We can write I as

$$I = \underbrace{\int \varphi(\mathcal{T}_\omega^n x, \mathcal{T}_\omega^n y)\left[\psi(x,y) - \overline{\psi}(x,y)\right] d\mu(x)d\mu(y)}_{:=I_1} + \underbrace{\int \varphi(\mathcal{T}_\omega^n x, \mathcal{T}_\omega^n y)\overline{\psi}(x,y)d\mu(x)d\mu(y)}_{:=I_2}.$$

Define now $g_{\omega,x}(y) := \varphi(\mathcal{T}_\omega^n x, y)$. Then (note that $\int h_x(y)h(y)dm(y) = 0$)

$$\begin{aligned}|I_1| = \left|\int\left(\int g_{\omega,x}(\mathcal{T}_\omega^n y)h_x(y)h(y)dm(y)\right)d\mu(x)\right| &= \left|\int\left(\int g_{\omega,x}(y)\mathcal{P}_\omega^n(h_x(y)h(y))dm(y)\right)d\mu(x)\right| \\ &\le \|\varphi\|_{L^\infty}\sup_x \|\mathcal{P}_\omega^n(h_x(y)h(y))\|_{L^1(m(y))} \\ &\le C'\|\varphi\|_{L^\infty}(\|\psi\|_{\mathcal{C}^1} + m(h))\rho(n).\end{aligned}$$

by the hypothesis.

Similarly, define $k_{\omega,y}(x) := \varphi(x, \mathcal{T}_\omega^n y)$ so then (again, $\int \overline{\psi}(x)h(x)dm(x) = 0$)

$$\begin{aligned}|I_2| &= \left|\int\left(\int k_{\omega,y}(\mathcal{T}_\omega^n x)\overline{\psi}(x)d\mu(x)\right)d\mu(y)\right| \\ &= \left|\int\left(\int k_{\omega,y}(x)\mathcal{P}_\omega^n(\overline{\psi}(x)h(x))dm(x)\right)d\mu(y)\right| \\ &\le \|\varphi\|_{L^\infty}\|\mathcal{P}_\omega^n(\overline{\psi}(x)h(x))\|_{L^1(m(x))} \\ &\le C'\|\varphi\|_{L^\infty}(\|\psi\|_{\mathcal{C}^1} + m(h))\rho(n).\end{aligned}$$

These imply that $|I| \le 2C'\|\varphi\|_{L^\infty}(\|\psi\|_{\mathcal{C}^1} + m(h))\rho(n)$. $\square$

Corollary 6.2. *Under the assumptions of Theorem 6.1, for $\psi \in \mathcal{C}^1(X \times X)$ with $(\mu \otimes \mu)(\psi) = 0$, $\frac{1}{\sqrt{n}}\sum_{k=1}^n \psi \circ \widetilde{F}^k(\omega, x, y)$ satisfies a CLT with respect to $\nu \otimes \mu \otimes \mu$, that is*

$$\frac{1}{\sqrt{n}}\sum_{k=1}^n \psi \circ \widetilde{F}^k(\omega, x, y) \to^d N(0, \widetilde{\sigma}^2)$$

in distribution for some $\widetilde{\sigma}^2 \ge 0$.

Proof. Let Q be the adjoint of $\tilde{F}(\omega, x, y) = (\sigma\omega, T_{\omega_1}x, T_{\omega_1}y)$ with respect to the invariant measure $\nu \otimes \mu \otimes \mu$ on $\Sigma \times X^2$ so that

$$\int \varphi \circ \tilde{F}(\omega, x, y)\psi(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega) = \int \varphi(\omega, x, y)(Q\psi)(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega).$$

for $\varphi \in L^\infty(\Sigma \times X \times X)$. Iterating we have

$$\int \varphi \circ \tilde{F}^n(\omega, x, y)\psi(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega) = \int \varphi(\omega, x, y)(Q^n\psi)(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega).$$

Taking $\varphi = \text{sign}(Q^n\psi)$, we see from Theorem 6.1 that $\|Q^n\psi\|_{L^1} \leq C'\rho(n)$.

The proof now follows, as in Proposition 4.7, from [Liv96, Theorem 1.1] (see Theorem 6.3 in the Appendix). □

Proof of Lemma 3.4. Let $f_1 = (\varphi + \lambda x + A)h + B$ and $f_2 = (A + \lambda x)h + B$.

First we show that $f_1 \in \mathcal{C}_2$. It is clear that $f_1 \in \mathcal{C}^0(0, 1] \cap L^1(m)$. Choose $\lambda < 0$ such that $|\lambda| > \|\varphi'\|_{L^\infty}$ and $A > 0$ large enough so that

$$\varphi + \lambda x + A > 0.$$

This ensures that $f_1 \geq 0$ for any value of $B \geq 0$. Note now that

$$(\varphi + \lambda x + A)' = \varphi' + \lambda \leq 0$$

so $\varphi + \lambda x + A$ is decreasing. Since both $\varphi + \lambda x + A$ and h are positive and decreasing, we obtain that f_1 is decreasing as well. We show now that $x^{\alpha+1}f_2$ is increasing. Since $h \in \mathcal{C}_2$, h is non-increasing so h' exists m-a.e. and $h' \leq 0$ m-a.e. Then $(x^{\alpha+1}h)'$ exists m-a.e. as well, and we can compute this derivative as

$$(x^{\alpha+1}h)' = (\alpha + 1)x^\alpha h + x^{\alpha+1}h' \geq 0$$

because it is increasing.

We compute now the derivative of $x^{\alpha+1}f_2$:

$$\begin{aligned}(x^{\alpha+1}[(\varphi + \lambda x + A)h + B])' = (\alpha + 1)x^\alpha\varphi h + x^{\alpha+1}\varphi' h + x^{\alpha+1}\varphi h' + (\alpha + 2)x^{\alpha+1}h\lambda + \\ \lambda x^{\alpha+2}h' + (\alpha + 1)Ax^\alpha h + Ax^{\alpha+1}h' + (\alpha + 1)x^\alpha B.\end{aligned}$$

We group terms conveniently: note that

$$(\alpha + 1)x^\alpha\varphi h + (\alpha + 1)Ax^\alpha h + x^{\alpha+1}\varphi h' + Ax^{\alpha+1}h' = (\varphi + A)[(\alpha + 1)x^\alpha h + h'x^{\alpha+1}] \geq 0$$

m-a.e., since the term in the square brackets corresponds to $(x^{\alpha+1}h)' \geq 0$. The term $\lambda x^{\alpha+2}h'$ is non-negative m-a.e. since $\lambda, h' \leq 0$. Since $0 \leq h(x)x^\alpha \leq am(h)$, we have $0 \leq -x^{\alpha+1}h' \leq (\alpha+1)x^\alpha h \leq (\alpha+1)am(h)$ and then the terms $(\alpha+2)\lambda x^{\alpha+1}h + x^{\alpha+1}h\varphi'$ are bounded. Thus, we can take $B > 0$ big enough so that

$$(\alpha + 1)x^\alpha B \geq (\alpha + 2)\lambda x^{\alpha+1}h + x^{\alpha+1}h\varphi'.$$

With this, we have that $(x^{\alpha+1}h)' \geq 0$ and so $x^{\alpha+1}h$ is increasing.

Finally, we check that $f_1(x)x^\alpha \leq am(f_1)$. Using that $h(x)x^\alpha \leq am(h)$,

$$[(\varphi + \lambda x + A)h + B]x^\alpha \leq (\varphi + \lambda x + A)hx^\alpha + B \leq \sup(\varphi + \lambda x + A)am(h) + B.$$

On the other hand, $am((\varphi + \lambda x + A)h + B) \geq a \inf(\varphi + \lambda x + A)m(h) + aB$, so it suffices to have

$$\begin{aligned}&\sup(\varphi + \lambda x + A)am(h) + B \leq a \inf(\varphi + \lambda x + A)m(h) + aB \\ \iff\ & B \geq \frac{a}{a-1}\big[\sup(\varphi + \lambda x + A) - \inf(\varphi + \lambda x + A)\big]m(h).\end{aligned}$$

Thus, we see that $f_1 \in \mathcal{C}_2$. The proof that $f_2 \in \mathcal{C}_2$ is the same, take $\varphi(x) \equiv 0$. □

Theorem 6.3 (special case of [Liv96, Theorem 1.1]). *Assume $T : Y \to Y$ preserves the probability measure η on the σ-algebra $\mathcal{B}$. Denote by P its transfer operator.*

If $\varphi \in L^\infty(\eta)$ with $\eta(\varphi) = 0$ and $\sum_k \|P^k\varphi\|_{L^1(\eta)} < \infty$ then a central limit theorem holds for $S_n\varphi := \sum_{k=1}^n \varphi \circ T^k$ with respect to the measure η, that is, $\frac{1}{\sqrt{n}} S_n\varphi$ converges in distribution to $\mathcal{N}(0, \sigma^2)$. The variance is given by

$$\sigma^2 = -\eta(\varphi^2) + 2\sum_{k=0}^{\infty} \eta(\varphi \cdot \varphi \circ T^k).$$

In addition, $\sigma^2 = 0$ iff $\varphi \circ T$ is a measurable coboundary, that is $\varphi \circ T = g - g \circ T$ for a measurable g.

Acknowledgements. We thank Ian Melbourne and the anonymous referee for many helpful comments and suggestions. MN was supported in part by NSF Grant DMS 1600780. FPP thanks the University of Houston for hospitality while this work was completed. FPP was partially supported by the Becas Chile scholarship scheme from CONICYT. AT was supported in part by NSF Grant DMS 1816315.

References

[AA16] Mohamed Abdelkader and Romain Aimino. On the quenched central limit theorem for random dynamical systems. *J. Phys. A*, 49(24):244002, 13, 2016.

[AHN$^+$15] Romain Aimino, Huyi Hu, Matthew Nicol, Andrei Török, and Sandro Vaienti. Polynomial loss of memory for maps of the interval with a neutral fixed point. *Discrete Contin. Dyn. Syst.*, 35(3):793–806, 2015.

[ALS09] Arvind Ayyer, Carlangelo Liverani, and Mikko Stenlund. Quenched CLT for random toral automorphism. *Discrete Contin. Dyn. Syst.*, 24(2):331–348, 2009.

[AM18] Romain Aimino and Jorge Milhazes Freitas. Large deviations for dynamical systems with stretched exponential decay of correlations. *arXiv e-prints*, page arXiv:1812.09742, Dec 2018.

[ANV15] Romain Aimino, Matthew Nicol, and Sandro Vaienti. Annealed and quenched limit theorems for random expanding dynamical systems. *Probab. Theory Related Fields*, 162(1-2):233–274, 2015.

[BB16a] Wael Bahsoun and Christopher Bose. Corrigendum: Mixing rates and limit theorems for random intermittent maps (2016 *nonlinearity* 29 1417) [MR3476513]. *Nonlinearity*, 29(12):C4, 2016.

[BB16b] Wael Bahsoun and Christopher Bose. Mixing rates and limit theorems for random intermittent maps. *Nonlinearity*, 29(4):1417–1433, 2016.

[BBR17] Wael Bahsoun, Christopher Bose, and Marks Ruziboev. Quenched decay of correlations for slowly mixing systems. *arXiv e-prints*, page arXiv:1706.04158, Jun 2017.

[BG97] Abraham Boyarsky and PawełGóra. *Laws of chaos.* Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1997. Invariant measures and dynamical systems in one dimension.

[CR07] Jean-Pierre Conze and Albert Raugi. Limit theorems for sequential expanding dynamical systems on $[0, 1]$. In *Ergodic theory and related fields*, volume 430 of *Contemp. Math.*, pages 89–121. Amer. Math. Soc., Providence, RI, 2007.

[Gor69] M. I. Gordin. The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR*, 188:739–741, 1969.

[HL19] Olli Hella and Juho Leppänen. Central limit theorems with a rate of convergence for time-dependent intermittent maps. *Stoch. Dyn.*, 2019.

[HS20] Olli Hella and Mikko Stenlund. Quenched Normal Approximation for Random Sequences of Transformations. *J. Stat. Phys.*, 178(1):1–37, 2020.

[Kar93] Alan F. Karr. *Probability.* Springer Texts in Statistics. Springer-Verlag, New York, 1993.

[KL20] Alexey Korepanov and Juho Leppänen. Loss of memory and moment bounds for nonstationary intermittent dynamical systems. Preprint, 2020.

[Liv96] Carlangelo Liverani. Central limit theorem for deterministic systems. In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 56–75. Longman, Harlow, 1996.

[LSV99] Carlangelo Liverani, Benoît Saussol, and Sandro Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems*, 19(3):671–685, 1999.

[Mel09] Ian Melbourne. Large and moderate deviations for slowly mixing dynamical systems. *Proc. Amer. Math. Soc.*, 137(5):1735–1741, 2009.

[MN08] Ian Melbourne and Matthew Nicol. Large deviations for nonuniformly hyperbolic systems. *Trans. Amer. Math. Soc.*, 360(12):6661–6676, 2008.

[MPU06] Florence Merlevède, Magda Peligrad, and Sergey Utev. Recent advances in invariance principles for stationary sequences. *Probab. Surv.*, 3:1–36, 2006.

[NTV18] Matthew Nicol, Andrew Török, and Sandro Vaienti. Central limit theorems for sequential and random intermittent dynamical systems. *Ergodic Theory Dynam. Systems*, 38(3):1127–1153, 2018.

[Rio17] Emmanuel Rio. *Asymptotic theory of weakly dependent random processes*, volume 80 of *Probability Theory and Stochastic Modelling*. Springer, Berlin, 2017. Translated from the 2000 French edition [MR2117923].

[VO16] Marcelo Viana and Krerley Oliveira. *Foundations of Ergodic Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.

Matthew Nicol, Department of Mathematics, University of Houston, Houston, TX 77204, USA

E-mail address: nicol@math.uh.edu

URL: http://www.math.uh.edu/~nicol/

Felipe Perez Pereira, School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK

E-mail address: fp16987@bristol.ac.uk

URL: https://felperez.github.io

Andrew Török, Department of Mathematics, University of Houston, Houston, TX 77204, USA and Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

E-mail address: torok@math.uh.edu

URL: http://www.math.uh.edu/~torok/