BASIC CONSTRUCTIONS AND EXAMPLES

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Contents

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1. Glossary.

• A transformation T of a measure space $(X, \mathcal{B}, \mu)$ is *measure-preserving* if $\mu(T^{-1}A) = \mu(A)$ for all measurable $A \in \mathcal{B}$.

• A measure-preserving transformation $(X, \mathcal{B}, \mu, T)$ is *ergodic* if $T^{-1}(A) = A$ (mod μ) implies $\mu(A) = 0$ or $\mu(A^c) = 0$ for each measurable set $A \in \mathcal{B}$.

• A measure-preserving transformation $(X, \mathcal{B}, \mu, T)$ of a probability space is *weak-mixing* if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0$ for all measurable sets $A, B \in \mathcal{B}$.

• A measure-preserving transformation $(X, \mathcal{B}, \mu, T)$ of a probability space is *strong-mixing* if $\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$ for all measurable sets $A, B \in \mathcal{B}$.

• A continuous transformation T of a compact metric space X is *uniquely ergodic* if there is only one T-invariant Borel probability measure on X.

• Suppose $(X, \mathcal{B}, \mu)$ is a probability space. A *finite partition* $\mathcal{P}$ of X is a finite collection of disjoint (mod μ, i.e., up to sets of measure 0) measurable sets $\{P_1, \dots, P_n\}$ such that $X = \cup P_i$ (mod μ). The *entropy* of $\mathcal{P}$ with respect to μ is $H(\mathcal{P}) = -\sum_i \mu(P_i) \ln \mu(P_i)$ (other bases are sometimes used for the logarithm).

• The *metric (or measure-theoretic) entropy of* T *with respect to* $\mathcal{P}$ is $h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee \ldots \vee T^{-n+1}(\mathcal{P}))$, where $\mathcal{P} \vee \ldots \vee T^{-n+1}(\mathcal{P})$ is the partition of X into sets of points with the same coding with respect to $\mathcal{P}$ under T^i, $i = 0, \dots, n-1$. That is x, y are in the same set of the partition $\mathcal{P} \vee \ldots \vee T^{-n+1}(\mathcal{P})$ if and only if $T^i(x)$ and $T^i(y)$ lie in the same set of the partition $\mathcal{P}$ for $i = 0, \dots, n-1$.

• The *metric entropy* $h_\mu(T)$ *of* $(X, \mathcal{B}, \mu, T)$ is the supremum of $h_\mu(T, \mathcal{P})$ over all finite measurable partitions $\mathcal{P}$.

• If T is a continuous transformation of a compact metric space X, then the *topological entropy* of T is the supremum of the metric entropies $h_\mu(T)$, where the supremum is taken over all T-invariant Borel probability measures.

• A system $(X, \mathcal{B}, \mu, T)$ is *loosely Bernoulli* if it is isomorphic to the first-return system to a subset of positive measure of an irrational rotation or a (positive or infinite entropy) Bernoulli system.

• Two systems are *spectrally isomorphic* if the unitary operators that they induce on their L^2 spaces are unitarily equivalent.

• A *smooth dynamical system* consists of a differentiable manifold M and a differentiable map $f : M \to M$. The degree of differentiablity may be specified.

• Two submanifolds S_1, S_2 of a manifold M *intersect transversely* at $p \in M$ if $T_p(S_1) + T_p(S_2) = T_p(M)$.

• An (ϵ-) small C^r *perturbation* of a C^r map f of a manifold M is a map g such that $d_{C^r}(f,g) < \epsilon$ i.e. the distance between f and g is less than ϵ in the C^r topology.

• A map T of an interval $I = [a,b]$ is *piecewise smooth* (C^k for $k \geq 1$) if there is a finite set of points $a = x_1 < x_2 < \ldots < x_n = b$ such that $T|(x_i, x_{i+1})$ is C^k for each i. The degree of differentiability may be specified.

• A measure μ on a measure space $(X, \mathcal{B})$ is *absolutely continuous* with respect to a measure ν on $(X, \mathcal{B})$ if $\nu(A) = 0$ implies $\mu(A) = 0$ for all measurable $A \in \mathcal{B}$.

• A Borel measure μ on a Riemannian manifold M is *absolutely continuous* if it is absolutely continuous with respect to the Riemannian volume on M.

• A measure μ on a measure space $(X, \mathcal{B})$ is equivalent to a measure ν on $(X, \mathcal{B})$ if μ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to μ.

2. Definition of the Subject and its Importance.

Measure-preserving systems are a common model of processes which evolve in time and for which the rules governing the time evolution don't change. For example, in Newtonian mechanics the planets in a solar system undergo motion according to Newton's laws of motion: the planets move but the underlying rule governing the planets' motion remains constant. The model adopted here is to consider the time-evolution as a transformation (either a map in discrete time or a flow in continuous time) on a probability space or more generally a measure space. This is the setting of the subject called ergodic theory. Applications of this point of view include the areas of statistical physics, classical mechanics, number theory, population dynamics, statistics, information theory and economics. The purpose of this chapter is to present a flavor of the diverse range of examples of measure-preserving transformations which have played a role in the development and application of ergodic theory and smooth dynamical systems theory. We also present common constructions involving measure-preserving systems. Such constructions may be considered a way of putting 'building-block' dynamical systems together to construct examples or decomposing a complicated system into simple 'building-blocks' to understand it better.

3. Introduction.

In this chapter we collect a brief list of some important examples of measure-preserving dynamical systems, which we denote typically by $(X, \mathcal{B}, \mu, T)$ or $(T, X, \mathcal{B}, \mu)$ or slight variations. These examples have played a formative role in the development of dynamical systems theory, either because they occur often in applications in one guise or another or because they have been useful simple models to understand certain features of dynamical systems. There is a fundamental difference in the dynamical properties of those systems which display hyperbolicity: roughly speaking there is some exponential divergence of nearby orbits under iteration of the transformation. In differentiable systems this is associated with the derivative of the transformation possessing eigenvalues of modulus greater than one on a 'dynamically significant' subset of phase space. Hyperbolicity leads to complex dynamical behavior such as positive topological entropy, exponential divergence of nearby orbits ("sensitivity to initial conditions") often coexisting with a dense set of periodic orbits. If ϕ, ψ are sufficiently regular functions on the phase space X of a hyperbolic measure-preserving transformation (T, X, μ), then typically we have fast decay of correlations in the sense that

$$|\int_X \phi(T^n x)\psi(x) d\mu - \int \phi d\mu \int \psi d\mu| \leq C a(n)$$

where $a(n) \to 0$. If $a(n) \to 0$ at an exponential rate we say that the system has exponential decay of correlations. A theme in dynamical systems is that the time series formed by sufficiently regular observations on systems with some degree of hyperbolicity often behave statistically like independent identically distributed random variables.

At this point it is appropriate to point out two pervasive differences between the usual probabilistic setting of a stationary stochastic process $\{X_n\}$ and the (smooth) dynamical systems setting of a time series of observations on a measure-preserving system $\{\phi \circ T^n\}$. The most crucial is that for deterministic dynamical systems the time series is usually not an independent process, which is a common assumption in the strictly probabilistic setting. Even if some weak-mixing is assumed in the probabilistic setting it is usually a mixing condition on the σ-algebras $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ generated by successive random variables, a condition which is not natural (and usually very difficult to check) for dynamical systems. Mixing conditions on dynamical systems are given more naturally in terms of the mixing of the sets of the σ-algebra $\mathcal{B}$ of the probability space $(X, \mathcal{B}, \mu)$ under the action of T and not by mixing properties of the σ-algebras generated by the random variables $\{\phi \circ T^n\}$. The other difference is that in the probabilistic setting, although $\{X_n\}$ satisfy moment conditions, usually no regularity properties, such as the Hölder property or smoothness, are assumed. In contrast in dynamical systems theory the transformation T is often a smooth or piecewise smooth transformation of a Riemannian manifold X and the observation $\phi : X \to \mathbb{R}$ is often assumed continuous or Hölder. The regularity of the observation ϕ turns out to play a crucial role in proving properties such as rates of decay of correlation, central limit theorems and so on.

An example of a hyperbolic transformation is an expanding map of the unit interval $T(x) = (2x)$ (where (x) is x modulo the integers). Here the derivative has modulus 2 at all points in phase space. This map preserves Lebesgue measure, has positive topological entropy, Lebesgue almost every point x has a dense orbit and periodic points for the map are dense in $[0, 1)$.

Non-hyperbolic systems are of course also an important class of examples, and in contrast to hyperbolic systems they tend to model systems of 'low complexity', for example systems displaying quasiperiodic behavior. The simplest non-trivial example is perhaps an irrational rotation of the unit interval $[0,1)$ given by a map $T(x) = (x+\alpha)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. T preserves Lebesgue measure, every point has a dense orbit (there are no periodic orbits), yet the topological entropy is zero and nearby points stay the same distance from each other under iteration under T.

There is a natural notion of equivalence for measure-preserving systems. We say that measure-preserving systems $(T, X, \mathcal{B}, \mu)$ and $(S, Y, \mathcal{C}, \nu)$ are *isomorphic* if (possibly after deleting sets of measure 0 from X and Y) there is a one-to-one onto measurable map $\phi : X \to Y$ with measurable inverse ϕ^{-1} such that $\phi \circ T = S \circ \phi$ μ a.e. and $\mu(\phi^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{C}$. If X, Y are compact topological spaces we say that T is *topologically conjugate* to S if there exists a homeomorphism $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$. In this case we call ϕ a *conjugacy.* If ϕ is C^r for some $r \geq 1$ we will call ϕ a C^r*-conjugacy* and similarly for other degrees of regularity.

We will consider $X = [0,1) \pmod 1$ as a representation of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (under the map $x \to e^{2\pi i x}$) and similarly represent the k-dimensional torus $T^k = S^1 \times \ldots \times S^1$ (k-times). If the σ-algebra is clear from the context we will write (T, X, μ) instead of $(T, X, \mathcal{B}, \mu)$ when denoting a measure-preserving system.

4. Examples.

4.1. **Rigid rotation of a compact group.** If G is a compact group equipped with Haar measure and $a \in G$, then the transformation $T(x) = ax$ preserves Haar measure and is called a *rigid rotation* of G. If G is abelian and the transformation is ergodic (in this setting transitivity implies ergodicity), then the transformation is uniquely ergodic. Such systems always have zero topological entropy.

The simplest example of such a system is a circle rotation. Take $X = [0,1) \pmod 1$, with

$$T(x) = (x+\alpha) \quad \text{where } \alpha \in \mathbb{R}.$$

Then T preserves Lebesgue (Haar) measure and is ergodic (in fact uniquely ergodic) if and only if α is irrational. Similarly, the map

$$T(x_1, \ldots, x_k) = (x_1 + \alpha_1, \ldots, x_k + \alpha_k) \text{ , where } \alpha_1, \ldots, \alpha_k \in \mathbb{R},$$

preserves k-dimensional Lebesgue (Haar) measure and is ergodic (uniquely ergodic) if and only if there are no integers $m_1, \ldots, m_k$, not all 0, which satisfy $m_1\alpha_1 + \ldots + m_k\alpha_k \in \mathbb{Z}$.

4.2. **Adding machines.** Let $\{k_i\}_{i\in\mathbb{N}}$ be a sequence of integers with $k_i \geq 2$. Equip each cyclic group $\mathbb{Z}_{k_i}$ with the discrete topology and form the product space $\Sigma = \prod_{i=1}^{\infty} \mathbb{Z}_{k_i}$ equipped with the product topology. An *adding machine* corresponding to the sequence $\{k_i\}_{i\in\mathbb{N}}$ is the topological space $\Sigma = \prod_{i=1}^{\infty} \mathbb{Z}_{k_i}$ together with the map

$$\sigma : \Sigma \to \Sigma$$

defined by

$$\sigma(k_1-1,k_2-1,\dots)=(0,0,\dots) \text{ if each entry in the } \mathbb{Z}_{k_i} \text{ component is } k_i-1,$$

while

$$\sigma(k_1-1,k_2-1,\dots,k_n-1,x_1,x_2,\dots)=(0,\overbrace{0,\dots,0}^{n\ times},x_1+1,x_2,x_3,\dots)$$

when $x_1 \neq k_{n+1}-1$.

The map σ may be thought of as "add one and carry" and also as mapping each point to its successor in a certain order. See Section 5.7 for generalizations. If each $k_i = 2$ then the system is called the *dyadic (or von Neumann-Kakutani) adding machine* or *2-odometer*. Adding machines give examples of naturally occurring minimal systems of low orbit complexity in the sense that the topological entropy of an adding machine is zero. In fact if f is a continuous map of an interval with zero topological entropy and S is a closed, topologically transitive invariant set without periodic orbits, then the restriction of f to S is topologically conjugate to the dyadic adding machine [47, Theorem 11.3.13].

We say a non-empty set Λ is an *attractor* for a map T if there is an open set U containing Λ such that $\Lambda = \cap_{n\geq 0} T^n(U)$ (other definitions are found in the literature). The dyadic adding machine is topologically conjugate to the Feigenbaum attractor at the limit point of period doubling bifurcations (see Section 4.13.1). Furthermore, attractors for continuous unimodal maps of the interval are either periodic orbits, transitive cycles of intervals, or Cantor sets on which the dynamics is topologically conjugate to an adding machine [33].

4.3. **Interval Exchange Maps.** A map $T:[0,1]\to[0,1]$ is an *interval exchange transformation* if it is defined in the following way. Suppose that π is a permutation of $\{1,\dots,n\}$ and $l_i > 0$, $i=1,\dots,n$, is a sequence of subintervals of I (open or closed) with $\sum_i l_i = 1$. Define t_i by $l_i = t_i - t_{i-1}$ with $t_0 = 0$. Suppose also that σ is an n-vector with entries ± 1. T is defined by sending the interval $t_{i-1} \leq x < t_{i-1}$ of length l_i to the interval

$$\sum_{\pi(j)<\pi(i)} l_{\pi(j)} \leq x < \sum_{\pi(j)>\pi(i)} l_{\pi(j)}$$

with orientation preserved if the i'th entry of σ is $+1$ and orientation reversed if the i'th entry of σ is -1. Thus on each interval l_i, T has the form $T(x) = \sigma_i x + a_i$, where σ_i is ± 1. If $\sigma_i = 1$ for each i, the transformation is called *orientation preserving.*

The transformation T has finitely many discontinuities (at the endpoints of each l_i), and modulo this set of discontinuities is smooth. T is also invertible (neglecting the finite set of discontinuities) and preserves Lebesgue measure. These maps have zero topological entropy and arise naturally in studies of polygonal billiards and more generally area-preserving flows. There are examples of minimal but non-ergodic interval exchange maps [58, 72].

4.4. **Full shifts and shifts of finite type.** Given a finite set (or alphabet) $A = \{0, \dots, d-1\}$, take $X = \Omega^+(A) = A^{\mathbb{N}}$ (or $X = A^{\mathbb{Z}}$) the sets of one-sided (two-sided) sequences, respectively, with entries from A. For example sequences in $A^{\mathbb{N}}$ have the form $x = \cdot x_0 x_1 \dots x_n \dots$. A *cylinder set* $C(y_{n_1}, \dots, y_{n_k})$, $y_{n_i} \in A$, of length k is a subset of X defined by fixing k entries; for example,

$$C(y_{n_1}, \dots, y_{n_k}) = \{x : x_{n_1} = y_{n_1}, \dots, x_{n_k} = y_{n_k}\}.$$

We define the set A^k to consist of all cylinders $C(y_1, \dots, y_k)$ determined by fixing the first k entries, i.e. an element of A^k is specified by fixing the first k entries of a sequence $\cdot x_0 \dots x_k$ by requiring $x_i = y_i$, $i = 0, \dots, k$.

Let $p = (p_0, \dots, p_{d-1})$ be a probability vector: all $p_i \geq 0$ and $\sum_{i=0}^{d-1} p_i = 1$. For any cylinder $B = C(b_1, \dots, b_k) \in A^k$, define

$$g_k(B) = p_{b_1} \dots p_{b_k}. \tag{4.1}$$

It can be shown that these functions on A^k extend to a shift-invariant measure μ_p on $A^{\mathbb{N}}$ (or $A^{\mathbb{Z}}$) called product measure. (See the article on Measure-Preserving Systems.) The space $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ may be given a metric by defining

$$d(x,y) = \begin{cases} 1 & \text{if } x_0 \neq y_0; \\ \frac{1}{2^{|n|}} & \text{if } x_n \neq y_n \text{ and } x_i = y_i \text{ for } |i| < n. \end{cases}$$

The *shift* $\sigma(.x_0 x_1 \dots x_n \dots) = .x_1 x_2 \dots x_n \dots$ is ergodic with respect to μ_p. The measure-preserving system $(\Omega, \mathcal{B}, \mu, \sigma)$ (with $\mathcal{B}$ the σ-algebra of Borel subsets of $\Omega(A)$, or its completion), is denoted by $\mathcal{B}(p)$ and is called the *Bernoulli shift* determined by p. This system models an infinite number of independent repetitions of an experiment with finitely many outcomes, the i'th of which has probability p_i on each trial.

These systems are mixing of all orders (i.e. σ^n is mixing for all $n \geq 1$) and have countable Lebesgue spectrum (hence are all spectrally isomorphic). Kolmogorov and Sinai showed that two of them cannot be isomorphic unless they have the same entropy; Ornstein [82] showed the converse. $\mathcal{B}(1/2, 1/2)$ is isomorphic to the Lebesgue-measure-preserving transformation $x \to 2x$ mod 1 on $[0,1]$; similarly, $\mathcal{B}(1/3, 1/3, 1/3)$ is isomorphic to $x \to 3x$ mod 1. Furstenberg asked whether the only nonatomic measure invariant for both $x \to 2x$ mod 1 and $x \to 3x$ mod 1 on $[0,1]$ is Lebesgue measure. Lyons [69] showed that if one of the actions is K, then the measure must be Lebesgue, and Rudolph [101] showed the same thing under the weaker hypothesis that one of the actions has positive entropy. For further work on this question, see [51, 88].

This construction can be generalized to model one-step finite-state Markov stochastic processes as dynamical systems. Again let $A = \{0, \dots, d-1\}$, and let $p = (p_0, \dots, p_{d-1})$ be a probability vector. Let P be a $d \times d$ *stochastic matrix* with rows and columns indexed by A. This means that all entries of P are nonnegative, and the sum of the entries in each row is 1. We regard P as giving the transition probabilities between pairs of elements of A. Now we define for any cylinder $B = C(b_1, \dots, b_k) \in A^k$

$$\mu_{p,P}(B) = p_{b_1} P_{b_1 b_2} P_{b_2 b_3} \dots P_{b_{k-1} b_k}. \tag{4.2}$$

It can be shown that $\mu_{p,P}$ extends to a measure on the Borel σ-algebra of $\Omega^+(A)$, and its completion. (See the article on Measure-Preserving Systems.) The resulting stochastic process is a (one-step, finite-state) *Markov process*. If p and P also satisfy

$$pP = p, \tag{4.3}$$

then the Markov process is stationary. In this case we call the (one or two-sided) measure-preserving system the *Markov shift* determined by p and P.

Aperiodic and irreducible Markov chains (those for which a power of the transition matrix P has all entries positive) are strongly mixing, in fact are isomorphic to Bernoulli shifts (usually by means of a complicated measure-preserving recoding).

More generally we say a set $\Lambda \subset A^{\mathbb{Z}}$ is a *subshift* if it is compact and invariant under σ. A subshift Λ is said to be of *finite type* (SFT) if there exists an $d \times d$ matrix $M = (a_{ij})$ such that all entries are 0 or 1 and $x \in \Lambda$ if and only if $a_{x_i x_{i+1}} = 1$ for all $i \in \mathbb{Z}$. Shifts of finite type are also called *topological Markov chains*. There are many invariant measures for a non-trivial shift of finite type. For example the orbit of each periodic point is the support of an invariant measure. An important role in the theory, derived from motivations of statistical mechanics, is played by *equilibrium measures* (or *equilibrium states*) for continuous functions $\phi : \Lambda \to \mathbb{R}$, i.e. those measures μ which maximize $\{h_\sigma(\mu) + \int_\Lambda \phi d\mu\}$ over all shift-invariant probability measures, where $h_\sigma(\mu)$ is the measure-theoretic entropy of σ with respect to μ. The study of full shifts or shifts of finite type has played a prominent role in the development of the hyperbolic theory of dynamical systems as physical systems with 'chaotic' dynamics 'typically' possess an invariant set with induced dynamics topologically conjugate to a shift of finite type (see the discussion by Smale in [110, p. 147]). Dynamical systems in which there are transverse homoclinic connections are a common example [45, Theorem 5.3.5]. Furthermore in certain settings positive metric entropy implies the existence of shifts of finite type. One result along these lines is a theorem of Katok [54]. Let $h_{top}(f)$ denote the topological entropy of a map f and $h_\mu(f)$ denote metric entropy with respect to an invariant measure μ.

Theorem 4.1 (Katok)**.** *Suppose $T : M \to M$ is a $C^{1+\epsilon}$ diffeomorphism of a closed manifold and μ is an invariant measure with positive metric entropy (i.e. $h_\mu(T) > 0$). Then for any $0 < \epsilon < h_\mu(T)$ there exists an invariant set Λ topologically conjugate to a transitive shift of finite type with $h_{top}(T|\Lambda) > h_\mu(T) - \epsilon$.*

4.5. **More examples of subshifts.** We consider some further examples of systems that are given by the shift transformation on a subset of the set of (usually doubly-infinite) sequences on a finite alphabet, usually $\{0, 1\}$. Associated with each subshift is its *language*, the set of all finite blocks seen in all sequences in the subshift. These languages are *extractive* (or *factorial*) (every subword of a word in the language is also in the language) and *insertive* (or *extendable*) (every word in the language extends on both sides to longer words in the language). In fact these two properties characterize the languages (subsets of the set of finite-length words on an alphabet) associated with subshifts.

4.5.1. *Prouhet-Thue-Morse.* An interesting (and often rediscovered) element of $\{0,1\}^{\mathbb{Z}_+}$ is produced as follows. Start with 0 and at each stage write down the opposite ($0' = 1$, $1' = 0$) or mirror image of what is available so far. Or, repeatedly apply the *substitution* $0 \to 01$, $1 \to 10$.

0
0 1
0 1 10
0 1 10 0110
⋮

The n'th entry is the sum, mod 2, of the digits in the dyadic expansion of n. Using Keane's *block multiplication* [57] according to which if B is a block, $B \times 0 = B, B \times 1 = B'$, and $B \times (\omega_1 \dots \omega_n) = (B \times \omega_1) \dots (B \times \omega_n)$, we may also obtain this sequence as

$$0 \times 01 \times 01 \times 01 \times \dots.$$

The orbit closure of this sequence is uniquely ergodic (there is a unique shift-invariant Borel probability measure, which is then necessarily ergodic). It is isomorphic to a skew product (see Section 5.3) over the von Neumann-Kakutani adding machine, or odometer (see Section 4.2). *Generalized Morse systems*, that is, orbit closures of sequences like $0 \times 001 \times 001 \times 001 \times \dots$, are also isomorphic to skew products over compact group rotations.

4.5.2. *Chacon system.* This is the orbit closure of the sequence generated by the substitution $0 \to 0010, 1 \to 1$. It is uniquely ergodic and is one of the first systems shown to be weakly mixing but not strongly mixing. It is *prime* (has no nontrivial factors) [31], and in fact has *minimal self joinings* [32]. It also has a nice description by means of cutting up the unit interval and stacking the pieces, using spacers (see Section 5.6). This system has singular spectrum. It is not known whether or not its Cartesian square is loosely Bernoulli (see [22]).

4.5.3. *Sturmian systems.* Take the orbit closure of the sequence $\omega_n = \chi_{[1-\alpha,1)}(n\alpha)$, where α is irrational. This is a uniquely ergodic system that is isomorphic to rotation by α on the unit interval. These systems have *minimal complexity* in the sense that the number of n-blocks grows as slowly as possible $(n+1)$ [29].

4.5.4. *Toeplitz systems.* A bi-infinite sequence (x_i) is a Toeplitz sequence if the set of integers can be decomposed into arithmetic progressions such that each x_i is constant on each arithmetic progression. A shift space X is a Toeplitz shift if it is the closure of the orbit of a Toeplitz sequence. It is possible to construct Toeplitz shifts which are uniquely ergodic and isomorphic to a rotation on a compact abelian group. [34]

4.5.5. *Sofic systems.* These are images of SFT's under continuous factor maps (finite codes, or block maps). They correspond to *regular languages*—languages whose words are recognizable by finite automata. These are the same as the languages defined by *regular expressions*—finite expressions built up from $\emptyset$ (empty set), ϵ

(empty word), + (union of two languages), $\cdot$ (all concatenations of words from two languages), and * (all finite concatenations of elements). They also have the characteristic property that the family of all *follower sets* of all blocks seen in the system is a finite family; similarly for *predecessor sets*. These are also generated by *phase-structure grammars* which are *linear*, in the sense that every production is either of the form $A \to Bw$ or $A \to w$, where A and B are variables and w is a string of terminals (symbols in the alphabet of the language).

[A *phase-structure grammar* consists of alphabets V of *variables* and A of *terminals*, a set of *productions*, which is finite set of pairs of words (α, w), usually written $\alpha \to w$, of words on $V \cup A$, and a *start symbol* S. The associated language consists of all words on the alphabet A of terminals which can be made by starting with S and applying a finite sequence of productions.]

Sofic systems typically support many invariant measures (for example they have many periodic points) but topologically transistive ones (those with a dense orbit) have a unique measures of maximal entropy. See [67].

4.5.6. *Context-free systems.* These are generated by phase-structure grammars in which all productions are of the form $A \to w$, where A is a variable and w is a string of variables and terminals.

4.5.7. *Coded systems.* These are systems all of whose blocks are concatenations of some (finite or infinite) list of blocks. These are the same as the closures of increasing sequences of SFT's [62]. Alternatively, they are the closures of the images under finite edge-labelings of irreducible countable-state topological Markov chains. They need not be context-free. Squarefree languages are not coded, in fact do not contain any coded systems of positive entropy. See [13–15].

4.6. **Smooth expanding interval maps.** Take $X = [0,1) \pmod 1$, $m \in \mathbb{N}$, $m > 1$ and

$$T(x) = (mx).$$

Then T preserves Lebesgue measure μ (recall that T preserves μ if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$). Furthermore it can be shown that T is ergodic.

This simple map exemplifies many of the characteristics of systems with some degree of hyperbolicity. It is isomorphic to a Bernoulli shift. The map has positive topological entropy and exponential divergence of nearby orbits, and Hölder functions have exponential decay of correlations and satisfy the central limit theorem and other strong statistical properties [20].

If $m = 2$ the system is isomorphic to a model of tossing a fair coin, which is a common example of randomness. To see this let $\mathcal{P} = \{P_0 = [0, 1/2), P_1 = [1/2, 1]\}$ be a partition of $[0, 1]$ into two subintervals. We code the orbit under T of any point $x \in [0, 1)$ by 0's and 1's by letting $x_k = i$ if $T^k x \in P_i, k = 0, 1, 2, \ldots$. The map $\phi : X \to \{0, 1\}^{\mathbb{N}}$ which associates a point x to its *itinerary* in this way is a measure-preserving map from (X, μ) to $\{0, 1\}^{\mathbb{N}}$ equipped with the Bernoulli measure from

$p_0 = p_1 = \frac{1}{2}$. The map ϕ satisfies $\phi \circ T = \sigma \circ \phi$, μ a.e. and is invertible a.e., hence is an isomorphism. Furthermore, reading the binary expansion of x is equivalent to following the orbit of x under T and noting which element of the partition $\mathcal{P}$ is entered at each time. Borel's theorem on normal numbers (base m) may be seen as a special case of the Birkhofff Ergodic Theorem in this setting.

4.6.1. *Piecewise C^2 expanding maps.* The main statistical features of the examples in Section 4.6 generalize to a broader class of expanding maps of the interval. For example:

Let $X = [0,1]$ and let $\mathcal{P} = \{I_1, \ldots, I_n\}$ $(n \geq 2)$ be a partition of X into intervals (closed, half-open or open) such that $I_i \cap I_j = \emptyset$ if $i \neq j$. Let I_i^o denote the interior of I_i. Suppose $T : X \to X$ satisfies:

(a) For each $i = 1, \ldots, n$, $T|I_i$ has a C^2 extension to the closure $\bar{I}_i$ of I_i and $|T^{'}(x)| \geq \alpha > 1$ for all $x \in I_i^o$.

(b) $T(I_j) = \cup_{i \in P_j} I_i$ Lebesgue a.e. for some non-empty subset $P_j \subset \{1, \ldots, n\}$.

(c) For each I_j there exists n_j such that $T^{n_j}(I_j) = [0,1]$ Lebesgue a.e.

Then T has an invariant measure μ which is absolutely continuous with respect to Lebesgue measure m, and there exists $C > 0$ such that $\frac{1}{C} \leq \frac{d\mu}{dm} \leq C$. Furthermore T is ergodic with respect to μ and displays the same statistical properties listed above for the C^2 expanding maps [20]. (See the "Folklore Theorem" in the article on Measure-Preserving Systems.)

4.7. **More interval maps.**

4.7.1. *Continued fraction map.* This is the map $T : [0,1] \to [0,1]$ given by $Tx = 1/x$ mod 1, and it corresponds to the shift $[0; a_1, a_2, \ldots] \to [0; a_2, a_3, \ldots]$ on the continued fraction expansions of points in the unit interval (a map on $\mathbb{N}^{\mathbb{N}}$). It preserves a unique finite measure equivalent to Lebesgue measure, the *Gauss measure* $dx/(\log 2)(1+x)$. It is Bernoulli with entropy $\pi^2/6\log 2$ (in fact the natural partition into intervals is a weak Bernoulli generator, for the definition and details see [92]). By using the Ergodic Theorem, Khintchine and Lévy showed that

$$(a_1 \ldots a_n)^{1/n} \to \prod_{k=1}^{\infty} \left[1 + \frac{1}{k^2 + 2k}\right]^{\log k/\log 2} \quad \text{a.e. as } n \to \infty;$$

$$\text{if } [0; a_1, \ldots, a_n] = \frac{p_n}{q_n}, \text{ then } \frac{1}{n}\log q_n \to \frac{\pi^2}{12\log 2} \text{ a.e.};$$

$$\frac{1}{n}\log\left|x - \frac{p_n(x)}{q_n(x)}\right| \to \frac{\pi^2}{6\log 2} \text{ a.e.};$$

and if m is Lebesgue measure (or any equivalent measure) and μ is Gauss measure, then for each interval I, $m(T^{-n}I) \to \mu(I)$, in fact exponentially fast, with a best constant $0.30366\ldots$ See [10, 76].

4.7.2. *The Farey map.* This is the map $U : [0,1] \to [0,1]$ given by $Ux = x/(1-x)$ if $0 \le x \le 1/2$, $Ux = (1-x)/x$ if $1/2 \le x \le 1$. It is ergodic for the σ-finite infinite measure dx/x (Rényi and Parry). It is also ergodic for the *Minkowski measure* d, which is a measure of maximal entropy. This map corresponds to the shift on the *Farey tree* of rational numbers which provide the *intermediate convergents* (best one-sided) as well as the continued fraction (best two-sided) rational approximations to irrational numbers. See [63, 64].

4.7.3. *f-expansions.* Generalizing the continued fraction map, let $f : [0,1] \to [0,1]$ and let $\{I_n\}$ be a finite or infinite partition of $[0,1]$ into subintervals. We study the map f by coding itineraries with respect to the partition $\{I_n\}$. For many examples, absolutely continuous (with respect to Lebesgue measure) invariant measures can be found and their dynamical properties determined. See [106].

4.7.4. *β-shifts.* This is the special case of f-expansions when $f(x) = \beta x \bmod 1$ for some fixed $\beta > 1$. This map of the interval is called the *β-transformation.* With a proper choice of partition, it is represented by the shift on a certain subshift of the set of all sequences on the alphabet $D = \{0, 1, \dots, \lfloor \beta \rfloor\}$., called the *$\beta$-shift.* A point x is expanded as an infinite series in negative powers of β with coefficients from this set; $d_\beta(x)_n = \lfloor \beta f^n(x) \rfloor$. (By convention terminating expansions are replaced by eventually periodic ones.) A one-sided sequence on the alphabet D is in the β-shift if and only if all of its shifts are lexicographically less than or equal to the expansion $d_\beta(1)$ of 1 base β. A one-sided sequence on the alphabet D is the valid expansion of 1 for some β is and only if it lexicographically dominates all its shifts. These were first studied by Bissinger [11], Everett [35], Rényi [95] and Parry [85, 86]; there are good summaries by Bertrand-Mathis [9] and Blanchard [12].

For $\beta = \frac{1+\sqrt{5}}{2}$, $d_\beta(1) = 10101010\dots$.

For $\beta = \frac{3}{2}$, $d_\beta(1) = 101000001\dots$ (not eventually periodic).

Every β-shift is coded.

The topological entropy of a β-shift is $\log \beta$. There is a unique measure of maximal entropy $\log \beta$.

A β-shift is a shift of finite type if and only if the β-expansion of 1 is finite. It is sofic if and only if the expansion of 1 is eventually periodic. If β is a Pisot-Vijayaragavhan number (algebraic integer all of whose conjugates have modulus less than 1), then the β-shift is sofic. If the β-shift is sofic, then β is a Perron number (algebraic integer of maximum modulus among its conjugates).

Theorem 4.2 Parry [87]**.** *Every strongly transitive (for every nonempty open set U, $\cup_{n>0} T^n U = X$) piecewise monotonic map on $[0,1]$ is topologically conjugate to a β-transformation.*

4.8. **Gaussian systems.** Consider a real-valued stationary process $\{f_k : -\infty < k < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$. The process (and the associated measure-preserving system consisting of the shift and a shift-invariant measure on $\mathbb{R}^{\mathbb{Z}}$) is called *Gaussian* if for each $d \geq 1$, any d of the f_k form an $\mathbb{R}^d$-valued Gaussian random variable on Ω : this means that with $E(f_k) = m$ for all k and

$$A_{ij} = \int_\Omega (f_{k_i} - m)(f_{k_j} - m)dP = C(k_i - k_j) \text{ for } i, j = 1, \dots, d,$$

where $C(\cdot)$ is a function, for each Borel set $B \subset \mathbb{R}$,

$$P\{\omega : (f_{k_1}(\omega), \dots, f_{k_d}(\omega)) \in B\} = \frac{1}{2\pi^{d/2}\sqrt{\det A}} \int_B \exp\left[-\frac{1}{2}(x - (m, \dots, m))^{tr} A^{-1}(x - (m, \dots, m))\right] dx_1 \dots dx_d.$$

where A is a matrix with entries (A_{ij}). The function $C(k)$ is positive semidefinite and hence has an associated measure σ on $[0, 2\pi)$ such that

$$C(k) = \int_0^{2\pi} e^{ikt} d\sigma(t).$$

Theorem 4.3. *The Gaussian system is ergodic if and only if the "spectral measure" σ is continuous (i.e., nonatomic), in which case it is also weakly mixing. It is mixing if and only if $C(k) \to 0$ as $|k| \to \infty$. If σ is singular with respect to Lebesgue measure, then the entropy is 0; otherwise the entropy is infinite* [30].

For more details see [28].

4.9. **Hamiltonian systems.** (This paragraph is from the article on Measure-Preserving Systems.) Many systems that model physical situations can be studied by means of Hamilton's equations. The state of the entire system at any time is specified by a vector $(q, p) \in \mathbb{R}^{2n}$, the *phase space*, with q listing the coordinates of the positions of all of the particles, and p listing the coordinates of their momenta. We assume there is a time-independent *Hamiltonian function* $H(q, p)$ such that the time development of the system satisfies *Hamilton's equations*:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n. \tag{4.4}$$

Often in applications the Hamiltonian function is the sum of kinetic and potential energy:

$$H(q, p) = K(p) + U(q). \tag{4.5}$$

Solving these equations with initial state (q, p) for the system produces a flow $(q, p) \to T_t(q, p)$ in phase space which moves (q, p) to its position $T_t(q, p)$ t units of time later. According to *Liouville's fomula* [72, Theorem 3.2], this flow preserves Lebesgue measure on $\mathbb{R}^{2n}$. Calculating dH/dt by means of the Chain Rule

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt}\right)$$

and using Hamilton's equations shows that H is constant on orbits of the flow, and thus each set of constant energy $X(H_0) = \{(q, p) : H(q, p) = H_0\}$ is an invariant set. There is a natural invariant measure on a constant energy set $X(H_0)$ for the restricted flow, namely the measure given by rescaling the volume element dS on $X(H_0)$ by the factor $1/||\nabla H||$.

4.9.1. *Billiard Systems.* These form an important class of examples in ergodic theory and dynamical systems, motivated by natural questions in physics, particularly the behavior of gas models. Consider the motion of a particle inside a bounded region D in $\mathbb{R}^d$ with piecewise smooth (C^1 at least) boundaries. In the case of planar billiards we have $d = 2$. The particle moves in a straight line with constant speed until it hits the boundary, at which point it undergoes a perfectly elastic collision with the angle of incidence equal to the angle of reflection and continues in a straight line until it next hits the boundary. It is usual to normalize and consider unit speed, as we do in this discussion for convenience. We take coordinates (x, v) given by the Euclidean coordinates in $x \in D$ together with a direction vector $v \in S^{d-1}$. A flow ϕ_t is defined with respect to Lebesgue almost every (x, v) by translating x a distance t defined by the direction vector v, taking account of reflections at boundaries. ϕ_t preserves a measure absolutely continuous with respect to Riemannian volume on (x, v) coordinates. The flow we have described is called a *billiard flow*. The corresponding *billiard map* is formed by taking the Poincaré map corresponding to the cross-section given by the boundary ∂D. We will describe the planar billiard map; the higher dimensional generalization is clear. The billiard map is a map $T : \partial D \to \partial D$, where ∂D is coordinatized by (s, θ), $s \in [0, L)$, where L is the length of ∂D and $\theta \in (0, \pi)$ measures the angle that inward pointing vectors make with the tangent line to ∂D at s. Given a point (s, θ), the angle θ defines an oriented line $l(s, \theta)$ which intersects ∂D in two points s and s'. Reflecting l in the tangent line to ∂D at the point s' gives another oriented line passing through s' with angle θ' (measured with respect to the angular coordinate system based at s'). The billiard map is the map $T(s, \theta) = (s', \theta')$. T preserves a measure $\mu = \sin\theta ds \times d\theta$. The billiard flow may be modeled as a suspension flow over the billiard map (see Section 5.5).

If the region D is a polygon in the plane (or polyhedron in $\mathbb{R}^d$), then ∂D consists of the faces of the polyhedron. The dynamical behavior of the billiard map or flow in regions with only flat (non-curved) boundaries is quite different to that of billiard flows or maps in regions D with strictly convex or strictly concave boundaries. The topological entropy of a flat polygonal billiard is zero. Research interest focuses on the existence and density of periodic or transitive orbits. It is known that if all the angles between sides are rational multiples of π then there are periodic orbits [16, 42, 75] and they are dense in the phase space [17]. It is also known that a residual set of polygonal billiards are topologically transitive and ergodic [55, 60].

On the other hand, billiard maps in which ∂D has strictly convex components are physical examples of non-uniformly hyperbolic systems (with singularities). The meaning of concave or convex varies in the literature. We will consider a billiard flow inside a circle to be a system with a strictly concave boundary, while a billiard flow on the torus from which a circle has been excised to be a billiard flow with strictly convex boundary.

The class of billiards with some strictly convex boundary components , sometimes called *dispersing billiards* or *Sinai billiards* , was introduced by Sinai [107] who proved many of their fundamental properties. Lazutkin [65] proved that planar billiards with generic strictly concave boundary are not ergodic. Nevertheless Bunimovich [22, 23] produced a large of billiard systems, *Bunimovich billiards*, with strictly concave boundary segments (perhaps with some flat boundaries as well) which were ergodic and non-uniformly hyperbolic. For more details see [26, 56, 70, 111]. We will discuss possibly the simplest example of a dispersing billiard, namely a toral billiard with a single convex obstacle. Take the torus T^2 and consider a single strictly convex subdomain S with C^∞ boundary. The domain of the billiard map is $[0, L) \times (0, \pi)$, where L is the length of ∂S. The measure $\sin(\theta) ds \times d\theta$ is preserved. If the curvature of ∂S is everywhere non-zero, then the billiard map T has positive topological entropy, periodic points are dense, and in fact the system is isomorphic to a Bernoulli shift [41].

4.9.2. *KAM-systems and stably non-ergodic behavior.* A celebrated theorem of Kolmogorov, Arnold and Moser (the KAM theorem) implies that the set of ergodic area-preserving diffeomorphisms of a compact surface without boundary is not dense in the C^r topology for $r \geq 4$. This has important implications, in that there are natural systems in which ergodicity is not generic. The constraint of perturbing in the class of area-preserving diffeomorphisms is an appropriate imposition in many physical models. We will take the version of the KAM theorem as given in [72, Theorem 5.1] (original references include [61], [3] and [79]). An elliptic fixed point for an area-preserving diffeomorphism T of a surface M is called a *non-degenerate elliptic fixed point* if there is a local C^r, $r \geq 4$, change of coordinates h so that in polar coordinates

$$hTh^{-1}(r, \theta) = (r, \theta + \alpha_0 + \alpha_1 r) + F(r, \theta),$$

where all derivatives of F up to order 3 vanish, $\alpha_1 \neq 0$ and $\alpha_0 \neq 0, \frac{\pm\pi}{2}, \pi, \frac{\pm 2\pi}{3}$. A map of the form

$$\tau(r, \theta) = (r, \theta + \alpha_0 + \alpha_1 r),$$

where $\alpha_1 \neq 0$, is called a *twist map.* Note that a twist map leaves invariant the circle $r = k$, for any constant k, and rotates each invariant curve by a rigid rotation $\alpha_1 r$, the magnitude of the rotation depending upon r. With respect to two-dimensional Lebesgue measure a twist map is certainly not ergodic.

Theorem. *Suppose T is a volume-preserving diffeomorphism of class C^r, $r \geq 4$, of a surface M. If x is a non-degenerate elliptic fixed point, then for every $\epsilon > 0$ there exists a neighborhood U_ϵ of x and a set $U_{0,\epsilon} \subset U_\epsilon$ with the properties:*

(a) $U_{0,\epsilon}$ is a union of T-invariant simple closed curves of class C^{r-1} containing x in their interior.

(b) The restriction of T to each such invariant curve is topologically conjugate to an irrational rotation.

(c) $m(U_\epsilon - U_{0,\epsilon}) \leq \epsilon m(U_\epsilon)$, where m is Lebesgue measure on M.

It is possible to prove the existence of a C^r volume preserving diffeomorphism ($r \geq 4$) with a non-degenerate elliptic fixed point and also show that if T possesses a non-degenerate elliptic fixed point then there is a neighborhood V of T in the C^r topology on volume-preserving diffeomorphisms such that each $T^{'} \in V$ possesses a non-degenerate elliptic fixed point [72, Chapter II, Section 6]. As a corollary we have

Corollary. *Let M be a compact surface without boundary and $Diff^r(M)$ the space of C^r area-preserving diffeomorphisms with the C^r topology. Then the set of $T \in Diff^r(M)$ which are ergodic with respect to the probability measure determined by normalized area is not dense in $Diff^r(M)$ for $r \geq 4$.*

4.10. **Smooth uniformly hyperbolic diffeomorphisms and flows.** Time series of measurements on deterministic dynamical systems sometimes display limit laws exhibited by independent identically distributed random variables, such as the central limit theorem, and also various mixing properties. The models of hyperbolicity we discuss in this section have played a key role in showing how this phenomenon of 'chaotic behavior' arises in deterministic dynamical systems. Hyperbolic sets and their associated dynamics have also been pivotal in studies of structural stability. A smooth system is C^r *structurally stable* if a small perturbation in the C^r topology gives rise to a system which is topologically conjugate to the original. When modeling a physical system it is desirable that slight changes in the modeling parameters do not greatly affect the qualitative or quantitative behavior of the ensemble of orbits considered as a whole. The orbit of a point may change drastically under perturbation (especially if the system has sensitive dependence on initial conditions) but the collection of all orbits should ideally be 'similar' to the original unperturbed system. In the latter case one would hope that statistical properties also vary only slightly under perturbation. Structural stability is one, quite strong, notion of stability. The conclusion of a body of work on structural stability is that a system is C^1 structurally stable if and only if it is uniformly hyperbolic and satisfies a technical assumption called strong transversality (see below for details).

Suppose M is a C^1 compact Riemannian manifold equipped with metric d and tangent space TM with norm $\| \ \|$. Suppose also that $U \subset M$ is a non-empty open subset and $T : U \to T(U)$ is a C^1 diffeomorphism. A compact T invariant set $\Lambda \subset U$ is called a *hyperbolic set* if there is a splitting of the tangent space T_pM at each point $p \in \Lambda$ into two invariant subspaces, $T_pM = E^u(p) \oplus E^s(p)$, and a number $0 < \lambda < 1$ such that for $n \geq 0$

$$\|D_pT^n v\| \leq C\lambda^n \|v\| \quad \text{for } v \in E^s(p),$$

$$\|D_pT^{-n} v\| < C\lambda^n \|v\| \quad \text{for } v \in E^u(p).$$

The subspace E^u is called the *unstable* or *expanding subspace* and the subspace E^s the *stable* or *contracting subspace.* The stable and unstable subspaces may be integrated to produce *stable* and *unstable manifolds*

$$W^s(p) = \{y : d(T^n p, T^n y) \to 0\} \text{ as } n \to \infty,$$

$$W^u(p) = \{y : d(T^{-n} p, T^{-n} y) \to 0\} \text{ as } n \to \infty.$$

The stable and unstable manifolds are immersions of Euclidean spaces of the same dimension as $E^s(p)$ and $E^u(p)$, respectively, and are of the same differentiability as T. Moreover, $T_p(W^s(p)) = E^s(p)$ and $T_p(W^u(p)) = E^u(p)$. It is also useful to define *local stable manifolds* and *local unstable manifolds* by

$$W^s_\epsilon(p) = \{y \in W^s(p) : d(T^n p, T^n y) < \epsilon\} \text{ for all } n \geq 0,$$

$$W^u_\epsilon(p) = \{y \in W^u(p) : d(T^{-n} p, T^{-n} y) < \epsilon\} \text{ for all } n \geq 0.$$

Finally we discuss the notion of strong transversality. We say a point x is *non-wandering* if for each open neighborhood U of x there exists an $n > 0$ such that $T^n(U) \cap U \neq \emptyset$. The NW set of non-wandering points is called the *non-wandering set*. We say a dynamical system has the *strong transversal property* if $W^s(x)$ intersects $W^u(y)$ transversely for each pair of points $x, y \in NW$. In the C^r, $r \geq 1$ topology Robbin [96], de Melo [78] and Robinson [98, 99] proved that dynamical systems with the strong transversal property are structurally stable, and Robinson [97] in addition showed that strong transversality was also necessary. Mañé [71] showed that a C^1 structurally stable diffeomorphism must be uniformly hyperbolic and Hayashi [48] extended this to flows. Thus a C^1 diffeomorphism or flow on a compact manifold is structurally stable if and only it is uniformly hyperbolic and satisfies the strong transversality condition.

4.10.1. *Geodesic flow on manifold of negative curvature.* The study of the geodesic flow on manifolds of negative sectional curvature by Hedlund and Hopf was pivotal to the development of the ergodic theory of hyperbolic systems. Suppose that M is a geodesically complete Riemannian manifold. Let $\gamma_{p,v}(t)$ be the geodesic with $\gamma_{p,v}(0) = p$ and $\dot\gamma_{p,v}(0) = v$, where $\dot\gamma_{p,v}$ denotes the derivative with respect to time t. The geodesic flow is a flow ϕ_t on the tangent bundle TM of M, $\phi_t : \mathbb{R} \times TM \to TM$, defined by

$$\phi_t(p, v) = (\gamma_{p,v}(t), \dot\gamma_{p,v}(t)).$$

where $(p, v) \in TM$. Since geodesics have constant speed, if $\|v\| = 1$ then $\|\gamma_{p,v}(t)\| = 1$ for all t, and thus the unit tangent bundle $T^1M = \{(p, v) \in TM : \|v\| = 1\}$ is preserved under the geodesic flow. The geodesic flow and its restriction to the unit tangent bundle both preserve a volume form, Liouville measure. In 1934 Hedlund [49] proved that the geodesic flow on the unit tangent bundle of a surface of strictly negative constant sectional curvature is ergodic, and in 1939 Hopf [50] extended this result to manifolds of arbitrary dimension and strictly negative (not necessarily constant) curvature. Hopf's technique of proof of ergodicity (Hopf argument) was extremely influential and used the foliation of the tangent space into stable and unstable manifolds. For a clear exposition of this technique, and the property of absolute continuity of the foliations into stable and unstable manifolds, see [70]. The geodesic flow on manifolds of constant negative sectional curvature is an Anosov flow (see Section 4.10.4). We remark that for surfaces sectional curvature is the same as Gaussian curvature. Recently the time-one map of the geodesic flow on the unit tangent bundle of a surface with constant negative curvature, which is a partially hyperbolic system (see Section 4.11), was shown to be stably ergodic [46], so the geodesic flow is still playing a major role in the development of ergodic theory.

4.10.2. *Horocycle flow.* All surfaces endowed with a Riemannian metric of constant negative curvature are quotients of the upper half-plane $\mathcal{H}^+ := \{x+iy \in \mathbb{C} : y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, whose sectional curvature is -1. The orientation-preserving isometries of this metric are exactly the linear fractional (also known as Möbius) transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \qquad [z \in \mathcal{H}^+ \mapsto \frac{az+b}{cz+d} \in \mathcal{H}^+].$$

Since each matrix $\pm I$ corresponds to the identity transformation, we consider matrices in $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I\}$.

The unit tangent bundle, $S\mathcal{H}^+$, of the upper half-plane can be identified with $PSL(2,\mathbb{R})$. Then the geodesic flow corresponds to the transformations

$$t \in \mathbb{R} \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

seen as acting on $PSL(2,\mathbb{R})$. The unstable foliation of an element $A \in PSL(2,\mathbb{R}) \cong S\mathcal{H}^+$ is given by

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} A, \qquad t \in \mathbb{R},$$

and the flow along this foliation, given by

$$t \in \mathbb{R} \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

is called the *horocycle flow.* Similarly for the flow induced on the unit tangent bundle of each quotient of the upper half-plane by a discrete group of linear fractional transformations.

The geodesic and horocycle flows acting on a (finite-volume) surface of constant negative curvature form the fundamental example of a transverse pair of actions. The geodesic flow often has many periodic orbits and many invariant measures, has positive entropy, and is in fact Bernoulli with respect to the natural measure [83], while the horocycle flow is often uniquely ergodic [40, 74] and of entropy zero, although mixing of all orders [73]. See [47] for more details.

4.10.3. *Markov partitions and coding.* If $(X, T, \mathcal{B}, \mu)$ is a dynamical system then a finite partition of X always induces a coding of the orbits and a semi-conjugacy with a subshift on a symbol space (it may not of course be a full conjugacy). For hyperbolic systems a special class of partitions, *Markov partitions*, induce a conjugacy for the invariant dynamics to a subshift of finite type. A Markov partition $\mathcal{P}$ for an invariant subset Λ of a diffeomorphism T of a compact manifold M is a finite collection of sets R_i, $1 \le i \le n$ called *rectangles.* The rectangles have the property, for some $\epsilon > 0$, if $x, y \in R_i$ then $W^s_\epsilon(x) \cap W^u_\epsilon(y) \in R_i$. This is sometimes described as being closed under *local product structure.* We let $W^u(x, R_i)$ denote $W^u_\epsilon(x) \cap R_i$ and $W^s(x, R_i)$ denote $W^s_\epsilon(x) \cap R_i$. Furthermore we require for all i, j:

(1) Each R_i is the closure of its interior.
(2) $\Lambda \subset \cup_i R_i$

(3) $R_i \cap R_j = \partial R_i \cap \partial R_j$ if $i \neq j$
(4) if $x \in R_i^o$ and $T(x) \in R_j^o$ then $W^u(T(x), R_j) \subset T(W^u(x, R_i))$ and $W^s(x, R_i) \subset T^{-1}(W^u(T(x), R_j))$

4.10.4. *Anosov systems.* An *Anosov diffeomorphism* [2] is a uniformly hyperbolic system in which the entire manifold is a hyperbolic set. Thus an Anosov diffeomorphism is a C^1 diffeomorphism T of M with a DT-invariant splitting (which is a continuous splitting) of the tangent space $TM(x)$ at each point p into a disjoint sum

$$T_pM = E^u(p) \oplus E^s(p)$$

and $0 < \lambda < 1$, constant C such that $\|DT^n v\| < C\lambda^n \|v\|$ for all $v \in E^s(p)$ and $\|DT^{-n} w\| \leq C\lambda^n \|w\|$ for all $w \in E^u(p)$.

A similar definition holds for Anosov flows $\phi : \mathbb{R} \times M \to M$. A flow is *Anosov* if there is a splitting of the tangent bundle into flow-invariant subspaces E^u, E^s, E^c so $D_p\phi_t E^s_p = E^s_{\phi_t(p)}$, $D_p\phi_t E^u_p = E^u_{\phi_t(p)}$ and $D_p\phi_t E^c_p = E^c_{\phi_t(p)}$, and at each point $p \in M$

$$T_pM = E^s_p \oplus E^u_p \oplus E^c_p$$

$$\|(D_p\phi_t)v\| < C\lambda^t \|v\| \quad \text{for } v \in E^s(p)$$

$$\|(D_p\phi_{-t})v\| < C\lambda^t \|v\| \quad \text{for } v \in E^u(p)$$

for some $0 < \lambda < 1$. The tangent to the flow direction $E^c(p)$ is a neutral direction:

$$\|(D_p\phi_t)v\| = \|v\| \quad \text{for } v \in E^c(p).$$

Anosov proved that Anosov flows and diffeomorphisms which preserve a volume form are ergodic [2] and are also structurally stable. Sinai [108] constructed Markov partitions for Anosov diffeomorphisms and hence coded trajectories via a subshift of finite type. Using ideas from statistical physics in [109] Sinai constructed Gibbs measures for Anosov systems. An SRB measure (see Section 4.13) is a type of Gibbs measure corresponding to the potential $-\log |det(DT|_{E^u})|$ and is characterized by the property of absolutely continuous conditional measures on unstable manifolds.

The simplest examples of Anosov diffeomorphisms are perhaps the two-dimensional hyperbolic toral automorphisms (the $n > 2$ generalization is clear). Suppose A is a 2×2 matrix with integer entries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $\det(A) = 1$ and A has no eigenvalues of modulus 1. Then A defines a transformation of the two-dimensional torus $T^2 = S^1 \times S^1$ such that if $v \in T^2$,

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

then

$$Av = \begin{pmatrix} (av_1 + bv_2) \\ (cv_1 + dv_2) \end{pmatrix}.$$

A preserves Lebesgue (or Haar) measure and is ergodic. A prominent example of such a matrix is

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

which is sometimes called the Arnold Cat Map. Each point with rational coordinates $(p_1/q_1, p_2/q_2)$ is periodic. There are two eigenvalues $\frac{1}{\lambda} < 1 < \lambda = \frac{3+\sqrt{5}}{2}$ with orthogonal eigenvectors, and the projections of the eigenspaces from $\mathbb{R}^2$ to T^2 are the stable and unstable subspaces.

4.10.5. *Axiom A systems.* In the case of Anosov diffeomorphisms the splitting into contracting and expanding bundles holds on the entire phase space M. A system $T : M \to M$ is an *Axiom A system* if the non-wandering set NW is a hyperbolic set and periodic points are dense in the non-wandering set. $NW \subset M$ may have Lebesgue measure zero. A set $\Lambda \subset M$ is *locally maximal* if there exists an open set U such that $\Lambda = \cap_{n \in \mathbb{Z}} T^n(U)$. The solenoid and horseshoe discussed below are examples of locally maximal sets. Bowen [18] constructed Markov partitions for Axiom A diffeomorphisms . Ruelle imported ideas from statistical physics, in particular the idea of an equilibrium state and the variational principle, to the study of Axiom A systems (see [103, 104]) This work extended the notion of Gibbs measure and other ideas from statistical mechanics, introduced by Sinai for Anosov systems [109], into Axiom A systems.

One achievement of the Axiom A program was the Smale Decomposition Theorem, which breaks the dynamics of Axiom A systems into locally maximal sets and describes the dynamics on each [18, 19, 110].

Theorem (Spectral Decomposition Theorem)**.** *If T is Axiom A then there is a unique decomposition of the non-wandering set NW of T*

$$NW = \Lambda_1 \cup \ldots \cup \Lambda_k$$

as a disjoint union of closed, invariant, locally maximal hyperbolic sets Λ_i such that T is transitive on each Λ_i. Furthermore each Λ_i may be further decomposed into a disjoint union of closed sets Λ_i^j, $j = 1, \ldots, n_i$ such that T^{n_i} is topologically mixing on each Λ_i^j and T cyclically permutes the Λ_i^j.

Horseshoe maps.

This type of map was introduced by Steven Smale in the 1960's and has played a pivotal role in the development of dynamical systems theory. It is perhaps the canonical example of an Axiom A system [110] and is conjugate to a full shift on 2 symbols. Let S be a unit square in $\mathbb{R}^2$ and let T be a diffeomorphism of S onto its image such that $S \cap T(S)$ consists of two disjoint horizontal strips S_0 and S_1. Think of stretching S uniformly in the horizontal direction and contracting uniformly in the vertical direction to form a long thin rectangle, and then bending the rectangle into the shape of a horsehoe and laying the straight legs of the horseshoe back on the unit square S. This transformation may be realized by a diffeomorphism and we may also require that T restricted to $T^{-1}S_i$, $i = 0, 1$, acts as a linear map. The restriction of T to the maximal invariant set $H = \cap_{i=-\infty}^{\infty} T^i S$ is a Smale horseshoe map. H is a Cantor set, the product of a Cantor set in the horizontal direction and a Cantor set in the vertical direction. The conjugacy with the shift on two symbols is realized by mapping $x \in H$ to its itinerary with respect to the sets S_0 and S_1 under powers of T (positive and negative powers).

Solenoids

The solenoid is defined on the solid torus X in $\mathbb{R}^3$ which we coordinatize as a circle of two-dimensional solid disks, so that

$$X = \{(\theta, z) : \theta \in [0,1) \text{ and } |z| \leq 1, z \in \mathbb{C}\}$$

The transformation $T : X \to X$ is given by

$$T(\theta, z) = (2\theta \ (\text{mod } 1)\ , \frac{1}{4}z + \frac{1}{2}e^{2\pi i\theta})$$

Geometrically the transformation stretches the torus to twice its length, shrinks its diameter by a factor of 4, then twists it and doubles it over, placing the resultant object without self-intersection back inside the original solid torus. $T(X)$ intersects each disk $D_c = \{(\theta, z) : \theta = c\}$ in two smaller disks of $\frac{1}{4}$ the diameter. The transformation T contracts volume by a factor of 2 upon each application, yet there is expansion in the θ direction ($\theta \to 2\theta$). The solenoid $A = \cap_{n\geq 0} T^n(X)$ has zero Lebesgue measure, is T-invariant and is (locally) topologically a line segment cross a two-dimensional Cantor set (A intersects each disk D_c in a Cantor set). The set A is an *attractor*, in that all points inside X limit under iteration by T upon A. $T : A \to A$ is an Axiom A system.

4.11. **Partially hyperbolic dynamical systems.** Partially hyperbolic dynamical systems are a generalization of uniformly hyperbolic systems in that an invariant central direction is allowed but the contraction in the central direction is strictly weaker than the contraction in the contracting direction and the expansion in the central direction is weaker than the expansion in the expanding direction. More precisely, suppose M is a C^1 compact (adapted) Riemannian manifold equipped with metric d and tangent space TM with norm $\| \ \|$. A C^1 diffeomorphism T of M is a partially hyperbolic diffeomorphism if there is a nontrivial continuous DT invariant splitting of the tangent space T_pM at each point p into a disjoint sum

$$T_pM = E^u(p) \oplus E^c(p) \oplus E^s(p)$$

and continuous positive functions $m, M, \tilde{\gamma}, \gamma$ such that

- E^s is contracted: if $v^s \in E^s(x) \setminus \{0\}$ then $\frac{\|D_pT^n v^s\|}{\|v^s\|} \leq m(p) < 1$;
- E^u is expanded: if $v^u \in E^u(x) \setminus \{0\}$ then $\frac{\|D_pTv^u\|}{\|v^u\|} \geq M(p) > 1$;
- E^c is uniformly dominated by E^u and E^s: if $v^c \in E^c(x) \setminus \{0\}$ then there are numbers $\tilde{\gamma}(p), \gamma(p)$ such that $m(p) < \tilde{\gamma}(p) \leq \frac{\|D_pTv^c\|}{\|v^c\|} \leq \gamma(p) < M(p)$.

The notion of partial hyperbolicity was introduced by Brin and Pesin [21] who proved existence and properties, including absolute continuity, of invariant foliations in this setting. There has been intense recent interest in partially hyperbolic systems primarily because significant progress has been made in establishing that certain volume-preserving partially hyperbolic systems are 'stably ergodic'—that is, they are ergodic and under small (C^r topology) volume-preserving perturbations remain ergodic. This phenomenon had hitherto been restricted to uniformly hyperbolic systems. For recent developments, and precise statements, on stable ergodicity of partially hyperbolic systems see [24, 94].

4.11.1. *Compact group extensions of uniformly hyperbolic systems.* A natural example of a partially hyperbolic system is given by a compact group extension of an Anosov diffeomorphism. If the following terms are not familiar see section 5 on standard constructions. Suppose that (T, M, μ) is an Anosov diffeomorphism, G is a compact connected Lie group and $h : M \to G$ is a differentiable map. The skew product $F : M \times G \to M \times G$ given by

$$F(x, g) = (Tx, h(x)g)$$

has a central direction in its tangent space corresponding to the Lie algebra LG of G (as a group element h acts isometrically on G so there is no expansion or contraction) and uniformly expanding and contracting bundles corresponding to those of the tangent space of $T : M \to M$. Thus $T(M \times G) = E^u \oplus LG \oplus E^s$.

4.11.2. *Time-one maps of Anosov flows.* Another natural context in which partial hyperbolicity arises is in time-one maps of uniformly hyperbolic flows. Suppose $\phi_t : \mathbb{R} \times M \to M$ is an Anosov flow. The diffeomorphism $\phi_1 : M \to M$ is a partially hyperbolic diffeomorphism with central direction given by the flow direction. There is no expansion or contraction in the central direction.

4.12. **Non-uniformly hyperbolic systems.** The assumption of uniform hyperbolicity is quite restrictive and few 'chaotic systems' found in applications are likely to exhibit uniform hyperbolicity. A natural weakening of this assumption, and one that is non-trivial and greatly extends the applicability of the theory, is to require the hyperbolic splitting (no longer uniform) to hold only at almost every point of phase space. A systematic theory was built by Pesin [90, 91] on the assumption that the system has non-zero Lyapunov exponents μ almost everywhere, where μ is Lebesgue equivalent invariant probability measure . Recall that a number λ is a *Lyapunov exponent* for $p \in M$ if $\|D_p T^n v\| \sim e^{\lambda n}$ for some unit vector $v \in T_p M$. Oseledet's theorem [84] (see also [114, p. 232]), which is also called the Multiplicative Ergodic Theorem, implies that if T is a C^1 diffeomorphism of M then for any T-invariant ergodic measure μ almost every point has well-defined Lyapunov exponents. One of the highlights of Pesin theory is the following structure theorem: If $T : M \to M$ is a $C^{1+\epsilon}$ diffeomorphism with a T-invariant Lebesgue equivalent Borel measure μ such that T has non-zero Lyapunov exponents with respect to μ then T has at most a countable number of ergodic components $\{C_i\}$ on each of which the restriction of T is either Bernoulli or Bernoulli times a rotation (by which we mean the support of $\mu_i = \mu_{|C_i}$ consists of a finite number n_i of sets $\{S_1^i, \dots S_{n_i}^i\}$ cyclically permuted and T^{n_i} is Bernoulli when restricted to each S_j^i) [91, 117]. This structure theorem has been generalized to SRB measures with non-zero Lyapunov exponents [66, 91].

4.13. **Physically relevant measures and strange attractors.** (This paragraph is from the article on Measure-Preserving Systems.) For Hamiltonian systems and other volume-preserving systems it is natural to consider ergodicity (and other statistical properties) of the system with respect to Lebesgue measure. In dissipative systems a measure equivalent to Lebesgue may not be invariant (for example the solenoid). Nevertheless Lebesgue measure has a distinguished role since sampling by experimenters is done with respect to Lebesgue measure. The idea of a physically relevant measure μ is that it determines the statistical behavior of a positive Lebesgue measure set of orbits, even though the support of μ may have zero Lebesgue measure. An example of such a situation in the uniformly hyperbolic setting is the solenoid Λ, where the attracting set Λ has Lebesgue measure zero and is (locally) topologically the product of a two-dimensional Cantor set and a line segment. Nevertheless Λ determines the behavior of all points in a solid torus in $\mathbb{R}^3$. More generally, suppose that $T : M \to M$ is a diffeomorphism on a compact Riemannian manifold and that m is a version of Lebesgue measure on M, given by a smooth volume form. Although Lebesgue measure m is a distinguished physically relevant measure, m may not be invariant under T, and the system may even be volume contracting in the sense that $m(T^n A) \to 0$ for all measurable sets A. Nevertheless an experimenter might observe long-term "chaotic" behavior whenever the state of the system gets close to some compact invariant set X which attracts a positive m-measure of orbits in the sense that these orbits limit on X. Possibly $m(X) = 0$, so that X is effectively invisible to the observer except through its effects on orbits not contained in X. The dynamics of T restricted to X can in fact be quite complicated—maybe a full shift, or a shift of finite type, or some other complicated topological dynamical system. Suppose there is a T-invariant measure μ supported on X such that for all continuous functions $\phi : M \to \mathbb{R}$

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi \circ T^k(x) \to \int_X \phi \, d\mu, \tag{4.6}$$

for a positive m-measure of points $x \in M$. Then the long-term equilibrium dynamics of an observable set of points $x \in M$ (i.e. a set of points of positive m measure) is described by (X, T, μ). In this situation μ is described as a *physical measure*. There has been a great deal of research on the properties of systems with attractors supporting physical measures.

In the dissipative non-uniformly hyperbolic setting the theory of 'physically relevant' measures is best developed in the theory of SRB (for Sinai, Ruelle and Bowen) measures. These dynamically invariant measures may be supported on a set of Lebesgue measure zero yet determine the asymptotic behavior of points in a set of positive Lebesgue measure.

If T is a diffeomorphism of M and μ is a T-invariant Borel probability measure with positive Lyapunov exponents which may be integrated to unstable manifolds, then we call μ an *SRB measure* if the conditional measure μ induces on the unstable manifolds is absolutely continuous with respect to the Riemannian volume element on these manifolds. The reason for this definition is technical but is motivated by the following observations. Suppose that the diffeomorphism has no zero Lyapunov exponents with respect to μ. Since T is a diffeomorphism, this implies

T has negative Lyapunov exponents as well as positive Lyapunov exponents and corresponding local stable manifolds as well as local unstable manifolds. Suppose that a T-invariant set A consists of a union of unstable manifolds and is the support of an ergodic SRB measure μ and that $\phi : M \to \mathbb{R}$ is a continuous function. Since μ has absolutely continuous conditional measures on unstable manifolds with respect to conditional Lebesgue measure on the unstable manifolds, almost every point x in the union of unstable manifolds U satisfies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ T^j(x) = \int \phi \, d\mu \tag{4.7}$$

If $y \in W^s_\epsilon(x)$ for such an $x \in U$ then $d(T^n x, T^n y) \to \infty$ and hence (4.7) implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ T^j(y) = \int \phi \, d\mu$$

Furthermore, if the holonomy between unstable manifolds defined by sliding along stable manifolds is absolutely continuous (takes sets of zero Lebesgue measure on W^u to sets of zero Lebesgue measure on W^u), there is a positive Lebesgue measure of points (namely an unstable manifold and the union of stable manifolds through it) satisfying (4.7). Thus an SRB measure with absolutely continuous holonomy maps along stable manifolds is a physically relevant measure. If the stable foliation possesses this property it is called *absolutely continuous*. An Axiom A attractor for a C^2 diffeomorphism is an example of an SRB attractor [19, 103, 104, 109]. The examples we have given of SRB measures and attractors and measures have been uniformly hyperbolic.

Recently much progress has been made in understanding the statistical properties of non-uniformly hyperbolic systems by using a tower (see Section 5.4) to construct SRB measures. We refer to Young's original papers [115, 116], the book by Baladi [4] and to [117] for a recent survey on SRB measures in the non-uniformly setting.

4.13.1. *Unimodal maps.* Maps of an interval to itself are simple examples of non-uniformly hyperbolic systems that have played an important role in the development of dynamical systems theory. Suppose $I \subset \mathbb{R}$ is an interval; for simplicity we take $I = [0, 1]$. A *unimodal map* is a map $T : [0, 1] \to [0, 1]$ such that there exists a point $0 < c < 1$ and

- T is C^2;
- $T'(x) > 0$ for $x < c$, $T'(x) < 0$ for $x > c$;
- $T'(c) = 0$.

Such a map is clearly not uniformly expanding, as $|T'(x)| < 1$ for points in a neighborhood of c. The family of maps $T_\mu(x) = \mu x(1 - x)$, $0 < \mu \leq 4$, is a family of unimodal maps with $c = 1/2$ and $T_2(1/2) = 1/2$, $T_4(1/2) = 1$.

We could have taken the interval I to be $[-1, 1]$ or indeed any interval with an obvious modification of the definition above. A well-studied family of unimodal maps in this setting is the *logistic family* $f_a : [-1, 1] \to [-1, 1]$, $f_a(x) = 1 - ax^2$, $a \in$

$(0,2]$. The families are equivalent under a smooth coordinate change, so statements about one family may be translated into statements about the other.

Unimodal maps are studied because of the insights they offer into transitions from regular or periodic to chaotic behavior as a parameter (e.g. μ or a) is varied, the existence of absolutely continuous measures, and rates of decay of correlations of regular observations for non-uniformly hyperbolic systems.

A result of Jakobson [53] and Benedicks and Carleson [6] implies that in the case of the logistic family there is a positive Lebesgue measure set of a such that f_a has an absolutely continuous ergodic invariant measure μ_a. It has been shown by Young [116] and Keller and Nowicki [59] that if f_a is mixing with respect to μ_a then the decay of correlations for Lipshitz observations on I is exponential. It is also known that the set of a such that f_a is mixing with respect to μ_a has positive Lebesgue measure. There is a well-developed theory concerning the bifurcations the maps T_μ undergo as μ varies [27]. We briefly describe the period-doubling route to chaos in the family $T_\lambda(x) = \lambda x(1-x)$. For a nice account see [47]. We let c_λ denote the fixed point $\frac{\lambda-1}{\lambda}$. For $3 < \lambda \leq 1+\sqrt{6}$, all points in $[0,1]$ except for $0, c_\lambda$ and their preimages are attracted to a unique periodic orbit $O(p_\lambda)$ of period 2. There is a monotone sequence of parameter values λ_n ($\lambda_1 = 3$) such that for $\lambda_n < \lambda \leq \lambda_{n+1}$, T_λ has a unique attracting periodic orbit $O(\lambda_n)$ of period 2^n and for each $k = 1, 2, \ldots, n-1$ a unique repelling orbit of period 2^k. All points in the interval $[0,1]$ except for the repelling periodic orbits and their preimages are attracted to the attracting periodic orbit of period 2^n. At $\lambda = \lambda_n$ the periodic orbit $O(\lambda_n)$ undergoes a period-doubling bifurcation. Feigenbaum [36] found that the limit $\delta = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \sim 4.699\ldots$ exists and that in a wide class of unimodal maps this period-doubling cascade occurs and the differences between successive bifurcation parameters give the same limiting ratio, an example of *universality.* At the end of the period-doubling cascade at a parameter $\lambda_\infty \sim 3.569\ldots$, T_{λ_∞} has an invariant Cantor set C (the Feigenbaum attractor) which is topologically conjugate to the dyadic adding machine coexisting with isolated repelling orbits of period 2^n, $n = 0, 1, 2, \ldots$. There is a unique repelling orbit of period 2^n for $n \geq 1$ along with two fixed points. The Cantor set is the ω-limit set for all points that are not periodic or preimages of periodic orbits. C is the set of accumulation points of periodic orbits. Despite this picture of incredible complexity the topological entropy is zero for $\lambda \leq \lambda_\infty$. For $\lambda > \lambda_\infty$ the map T_λ has positive topological entropy and infinitely many periodic orbits whose periods are not powers of 2. For each $\lambda \geq \lambda_\infty$, T_λ possesses an invariant Cantor set which is repelling for $\lambda > \lambda_\infty$. We say that T_λ is *hyperbolic* if there is only one attracting periodic orbit and the only recurrent sets are the attracting periodic orbit, repelling periodic orbits and possibly a repelling invariant Cantor set. It is known that the set of $\lambda \in [0,4]$ for which T_λ is hyperbolic is open and dense [44]. Remarkably, by Jakobson's result [53] there is also a positive Lebesgue measure set of parameters λ for which T_λ has an absolutely continuous invariant measure μ_λ with a positive Lyapunov exponent.

4.13.2. *Intermittent maps.* Maps of the unit interval $T : [0,1] \to [0,1]$ which are expanding except at the point $x = 0$, where they are locally $x \sim x + x^{1+\alpha}$, $\alpha > 0$, have been extensively studied both for the insights they give into rates of decay of correlations for non-uniformly hyperbolic systems (hyperbolicity is lost at the point $x = 0$, where the derivative is 1) and for their use as models of intermittent behavior in turbulence [93]. A fixed point where the derivative is 1 is sometimes called an *indifferent fixed point.* It is a model of *intermittency* in the sense that orbits close to 1 will stay close for many iterates (since the expansion is very weak there) and hence a time series of observations will be quite uniform for long periods of time before displaying chaotic type behavior after moving away from the indifferent fixed into that part of the domain where the map is uniformly expanding.

A particularly simple model [68] is provided by

$$T(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } x \in [0,1/2); \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases}$$

For $\alpha = 0$ the map is uniformly expanding and Lebesgue measure is invariant. In this case the rate of decay of correlations for Hölder observations is exponential. For $0 < \alpha < 1$ the map has an SRB measure μ_α with support the unit interval. For $\alpha \geq 1$ there are no absolutely continuous invariant probability measures though there are σ-finite absolutely continuous measures. Upper and lower polynomial bounds on the rate of decay of observations on such maps have been given as a function of $0 < \alpha < 1$ and the regularity of the observable. For details see [52, 68, 105].

4.13.3. *Hénon diffeomorphisms.* The Henón family of diffeomorphisms was introduced and studied as Poincaré maps for the Lorenz system of equations. It is a two-parameter two-dimensional family which shares many characteristics with the logistic family and for small $b > 0$ may be considered a two-dimensional 'perturbation' of the logistic family. The parametrized mapping is defined as

$$T_{a,b}(x,y) = (1 - ax^2 + y, bx),$$

so $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ with $0 < a < 2$ and $b > 0$. Benedicks and Carleson [7] showed that for a positive-measure set of parameters (a,b), $T_{a,b}$ has a topologically transitive attractor $\Lambda_{a,b}$. Benedicks and Young [8] later proved that for a positive-measure set of parameters (a,b), $T_{a,b}$ has a topologically transitive SRB attractor $\Lambda_{a,b}$ with SRB measure $\mu_{a,b}$ and that $(T_{a,b}, \Lambda_{a,b}, \mu_{a,b})$ is isomorphic to a Bernoulli shift.

4.14. **Complex dynamics.** Complex dynamics is concerned with the behavior of rational maps

$$\frac{\alpha_1 z^d + \alpha_2 z^{d-1} + \dots \alpha_{d+1}}{\beta_1 z^d + \beta_2 z^{d-1} + \dots \beta_{d+1}}$$

of the extended complex plane $\bar{\mathbb{C}}$ to itself, in which the domain is $\mathbb{C}$ completed with the point at infinity (called the Riemann sphere). Recall that a family F of meromorphic functions is called *normal* on a domain D if every sequence possesses a subsequence that converges uniformly (in the spherical metric $\bar{\mathbb{C}} \sim S^2$) on compact subsets of D. A family is *normal at a point* $z \in \bar{\mathbb{C}}$ if it is normal on a neighborhood of z. The *Fatou set* $F(R) \subset \bar{\mathbb{C}}$ of a rational map $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is the set of points $z \in \bar{\mathbb{C}}$ such that the family of forward iterates $\{R^n\}_{n \geq 0}$ is normal at z. The *Julia set* $J(R)$ is the complement of the Fatou set $F(R)$. The Fatou set is open and hence the Julia set is a closed set. Another characterization in the case $d > 1$ is that $J(R)$ is the closure of the set of all repelling periodic orbits of $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$. Both $F(R)$

and $J(R)$ are invariant under R. The dynamics of greatest interest is the restriction $R : J(R) \to J(R)$. The Julia set often has a complicated fractal structure. In the case that $R_a(z) = z^2 - a$, $a \in \mathbb{C}$, the *Mandelbrot set* is defined as the set of a for which the orbit of the origin 0 is bounded. The topology of the Mandelbrot set has been the subject of intense research. The study of complex dynamics is important because of the fascinating and complicated dynamics displayed and also because techniques and results in complex dynamics have direct implications for the behavior of one-dimensional maps. For more details see [25].

4.15. **Infinite ergodic theory.** We may also consider a measure-preserving transformation (T, X, μ) of a measure space such that $\mu(X) = \infty$. For example X could be the real line equipped with Lebesgue measure. This setting also arises with compact X in applications. For example, suppose $T : [0,1] \to [0,1]$ is the simple model of intermittency given in Section 4.13.2 and $\gamma \in (1,2)$. Then T possesses an absolutely continuous invariant measure μ with support $[0,1]$, but $\mu([0,1]) = \infty$. The Radon-Nikodym derivative of μ with respect to Lebesgue measure m exists but is not in $L^1(m)$.

In this setting we say a measurable set A is a *wandering set* for T if $\{T^{-n}A\}_{n=0}^{\infty}$ are disjoint. Let $D(T)$ be the measurable union of the collection of wandering sets for T. The transformation T is *conservative* with respect to μ if $(X \setminus D(T)) = X$ (mod μ) (see the article on Measure-Preserving Systems). It is usually necessary to assume T conservative with respect to μ to say anything interesting about its behavior. For example if $T(x) = x + \alpha$, $\alpha > 0$, is a translation of the real line then $D(T) = X$. The definition of ergodicity in this setting remains the same: T is *ergodic* if $A \in \mathcal{B}$ and $T^{-1}A = A$ mod μ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$. However the equivalence of ergodicity of T with respect to μ and the equality of time and space averages for $L^1(\mu)$ functions no longer holds. Thus in general μ ergodic does not imply that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ T^j(x) = \int_X \phi \, d\mu \quad \mu \text{ a.e. } x \in X$$

for all $\phi \in L^1(\mu)$. In the example of the intermittent map with $\gamma \in (1,2)$ the orbit of Lebesgue almost every $x \in X$ is dense in X, yet the fraction of time spent near the indifferent fixed point $x = 0$ tends to one for Lebesgue almost every $x \in X$. In fact it may be shown [1, Section 2.4] that when $\mu(x) = \infty$ there are no constants $a_n > 0$ such that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=0}^{n-1} \phi \circ T^j(x) = \int_X \phi \, d\mu \quad \mu \text{ a.e. } x \in X$$

Nevertheless it is sometimes possible to obtain distributional limits, rather than almost sure limits, of Birkhoff sums under suitable normalization. We refer the reader to Aaronson's book [1] for more details.

5. Constructions.

We give examples of some of the standard constructions in dynamical systems. Often these constructions appear in modeling situations (for example skew products are often used to model systems which react to inputs from other systems, continuous time systems are often modeled as suspension flows over discrete-time dynamics) or to reduce systems to simpler components (often a factor system or induced system is simpler to study). Unless stated otherwise, in the sequel we will be discussing measure-preserving transformations on Lebesgue spaces (see the article on Measure-Preserving Systems).

5.1. **Products.** Given measure-preserving systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$, their *product* consists of their completed product measure space with the transformation $T \times S : X \times Y \to X \times Y$ defined by $(T \times S)(x, y) = (Tx, Sy)$ for all $(x, y) \in X \times Y$. Neither ergodicity nor transitivity is in general preserved by taking products; for example the product of an irrational rotation on the unit circle with itself is not ergodic. For a list of which mixing properties are preserved by forming a product see [114]. Given any countable family of measure-preserving transformations on probability spaces, their direct product is defined similarly.

5.2. **Factors.** We say that a measure-preserving system $(Y, \mathcal{C}, \nu, S)$ is a *factor* of a measure-preserving system $(X, \mathcal{B}, \mu, T)$ if (possibly after deleting a set of measure 0 from X) there is a measurable onto map $\phi : X \to Y$ such that

$$\begin{aligned} \phi^{-1}\mathcal{C} &\subset \mathcal{B}, \\ \phi T &= S\phi, \quad \text{and} \\ \mu T^{-1} &= \nu. \end{aligned} \tag{5.1}$$

For Lebesgue spaces, there is a correspndence of factors of $(X, \mathcal{B}, \mu, T)$ and T-invariant complete sub-σ-algebras of $\mathcal{B}$. According to Rokhlin's theory of Lebesgue spaces [100] factors also correspond to certain partitions of X (see the article on Measure-Preserving Systems). A factor map $\phi : X \to Y$ between Lebesgue spaces is an *isomorphism* if and only if it has a measurable inverse, or equivalently $\phi^{-1}\mathcal{C} = \mathcal{B}$ up to sets of measure 0.

5.3. **Skew products.** If $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system, $(Y, \mathcal{C}, \nu)$ is a measure-space, and $\{S_x : x \in X\}$ is a family of measure-preserving maps $Y \to Y$ such that the map that takes (x, y) to $S_x y$ is jointly measurable in the two variables x and y, then we may define a *skew product system* consisting of the product measure space of X and Y equipped with product measure $\mu \times \nu$ together with the measure-preserving map $T \ltimes S : X \times Y \to X \times Y$ defined by

$$(T \ltimes S)(x, y) = (Tx, S_x y). \tag{5.2}$$

The space Y is called the *fiber* of the skew product and the space X the *base*. Sometimes in the literature the word skew product has a more general meaning and refers to the structure $(T \ltimes S)(x, y) = (Tx, S_x y)$ (without any assumption of measure-preservation), where the action of the map on the fiber Y is determined or 'driven' by the map $T : X \to X$.

Some common examples of skew products include:

5.3.1. *Random dynamical systems.* Suppose the X indexes a collection of mappings $S_x : Y \to Y$. We may have a transformation $T : X \to X$ which is a full shift. Then the sequence of mappings $\{S_{T^n x}\}$ may be considered a (random) choice of a mapping $Y \to Y$ from the set $\{S_x : x \in X\}$. The projection onto Y of the orbits of $(Tx, S_x y)$ give the orbits of a point $y \in Y$ under a random composition of maps $S_{T^n x} \circ \ldots \circ S_{Tx} \circ S_x$. More generally we could consider the choice of maps S_x that are composed to come from any ergodic dynamical system, (T, X, μ) to model the effect of perturbations by a stationary ergodic 'noise' process.

5.3.2. *Group extensions of dynamical systems.* Suppose Y is a group, ν is a measure on Y invariant under a left group action, and $S_x y := g(x)y$ is given by a group-valued function $g : X \to Y$. In this setting g is often called a *cocycle*, since upon defining $g^{(n)}(x)$ by $(T \ltimes S)^{(n)}(x, y) = (T^n x, g^{(n)}(x)y)$ we have a cocycle relation, namely $g^{(m+n)}(x) = g^{(m)}(T^n x)g^{(n)}(x)$. Group extensions arise often in modeling systems with symmetry [37]. Common examples are provided by a random composition of matrices from a group of matrices (or more generally from a set of matrices which may form a group or not).

5.4. **Induced transformations.** Since by the Poincaré Recurrence Theorem a measure-preserving transformation $(T, X, \mu, \mathcal{B})$ on a probability space is recurrent, given any set B of positive measure, the return-time function

$$n_B(x) = \inf\{n \geq 1 : T^n x \in B\} \tag{5.3}$$

is finite μ a.e. We may define the *first-return map* by

$$T_B x = T^{n_B(x)} x. \tag{5.4}$$

Then (after perhaps discarding as usual a set of measure 0) $T_B : B \to B$ is a measurable transformation which preserves the probability measure $\mu_B = \mu/\mu(B)$. The system $(B, \mathcal{B} \cap B, \mu_B, T_B)$ is called an *induced*, *first-return* or *derived transformation.* If $(T, X, \mu, \mathcal{B})$ is ergodic then $(B, \mathcal{B} \cap B, \mu_B, T_B)$ is ergodic, but the converse is not in general true.

The construction of the transformation T_B allows us to represent the forward orbit of points in B via a *tower* or *skyscraper* over B. For each $n = 1, 2, \ldots,$ let

$$B_n = \{x \in B : n_B(x) = n\}. \tag{5.5}$$

Then $\{B_1, B_2, \ldots\}$ form a partition of B, which we think of as the bottom floor or base of the tower. The next floor is made up of $TB_2, TB_3, \ldots,$ which form a partition of $TB \setminus B$, and so on. All these sets are disjoint. A *column* is a part of the tower of the form $B_n \cup TB_n \cup \cdots \cup T^{n-1}B_n$ for some $n = 1, 2, \ldots.$ The action of T on the entire tower is pictured as mapping each x not at the top of its column straight up to the point Tx above it on the next level, and mapping each point on the top level to $T^{n_B} x \in B$. An equivalent way to describe the transformation on the tower is to write for each n and $j < n$, $T^j B_n$ as $\{(x, j) : x \in B_n\}$, and then the transformation F on the tower becomes

$$F(x,l) = \begin{cases} (x, l+1) & \text{if } l < n_B(x) - 1; \\ (T^{n_B(x)}x, 0) & \text{if } l = n_B(x) - 1. \end{cases}$$

If T preserves a measure μ, then F preserves $\mu \times dl$, where l is counting measure so that the measure $\mu \times dl$ can be naturally lifted to the tower.

Sometimes the process of inducing yields an induced map which is easier to analyse (perhaps it has stronger hyperbolicity properties) than the original system. Sometimes also it is possible to 'lift' ergodic or statistical properties from an induced system to the original system, so the process of inducing plays an important role in the study of statistical properties of dynamical systems [77].

It is possible to generalize the tower construction and relax the condition that $n_B(x)$ is the first-return time function. We may take a measurable set $B \subset X$ of positive μ measure and define for almost every point $x \in B$ a *height* or *ceiling* function $R : B \to \mathbb{N}$ and take a countable partition $\{X_n\}$ of B into the sets on which R is constant. We define the *tower* as the set $\Delta := \{(x, l) : x \in B, 0 \le l < R(x)\}$ and the *tower map* $F : \Delta \to \Delta$ by

$$F(x,l) = \begin{cases} (x, l+1) & \text{if } l < R(x) - 1; \\ (T^{R(x)}x, 0) & \text{if } l = R(x) - 1. \end{cases}$$

In this setting, if $\int_B R(x) d\mu < \infty$, we may define an F-invariant probability measure on Δ as $\frac{\mu}{C(R,B)} \times dl$, where dl is counting measure and $C(R, B)$ is the normalizing constant $C(R, B) = \mu(B) \int_B R(x) d\mu$. This viewpoint is connected with the construction of systems by cutting and stacking—see Section 5.6.

5.5. **Suspension flows.** The tower construction has an analogue in which the height function R takes values in $\mathbb{R}$ rather than $\mathbb{N}$. Such towers are commonly used to model dynamical systems with continuous time parameter. Let (T, X, μ) be a measure-preserving system and $R : X \to (0, \infty)$ a measurable "ceiling" function on X. The set

$$X^R = \{(x, t) : 0 \le R(x) < t\}, \tag{5.6}$$

with measure ν given locally by the product of μ on X with Lebesgue measure m on $\mathbb{R}$, is a measure space in a natural way. If μ is a finite measure and R is integrable with respect to μ then ν is a finite measure. We define an action of $\mathbb{R}$ on X^R by letting each point x flow at unit speed up the vertical lines $\{(x, t) : 0 \le t < R(x)\}$ under the graph of R until it hits the ceiling, then jump to Tx, and so on. More precisely, defining $R_n(x) = R(x) + \cdots + R(T^n x)$,

(5.7)
$$T_s(x,t) = \begin{cases} (x, s+t) & \text{if } 0 \le s+t < R(x), \\ (Tx, s+t-f(x)) & \text{if } R(x) \le s+t < R(x) + R(Tx) \\ \cdots & \\ (T^n x, s+t-[R(x)+\cdots+R(T^{n-1}x)] & \text{if } R_{n-1}(x) \le s+t < R_n(x). \end{cases}$$

Ergodicity of (T, X, μ) implies the ergodicity of (T_s, X^R, ν).

5.6. **Cutting and stacking.** Several of the most interesting examples in ergodic theory have been constructed by this method; in fact, because of Rokhlin's Lemma (see Section 5.8) every ergodic measure-preserving transformation on a Lebesgue space is isomorphic to one constructed by cutting and stacking. For example, the von Neumann-Kakutani adding machine (or 2-odometer) (Section 4.2), the Chacon weakly mixing but not strongly mixing system (Section 4.5.2), Ornstein's mixing rank one examples (see e.g. [81, p. 160 ff.]), and many more.

We construct a Lebesgue measure-preserving transformation T on an interval X (bounded or maybe unbounded) by defining it as a translation on each of a pairwise disjoint countable collection of subintervals. The construction proceeds by stages, at each stage defining T on an additional part of X, until eventually T is defined a.e.

At each stage X is represented as a *tower*, which is defined to be a disjoint union of *columns*. A *column* is defined to be a finite disjoint union of intervals of equal length, which are numbered from 0, for the "floor", to the last one, for the "roof", and which we picture as lying each above the preceding-numbered interval. T is defined on each *level* of a column (i.e. each interval in the column) except the roof by mapping it by translation to the next higher interval in the column.

At stage 0, we have just one column, consisting of all of X as the floor, and T is not defined anywhere. To pass from one stage to the next, the columns are *cut* and *stacked*. This means that each column is divided, by vertical cuts, into a disjoint union of subcolumns of equal height (but maybe not equal width), and then some of these subcolumns are stacked above others (of the same width) so as to form a new tower. This allows the definition of T to be extended to some parts of X that were previously tops of towers, since they now may have levels above them. (Sometimes columns of height 1 are thought of as forming a reservoir for "spacers" to be inserted between subcolumns that are being stacked.) If the measure of the union of the tops of the columns tends to 0, eventually T becomes defined a.e.. This description in words can be made precise with cumbersome notation, but the process can also be given a neater graphical description, which we sketch in the next section.

5.7. **Adic transformations.** A.M. Vershik has introduced a family of models, called *adic* or *Bratteli-Vershik* transformations, into ergodic theory and dynamical systems. One begins with a graph which is arranged in levels, finitely many vertices on each level, with connections only from each level to the adjacent ones. The space X consists of the set of all infinite paths in this graph; it is a compact metric space in a natural way. We are given an order on the set of edges into each vertex, and then X is partially ordered as follows: x and y are comparable if they agree from some point on, in which case we say that $x < y$ if at the last level n where they traverse different edges, the edge x_n of x is smaller than the edge y_n of y. A map T is defined by letting Tx be the smallest y that is larger than x, if there is one. In nice situations, T is a homeomorphism after defining it and its inverse on perhaps countably many maximal and minimal elements. Invariant measures can sometimes be defined by assigning weights to edges, which are then multiplied to define the measure of each cylinder set. This is a nice combinatorial way to present the cutting and stacking method of constructing m.p.t.'s, allows for more convenient analysis of questions such as orbit equivalence, and leads to the construction of many interesting examples, such as those based on the Pascal or Euler graphs [5, 38, 80]. Odometers and generalizations are natural examples of adic systems. Vershik showed that in fact every ergodic measure-preserving transformation on a Lebesgue space is isomorphic to a uniquely ergodic adic transformation. See [113].

5.8. **Rokhlin's Lemma.** The following result is the fundamental starting point for many constructions in ergodic theory, from representing arbitrary systems in terms of cutting and stacking or adic systems, to constructing useful partitions and symbolic codings of abstract systems, to connecting convergence theorems in abstract ergodic theory with those in harmonic analysis. It allows us to picture arbitrarily long stretches of the action of a measure-preserving transformation as a translation within the set of integers. In the ergodic nonatomic case the statement follows readily from the construction of derivative transformations.

Lemma 5.1 (Rokhlin's Lemma)**.** *Let $T : X \to X$ be a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$. Suppose that $(X, \mathcal{B}, \mu)$ is nonatomic and $T : X \to X$ is ergodic, or, more generally, $(T, X, \mathcal{B}, \mu)$ is aperiodic: that is to say, the set $\{x \in X : \text{ there is } n \in \mathbb{N} \text{ such that } T^n x = x\}$ of periodic points has measure 0. Then given $n \in \mathbb{N}$ and $\epsilon > 0$, there is a measurable set $B \subset X$ such that the sets $B, TB, \ldots, T^{n-1}B$ are pairwise disjoint and $\mu(\cup_{k=0}^{n-1} T^k B) > 1 - \epsilon$.*

5.9. **Inverse limits.** Suppose that for each $i = 1, 2, \ldots$ we have a Lebesgue probability space $(X_i, \mathcal{B}_i, \mu_i)$ and a measure-preserving transformation $T_i : X_i \to X_i$. Suppose also that for each $i \le j$ there is a factor map $\phi_{ji} : (T_j, X_j, \mathcal{B}_j, \mu_j) \to (T_i, X_i, \mathcal{B}_i, \mu_i,)$, such that each ϕ_{jj} is the identity on X_j and $\phi_{ji}\phi_{kj} = \phi_{ki}$ whenever $k \ge j \ge i$. Let

$$X = \{x \in \Pi_{i=1}^{\infty} X_i : \phi_{ji} x_j = x_i \text{ for all } j \ge i\}. \tag{5.8}$$

For each j, let $\pi_j : X \to X_j$ be the projection defined by $\pi_j x = x_j$.

Let $\mathcal{B}$ be the smallest σ-algebra of subsets of X which contains all the $\pi_j^{-1}\mathcal{B}_j$. Define μ on each $\pi_j^{-1}\mathcal{B}_j$ by

$$\mu(\pi_j^{-1} B) = \mu_j(B) \quad \text{ for all } B \in \mathcal{B}_j. \tag{5.9}$$

Because $\phi_{ji}\pi_j = \pi_i$ for all $j \ge i$, the $\pi_j^{-1}\mathcal{B}_j$ are increasing, and so their union is an algebra. The set function μ can, with some difficulty, be shown to be countably additive on this algebra: since we are dealing with Lebesgue spaces, by means of measure-theoretic isomorphisms it is possible to replace the entire situation by compact metric spaces and continuous maps, then use regularity of the measures involved–see [89, p. 137 ff.]. Thus by Carathéodory's Theorem (see the article on Measure-Preserving Systems) μ extends to all of $\mathcal{B}$.

Define $T : X \to X$ by $T(x_j) = (T_j x_j)$. Then $(T, X, \mathcal{B}, \mu)$ is a measure-preserving system such that any system which has all the $(T_j, X_j, \mathcal{B}_j, \mu_j)$ as factors, also has $(T, X, \mathcal{B}, \mu)$ a factor.

5.10. **Natural extension.** The natural extension is a way to produce an invertible system from a noninvertible system. The original system is a factor of its natural extension and its orbit structure and ergodic properties are captured by the natural extension, as will be seen from its construction. Let $(T, X, \mathcal{B}, \mu)$ be a measure-preserving transformation of a Lebesgue probability space. Define

$$\Omega\colon = \{(x_0, x_1, x_2, \ldots) : x_n = T(x_{n+1}), x_n \in X, n = 0, 1, 2, \ldots\}$$

with $\sigma : \Omega \to \Omega$ defined by $\sigma((x_0, x_1, x_2, \ldots)) = (T(x_0), x_0, x_1, \ldots)$. The map σ is invertible on Ω. Given the invariant measure μ we define the invariant measure for the natural extension $\tilde{\mu}$ on Ω by defining it first on cylinder sets $C(A_0, A_1, \ldots, A_k)$ by

$$\tilde{\mu}(C(A_0, A_1, \ldots, A_k)) = \mu(T^{-k}(A_0) \cap T^{-k+1}(A_1) \ldots \cap T^{-k+i}(A_i) \cap \ldots \cap A_k)$$

and then extending it to Ω using Kolmogorov's extension theorem. We think of $(x_0, x_1, x_2, \ldots)$ as being an inverse branch of $x_0 \in X$ under the mapping $T\colon X \to X$. The maps $\sigma, \sigma^{-1}\colon \Omega \to \Omega$ are ergodic with respect to $\tilde{\mu}$ if $(T, X, \mathcal{B}, \mu)$ is ergodic [114]. If $\pi : \Omega \to X$ is projection onto the first component i.e. $\pi(x_0, \ldots, x_n, \ldots) = x_0$ then $\pi \circ \sigma^n(x_0, \ldots, x_n, \ldots) = T^n(x_0)$ for all x_0 and thus the natural extension yields all information about the orbits of X under T.

The natural extension is an inverse limit. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and $T : X \to X$ a map such that $T^{-1}\mathcal{B} \subset \mathcal{B}$ and $\mu T^{-1} = \mu$. For each $i = 1, 2, \ldots$ let $(T_i, X_i, \mathcal{B}_i, \mu_i) = (T, X, \mathcal{B}, \mu)$, and $\phi_{ji} = T^{j-i}$ for each $j > i$. Then the inverse limit $(\hat{T}, \hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ of this system is an *invertible* measure-preserving system which is the natural extension of $(T, X, \mathcal{B}, \mu)$. We have

$$\hat{T}^{-1}(x_1, x_2, \ldots) = (x_2, x_3, \ldots). \tag{5.10}$$

The original system $(T, X, \mathcal{B}, \mu)$ is a factor of $(\hat{T}, \hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ (using any π_i as the factor map), and any factor mapping from an invertible system onto $(T, X, \mathcal{B}, \mu)$ consists of a factor mapping onto $(\hat{T}, \hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ followed by projection onto the first coordinate.

5.11. **Joinings.** Given measure-preserving systems $(T, X, \mathcal{B}, \mu)$ and $(S, Y, \mathcal{C}, \nu)$, a *joining* of the two systems is a $T \times S$-invariant measure P on their product measurable space that projects to μ and ν, respectively, under the projections of $X \times Y$ to X and Y, respectively. That is, if $\pi_1 : X \times Y \to X$ is the projection onto the first component i.e. $\pi_1(x, y) = x$ then $P(\pi_1^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$ and similarly for $\pi_2 : X \times Y \to Y$.

This concept is the ergodic-theoretic version of the notion in probability theory of a *coupling*. The product measure $\mu \times \nu$ is always a joining of the two systems. If product measure is the only joining of the two systems, then we say that they are disjoint and write $X \perp Y$ [39]. If $\mathcal{D}$ is any family of systems, we write $\mathcal{D}^{\perp}$ for the family of all measure-preserving systems which are disjoint from every system in $\mathcal{D}$. Extensive recent accounts of the use of joinings in ergodic theory are in [43, 102, 112].

6. Future Directions.

The basic examples and constructions presented here are idealized, and many of the underlying assumptions (such as uniform hyperbolicity) are seldom satisfied in applications, yet they have given important insights into the behavior of real-world physical systems. Recent developments have improved our understanding of the ergodic properties of non-uniformly and partially hyperbolic systems. The ergodic properties of deterministic systems will continue to be an active research area for the foreseeable future. The directions will include, among others: establishing statistical and ergodic properties under weakened dependence assumptions; the study of systems which display 'anomalous statistics'; the study of the stability and typicality of ergodic behavior and mixing in dynamical systems; the ergodic theory of infinite-dimensional systems; advances in number theory (see the sections on Szemerédi and Ramsey theory); research into models with non-singular rather than invariant measures; and infinite-measure systems. Other chapters in this Encyclopedia discuss in more detail these and other topics.

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