Section 3.1: Mathematical Induction

Let \( \mathbb{N} \) denote the set of natural numbers (positive integers).

\[
\mathbb{N} = \{1, 2, 3, 4, \ldots\}
\]

Axiom: If \( S \) is a nonempty subset of \( \mathbb{N} \), then \( S \) has a least element. That is, there is an element \( m \in S \) such that \( m \leq n \) for all \( n \in S \).
THEOREM 1: (Mathematical Induction)

Let $S$ be a subset of $\mathbb{N}$. If $S$ has the following properties:

(a) $1 \in S$, and

(b) $k \in S$ implies $k + 1 \in S$,

then $S = \mathbb{N}$. 
Examples:

1. Prove that

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \]

for all \( n \in \mathbb{N} \).

Let \( S \) be the set of positive integers for which the equation holds.

**Step 1.** Show that \( 1 \in S \).
Step 2. Assume that $k \in S$. 
Step 3. Prove that $k + 1 \in S$
2. Prove that

\[ 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1 \]

for all \( n \in \mathbb{N} \).
Let $\mathbb{R}$ be the set of real numbers.

**Important subsets of $\mathbb{R}$:**

\[ \mathbb{N} = \{1, 2, 3, \ldots, \} \]  
the natural numbers

\[ \mathbb{J} = \{0, 1, -1, 2, -2, 3, -3, \ldots, \} \]  
the integers

\[ \mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{J}, q \neq 0 \right\} \]  
the rational numbers

$\mathbb{R} - \mathbb{Q}$  
the irrational numbers
There is a one-to-one correspondence between the set of real numbers and the set of points on the line.

This line is called the real line or the number line.
Section 3.2. Ordered Fields

We consider \( \mathbb{R} \) together with the arithmetic operations: addition \((+\,\!)\) and multiplication \((\cdot\,\!)\).
Addition:

A1. For all \( x, y \in \mathbb{R} \), \( x + y \in \mathbb{R} \) (addition is a closed operation).

A2. For all \( x, y \in \mathbb{R} \), \( x + y = y + x \) (addition is commutative)

A3. For all \( x, y, z \in \mathbb{R} \),

\[
x + (y + z) = (x + y) + z
\]

(addition is associative).
A4. There is a unique number 0 such that

\[ x + 0 = 0 + x = x \]

for all \( x \in \mathbb{R} \). (0 is the additive identity.)

A5. For each \( x \in \mathbb{R} \), there is a unique number \( -x \in \mathbb{R} \) such that

\[ x + (-x) = (-x) + x = 0. \]

\( -x \) is the additive inverse of \( x \).
**Multiplication:**

M1. For all $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$.

(multiplication is a closed)

M2. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

(multiplication is commutative)

M3. For all $x, y, z \in \mathbb{R}$,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

(multiplication is associative).
M4. There is a unique number 1 such that \( x \cdot 1 = 1 \cdot x = x \) for all \( x \in \mathbb{R} \). (1 is the *multiplicative identity*.)

M5. For each \( x \in \mathbb{R}, \ x \neq 0 \), there is a unique number \( 1/x = x^{-1} \in \mathbb{R} \) such that

\[
x \cdot (1/x) = (1/x) \cdot x = 1.
\]

(1/x is the *multiplicative inverse* of \( x \).)
Distributive Law:

D. For all $x, y, z \in \mathbb{R}$,

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$
A non-empty set $S$ together with two operations, “addition” and “multiplication” which satisfies A1-A5, M1-M5, and D is called a field.

The set $\mathbb{R}$ of real numbers with ordinary addition and multiplication is a field.
**Other examples of fields**

1. The set of rational numbers \( \mathbb{Q} \), together with ordinary addition and multiplication, is also a field, a subfield of \( \mathbb{R} \).

2. The set of complex numbers \( \mathbb{C} \) is a field and \( \mathbb{R} \subset \mathbb{C} \).

\[
\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}
\]

\[
\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}, \ i^2 = -1\}
\]
**Theorem:** A subset $S$ of a field $\mathbb{F}$ is itself a field if it is closed with respect to addition and multiplication, if it contains the additive and multiplicative identities 0 and 1, and if it is closed with respect to additive and multiplicative inverses.

3. $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field, a sub-field of $\mathbb{R}$. 
Order

There is a subset $P$ of $\mathbb{R}$ that has the following properties:

(a) If $x, y \in P$, then $x + y \in P$.

(b) If $x, y \in P$, then $x \cdot y \in P$.

(c) For each $x \in \mathbb{R}$ exactly one of the following holds:

$$x \in P, \quad x = 0, \quad -x \in P.$$ 

$P$ is the set of positive numbers.
\( x \in P \) is written \( x > 0 \).

**Def.** \( x \) is less than \( y \) \((x < y)\) if and only if

\[
y - x \in P \quad \text{i.e.,} \quad y - x > 0.
\]

"less than or equal to" \( x \leq y \)

"greater than" \( y > x \)

"greater than or equal to" \( y \geq x \)
Properties of <

O1. For all \( x, y \in \mathbb{R} \), exactly one of the following holds:

\[ x < y, \quad x = y, \quad x > y. \]

(Trichotomy Law)

O2. For all \( x, y, z \in \mathbb{R} \),

if \( x < y \) and \( y < z \),

then \( x < z \).
O3. For all $x, y, z \in \mathbb{R}$,

if $x < y$, then $x + z < y + z$.

O4. For all $x, y, z \in \mathbb{R}$,

if $x < y$ and $z > 0$,

then $x \cdot z < y \cdot z$.

Note: These properties also hold for $>$, $\leq$, $\geq$. 
\{\mathbb{R}, +, \cdot, <\} \text{ is an ordered field. Any mathematical system } \{S, +, \cdot, <\} \text{ satisfying these 15 axioms is an ordered field. In particular, the set of rational numbers } \mathbb{Q}, \text{ together with ordinary addition, multiplication and “less than”, is an ordered field, a subfield of } \mathbb{R}.
THEOREM 1: Let $x, y \in \mathbb{R}$. If

$$x \leq y + \epsilon$$

for every positive number $\epsilon$, then

$x \leq y$.

Proof:
Def. Let \( x \in \mathbb{R} \). The absolute value of \( x \), denoted \( |x| \), is:

\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0
\end{cases}
\]
Properties of absolute value:

Let \( x, y \in \mathbb{R} \) and let \( a \geq 0 \). Then

(a) \( |x| \geq 0 \).

(b) \( |x| \leq a \) iff \( -a \leq x \leq a \).

(c) \( |xy| = |x| \cdot |y| \).

(d) \( |x + y| \leq |x| + |y| \). (Triangle inequality)
Section 3.3. Boundedness and the Completeness Axiom

There is a one-to-one correspondence between the set \( \mathbb{R} \) of real numbers and the points \( P \) on the number line.

There are two basic types of real numbers:

1. The rational numbers \( \mathbb{Q} \), and

2. the irrational numbers \( \mathbb{R} - \mathbb{Q} \).
$\sqrt{2}$ is a point on the number line, i.e., $\sqrt{2}$ is a real number.

$\sqrt{2}$ is not a rational number.
In general, if \( r = \frac{p}{q} \) is a rational number which is not a perfect square, then \( \sqrt{\frac{p}{q}} \) is an irrational number.
**Def.** Let \( S \subseteq \mathbb{R}, S \neq \emptyset \). \( S \) is **bounded above** if there exists a number \( m \) such that \( s \leq m \) for all \( s \in S \). \( m \) is called an **upper bound** for \( S \).

\( S \) is **bounded below** if there exists a number \( k \) such that \( s \geq k \) for all \( s \in S \). \( k \) is a **lower bound** for \( S \).

\( S \) is **bounded** if it is bounded above and below.
Examples:

1. \( S = (-\infty, 0] \cup (1, 2] \)

2. \( S = (-1, \infty] \)

3. \( S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \)

4. \( S = \{\sin(n\pi/4)\}, \ n = 1, 2, 3, \ldots \)
**Theorem 1**: Let $S \subseteq \mathbb{R}$. If $m$ is an upper bound for $S$, then any number $p > m$ is also an upper bound for $S$. If $k$ is a lower bound for $S$, then any number $q < k$ is also a lower bound for $S$.

**Proof:**
**Def.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$, be bounded above. A number $\alpha$ is the **supremum** or **least upper bound** for $S$ if:

1. $\alpha$ is an upper bound for $S$, and
2. $\alpha \leq m$ where $m$ is any other upper bound for $S$.

Notation: $\alpha = \sup S$ or $\alpha = \lub S$
**Def.** Let \( S \subseteq \mathbb{R}, \ S \neq \emptyset, \) be bounded below. A number \( \beta \) is the infimum (greatest lower bound) of \( S \) if:

1. \( \beta \) is a lower bound for \( S \), and

2. \( \beta \geq k \) where \( k \) is any other lower bound for \( S \).

**Notation:** \( \beta = \inf S \) or \( \beta = \text{glb} \ S \)
**Def.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. If there is an element $p \in S$ such that

$$s \leq p \quad \text{for all } s \in S,$$

then $p$ is the **maximum** (or largest element) of $S$. \((p = \max S)\)

**Def.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. If there is an element $q \in S$

$$s \geq q \quad \text{for all } s \in S,$$

then $q$ is the **minimum** (or smallest element) of $S$. \((q = \min S)\)
THEOREM 2: Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$, be bounded above. If

$$\alpha_1 = \sup S \quad \text{and} \quad \alpha_2 = \sup S,$$

then $\alpha_1 = \alpha_2$. That is, the supremum of a set is unique.

Proof:
The same result holds for inf, max, and min.

Examples:

1. $S = (0, 1]$

2. $S = \{-1\} \cup [0, 1)$

3. $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$
4. $S = \mathbb{N}$

5. $S = (-\infty, 2] \cup \{5\}$

6. $S = \left\{ 1 + \frac{(-1)^n}{n} \right\}$
7. \[ S = \left\{ \frac{1 + (-1)^n}{2} \right\} \]

8. \[ S = \{ \cos(n\pi/3) \} \]

9. \[ S = \left\{ \frac{1}{n} + \cos(n\pi/3) \right\} \]
THEOREM 3:  If 

\[ \alpha = \sup S \text{ and } m < \alpha, \]

then there exists an element \( x \in S \)

such that

\[ m < x \leq \alpha. \]

Proof:
**Theorem 3’** If $\alpha = \sup S$ and $\epsilon$ is any positive number, then there exists an element $x \in S$ such that

$$\alpha - \epsilon < x \leq \alpha.$$
THEOREM 4: If $\beta = \inf S$ and $k > \beta$, then there exists an element $x \in S$ such that

$$\beta \leq x < k.$$ 

THEOREM 4': If $\beta = \inf S$ and $\epsilon$ is any positive number, then there exists an element $x \in S$ such that

$$\beta \leq x < \beta + \epsilon.$$
The Completeness Axiom:

Let \( S \subseteq \mathbb{R}, S \neq \emptyset \). If \( S \) is bounded above, then \( S \) has a least upper bound. If \( S \) is bounded below, then \( S \) has a greatest lower bound.

The real number system \( \{\mathbb{R}, +, \cdot, <\} \) is a complete ordered field.
Notes:

• The rational number system

\[ \{\mathbb{Q}, +, \cdot, <\} \]

is not complete.

\[ S = \{ r \in \mathbb{Q} \mid r < \sqrt{2} \} \] is bounded above. \( S \) does not have a (rational) least upper bound.

• The complex number system \( \{\mathbb{C}, +, \cdot\} \) is complete; it is not ordered.
• Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. Suppose that $S$ is bounded above. $\sup S$ may or not be an element of $S$. Similarly for $\inf S$.

Examples:
THEOREM 5: Let $S \subseteq \mathbb{R}, S \neq \emptyset$, be bounded above and let $\alpha = \sup S$.

(a) If $\alpha \in S$, then $\alpha = \max S$.

Proof:
(b) If $\alpha \notin S$, then $S$ does not have a maximum element.

Proof:
THEOREM 6: (Archimedean Property) \( \mathbb{N} \) is not bounded above.

Proof:
Corollaries – re-statements of the theorem

1. For each $z \in \mathbb{R}$ there exists an $n_z \in \mathbb{N}$ such that $n_z > z$.

2. For each $x > 0$ and for each $y \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $nx > y$.

3. For each $x > 0$ there exists $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < x.$$
THEOREM 7: 

Thm 6 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow \text{Thm 6}

Proof:
THEOREM 8: Given any positive number \( y \), there is a corresponding positive integer \( n \) such that

\[ n - 1 \leq y < n. \]

Proof:
Density of the rationals

**THEOREM 9:** Let $x, y \in \mathbb{R}$, $x < y$. There exists a rational number $r$ such that

$$x < r < y.$$ 

**Proof:**
Corollary: Let $x, y \in \mathbb{R}, x < y$.

There exists an irrational number $t$ such that

$$x < t < y.$$

Proof:
Points and Neighborhoods

Let $x \in \mathbb{R}$.

**Def.** An $\epsilon$-neighborhood of $x$ is the set

$$N(x, \epsilon) = \{y \in \mathbb{R} | |y - x| < \epsilon\}$$

**Note:** $N(x, \epsilon)$ is the open interval $(x - \epsilon, x + \epsilon)$.

centered at $x$ with radius $\epsilon$. 
Neighborhoods and Deleted Neighborhoods

Let \( x \in \mathbb{R} \).

**Def.** An \( \epsilon \)-neighborhood of \( x \) is the set

\[
N(x, \epsilon) = \{ y \in \mathbb{R} \mid |y - x| < \epsilon \}
\]

**Note:** \( N(x, \epsilon) \) is the open interval \((x - \epsilon, x + \epsilon)\).

centered at \( x \) with radius \( \epsilon \).
Def. A deleted $\epsilon$-neighborhood of $x$ is the set

$$N^*(x, \epsilon) = \{ y \in \mathbb{R} \mid 0 < |y - x| < \epsilon \}$$

Note:

$$N^*(x, \epsilon) = (x - \epsilon, x) \cup (x, x + \epsilon).$$
Let $S \subseteq \mathbb{R}$.

**Def.** A point $x$ is an **interior point** of $S$ if there exists a neighborhood $N(x)$ such that $N(x) \subseteq S$.

The **interior of** $S$ is the set

$$\text{int} \ S = \{x \in S \mid x \text{ is an interior point of } S\}$$
Negation: Let $x \in S$. $x$ is not an interior point of $S$ if there is no neighborhood $N(x)$ which is contained in $S$. That is, for every neighborhood $N(x)$ of $x$,

$$N(x) \cap S^c \neq \emptyset.$$
1. $S = (0, 3)$

2. $S = \{-1\} \cup [0, 1)$

3. $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$
4. \( S = \mathbb{N} \)

5. \( S = (\infty, 2] \cup \{5\} \)

6. \[
S = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\right\} \cup (2, 4] \cup [5, \infty)
\]
**Def.** A point $x$ is a **boundary point of** $S$ if every neighborhood $N(x)$ of $x$ contains points of $S$ and points of $\mathbb{R}/S = S^c$ (the complement of $S$).

That is, $x$ is a boundary point of $S$ if

$$N(x) \cap S \neq \emptyset \quad \text{and} \quad N(x) \cap S^c \neq \emptyset$$

for **every** neighborhood $N(x)$ of $x$. 
The **boundary of** $S$ is the set

$$\text{bd } S = \{ x \in S \mid x \text{ is a boundary point of } S \}$$

**Negation:** A point $x$ is not a boundary point of $S$ if there is at least one neighborhood $N(x)$ such that either

$$N(x) \cap S = \emptyset \quad \text{or} \quad N(x) \cap S^c = \emptyset$$
1. $S = (0, 3)$

2. $S = \{-1\} \cup [0, 1)$

3. $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$
4. \( S = \mathbb{N} \)

5. \( S = (-\infty, 2] \cup (2, 3) \cup \{5\} \)

6. \( S = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots \right\} \cup (2, 4] \cup (4, \infty) \)
Open Sets/Closed Sets

Let \( S \subseteq \mathbb{R} \)

**Def.** \( S \) is open if every point \( x \in S \) is an interior point of \( S \).

**Def.** \( S \) is closed if \( S^c \) is open.

**Examples:**
1. \( S = (0, 3) \)

\[ S = \mathbb{R} = (-\infty, \infty) \]

\[ S = \emptyset \quad \text{the empty set.} \]

2. \( S = \{-1\} \cup (0, 2) \)

3. \( S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \)
4. \( S = \mathbb{N} \)

5. \( S = [0, 1] \cup [4, 5] \)

6. \( S = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots \right\} \cup (2, 4] \cup [5, \infty) \)
THEOREM 1: \( S \) is closed iff

\[ \text{bd } S \subseteq S. \]

Proof:
THEOREM 2: If $\mathcal{F}$ is a collection of open sets, then

$$S = \bigcup_{A \in \mathcal{F}} A$$

is open.

Proof:
THEOREM 3: \hspace{1em} \text{If}

\[ G = \{ A_1, A_2, \ldots, A_n \} \]

is a \textbf{finite} collection of open sets, then

\[ S = A_1 \cap A_2 \cap \cdots \cap A_n \]

is open.

Proof:
Note: The intersection of an infinite collection of collection of open sets may not be open; the union of an infinite collection of closed sets may not be closed.

Examples:

1. \( \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) = [0, 1] \)

2. \( \bigcup_{n=1}^{\infty} \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1) \)
DeMorgan’s Laws:

(a) \((A \cup B)^c = A^c \cap B^c\)

(b) \((A \cap B)^c = A^c \cup B^c\)

These laws extend to any collection of sets

\((\bigcup A_\alpha)^c = \bigcap A_\alpha^c\)

\((\bigcap A_\alpha)^c = \bigcup A_\alpha^c\)
Corollary: The intersection of any collection of closed sets is closed; the union of any finite collection of closed sets is closed.
Accumulation Points/Isolated Points

Let $S \subseteq \mathbb{R}$

**Def.** A point $x \in \mathbb{R}$ is an accumulation point of $S$ if every deleted neighborhood $N^*(x)$ contains a point $s \in S$.

Equivalently, $x$ is an accumulation point of $S$ if every neighborhood $N(x)$ contains a point $s \in S, s \neq x$.

$S' = \{ x \in \mathbb{R} \mid x \text{ is an accum. point of } S \}$
**Accumulation Points**  Let $S \subseteq \mathbb{R}$

**Def.** A point $x \in \mathbb{R}$ is an **accumulation point** of $S$ if every deleted neighborhood $N^*(x, \epsilon)$ contains a point $s \in S$.

Equivalently, $x$ is an accumulation point of $S$ if for every $\epsilon > 0$, the neighborhood $N(x, \epsilon)$ contains a point $s \in S$ such that $s \neq x$.

$S' = \{ x \in \mathbb{R} \mid x \text{ is an accum. point of } S \}$
**Negation:** \( x \) is not an accumulation point of \( S \) if there is at least one deleted neighborhood \( N^*(x) \) such that \( N^*(x) \cap S = \emptyset \).
Examples:

1. \( S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \)

2. \( S = \{-1\} \cup [0, 1) \)

3. \( S = \{\sin(n\pi/6)\}' \quad n = 1, 2, 3, \ldots \)

\[ S = \left\{ \frac{1}{n} + \sin\left(\frac{n\pi}{6}\right) \right\}, \quad n = 1, 2, 3, \ldots \]
4. $S = \mathbb{N}$

5. $S = (-\infty, 2] \cup (2, 3) \cup \{5\}$

6. $S = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n-1}{n}, \cdots \right\} \cup (2, 4] \cup (4, \infty)$

Note: an accumulation point of $S$ may or may not be an element of $S$. 
Def. A point \( x \in S \) is an isolated point of \( S \) if there is a deleted neighborhood \( N^*(x) \) such that

\[
N^*(x) \cap S = \emptyset.
\]

That is, \( x \in S \) but \( x \notin S' \).

Note: an isolated point of \( S \) is a boundary point of \( S \).
Negation: A point $x \in S$ is not an isolated point of $S$ if

$$N^*(x) \cap S \neq \emptyset$$

for every deleted neighborhood $N^*(x)$.
Examples:  1. \( S = (0, 3) \)

2. \( S = \{-1\} \cup [0, 1) \)

3. \( S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \)
4. \( S = \{ r \in \mathbb{Q} : 0 < r < \sqrt{5} \} \)

5. \( S = (-\infty, 2] \cup (2, 3) \cup \{5\} \)

6. \( S = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots \right\} \cup (2, 4] \cup (4, \infty) \)
Def. The closure of $S$ is the set

$$\text{cl } S = S \cup S'$$

Note: If $x \in \text{cl } S$ and $N(x)$ is any neighborhood of $x$, then

$$N(x) \cap S \neq \emptyset.$$
Examples:

1. \( S = (0, 3) \)

2. \( S = \{ -1 \} \cup [0, 1) \)

3. \( S = \{ 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \} \)
4. \( S = \mathbb{N} \)

5. \( S = (-\infty, 2] \cup (2, 3) \cup \{5\} \)

6. \( S = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots \right\} \cup (2, 4] \cup (4, \infty) \)
THEOREM 4: $S$ is closed iff $S' \subseteq S$.

Proof:
THEOREM 5: \( \text{cl} \ S \) is a closed set.

Proof:
Let $S$ be bounded above and let $\alpha = \sup S$. True/False:

1. If $\alpha \notin S$, then $\alpha$ is an accumulation point of $S$.

2. If $\alpha \notin S$, then $S$ does not have a maximum.

3. If $\alpha \in S$, then $\alpha$ is an accumulation point of $S$. 
4. If $\alpha \in S$, then $\alpha = \max S$.

5. If $p$ is an accumulation point of $S$, then $p \leq \alpha$.

Restate these in terms of “bounded below” and inf.
Section 3.5. Compact Sets

Let \( S \subseteq \mathbb{R} \).

**Def.** A collection \( G \) of open sets such that

\[
S \subseteq \bigcup_{G \in G} G
\]

is called an open cover of \( S \).

A subset \( F \) of \( G \) which also covers \( S \) is called a subcover.
Def. $S$ is **compact** if for every open cover $\mathcal{G}$ of $S$ there is a finite set $G_1, G_2, \cdots, G_n$ of elements of $\mathcal{G}$ such that

$$S \subseteq \bigcup_{i=1}^{n} G_i$$

That is, $S$ is compact iff every open cover of $S$ has a finite subcover.
Examples:
THEOREM 1: If $S \neq \emptyset$ is closed and bounded, then has a maximum $m$ and a minimum $k$.

Proof:
THEOREM 2: (Heine-Borel)

$S \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Proof:
THEOREM 3: (Bolzano-Weierstrass)
If $S$ is a bounded infinite set, then $S$ has at least one accumulation point.

Proof: