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ASYMPTOTICS OF A SPECTRAL PROBLEM ASSOCIATED WITH THE NEUMANN SIEVE

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In this paper, we analyze the asymptotic behavior of a Stekloff spectral problem associated with the Neumann Sieve model, i.e. a three-dimensional set Ω, cut by a hyperplane Σ where each of the two-dimensional holes, ϵ-periodically distributed on Σ, have diameter r_ϵ. Depending on the asymptotic behavior of the ratios $\frac{r_\epsilon}{\epsilon^2}$ we find the limit problem of the ϵ spectral problem and prove that the sequences λ_n^ϵ, formed by the nth eigenvalue of the ϵ problem, converge to λ_n, the nth eigenvalue of the limit problem, for any $n \in \mathbf{N}$. We also prove the weak convergence, on a subsequence, of the associated sequence of eigenvectors u_n^ϵ, to an eigenvector associated with λ_n. When λ_n is a simple eigenvalue, we show that the entire sequence of the eigenvectors converges.

As a consequence, similar results hold for the spectrum of the DtN map associated to this model.

Keywords: DtN map; homogenization; Stekloff problem; Γ-convergence.

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1. Introduction

In this paper we study a spectral problem associated with the Neumann sieve. Consider a plane Σ that separates a three dimensional domain Ω into two subdomains Ω_1 and Ω_2 and distributes ϵ-periodically on Σ two dimensional small holes of diameter $r_\epsilon < \epsilon$, denoted by T_ϵ.

Set

$$V = \{u \in H^1(\Omega_1) \cup H^1(\Omega_2) | u = 0 \text{ on } \partial\Omega\}$$

and

$$V^\epsilon = \{u \in V | [u] = 0 \text{ on } T_\epsilon\},$$

where $[u] = u^+ - u^-$ and $u^+ = u$ on Ω_1 and $u^- = u$ on Ω_2.

The problem can be formulated as a Steklov-type spectral problem associated with the Neumann sieve problem presented in Damlamian [5] (see also Attouch [1]), i.e.

$$\begin{cases} -\Delta u^\epsilon = 0, & \text{in } \Omega_1 \cup \Omega_2 \cup T_\epsilon, \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = \lambda^\epsilon [u^\epsilon], & \text{on } \Sigma - T_\epsilon, \\ u^\epsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

An equivalent formulation exists in terms of the DtN map. For this, we consider for any z in $H^{\frac{1}{2}}(\Sigma)$ the solution v of the following problem

$$\inf \left\{ \frac{1}{2} |\nabla v|^2_{L^2(\Omega_1 \cup \Omega_2)} | v \in V \text{ with } [v]_\Sigma = z \right\}.$$

Let n be the unit normal to Σ towards Ω_1. Then denote by L the map from z to $Lz \doteq \frac{\partial v^+}{\partial n} = -\frac{\partial v^-}{\partial n}$. L is a well defined fixed operator from $H^{\frac{1}{2}}(\Sigma)$ to $H^{-\frac{1}{2}}(\Sigma)$. It is known that L^{-1} is onto V and is compact (see [13]).

Let TV_ϵ be the subspace of $H^{\frac{1}{2}}(\Sigma)$ of elements which vanish on T_ϵ, i.e. TV_ϵ is the trace subspace of V_ϵ. The ϵ-spectral problem is then, find $z_\epsilon \in TV_\epsilon$ and $\lambda^\epsilon \in \mathbf{R}$, such that

$$L(z_\epsilon) = \lambda^\epsilon z_\epsilon. \tag{1.2}$$

The spectral problem (1.1) is associated with the Neumann sieve problem:

$$\begin{cases} -\Delta u^\epsilon = f, & \text{in } \Omega_1 \cup \Omega_2 \cup T_\epsilon, \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = 0, & \text{on } \Sigma - T_\epsilon, \\ u^\epsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Depending on the order of r_ϵ with respect to ϵ it has been proved in Damlamian [5] that the homogenized problem is of the form

$$\begin{cases} -\Delta u = f, & \text{on } \Omega_1 \cup \Omega_2, \\ \dfrac{\partial u^+}{\partial n} = -\dfrac{\partial u^-}{\partial n} = \dfrac{C}{4}[u], & \text{on } \Sigma. \end{cases}$$

where $C = 0$ if $r_\epsilon \ll \epsilon^2$, C is the capacity in $\mathbf{R}^3$ of the holes if $r_\epsilon = \epsilon^2$ or $C = \infty$ if $r_\epsilon \gg \epsilon^2$ and $[u]$ is defined above.

This type of behavior was first observed in the work of Cioranescu and Murat [4], where the same problem, but with three-dimensional holes periodically distributed in the entire domain or on a hyperplane, was studied.

Homogenization of a Stekloff type problem for perforated domains with three-dimensional ϵ sized holes distributed in the entire domain has been studied in Vanninathan [14], using multiscale analysis and Tartar's method.

In Sec. 2, we set the functional framework and the problem to be analyzed. By using G-convergence techniques and the homogenization result of (1.3) obtained by Damlamian [5], we obtain in Sec. 3 the limit problem for (1.1),

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega_1 \cup \Omega_2, \\ \dfrac{\partial u^+}{\partial n} = -\dfrac{\partial u^-}{\partial n} = \left(\lambda - \dfrac{C}{4}\right)[u], & \text{on } \Sigma, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

or, equivalently, the limit problem for (1.2),

$$Lz = \left(\lambda - \frac{C}{4}\right) z,$$

where $L: H^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma)$ is the DtN operator defined above, and λ is a limit point of a sequence of eigenvalues $\{\lambda^\epsilon\}_{\epsilon>0}$ of (1.2) or (1.1).

We show that the entire sequence formed by the nth eigenvalue of the ϵ-problem, i.e. $\{\lambda_n^\epsilon\}_\epsilon$ converges to the nth eigenvalue of the limit problem (1.4). When λ_n is a simple eigenvalue we can prove that the entire sequence of eigenvectors, u_n^ϵ associated with λ_n^ϵ for the problem (1.1) will converge to the eigenvector u_n associated with λ_n. Sections 3.1 and 3.2 present the cases when $\frac{r_\epsilon}{\epsilon^2} = \infty$ and $\frac{r_\epsilon}{\epsilon^2} = 0$, respectively.

In the form (1.1), our problem is related to the modelling of earthquake initiation phase where one has a periodic system of faults on which slip-weakening friction is considered. The eigenvalues λ^ϵ provide stability properties of the solutions of the dynamic problem (see [7] and [9]). Also (1.1) can be considered as the spectral problem associated with a heat conduction problem, where imperfectly conducting interfaces are present (see Sanchez-Palencia [12], Lipton and Vernescu [11] and Belyaev *et al.* [2]).

In the form (1.2), the problem is a spectral problem for DtN operator in domains perforated along a hyperplane. The asymptotic behavior of the spectrum is similar and is obtained as a consequence of the analysis for (1.1).

2. Problem Statement

Consider an open set $\Omega \subset \mathbf{R}^3$ and a plane Σ that separates Ω into two open subsets Ω_1, Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma.$$

For simplicity we will consider in the sequel $\Sigma = \{z = 0\}$.

We define $Y = [0,1]^2$ as the reference square and an open set $T \subset Y$. With $0 < r_\epsilon \leq \epsilon < 1$, we construct on Σ ϵ-periodically distributed obstacles obtained by r_ϵ-homothety from T and denote by T_ϵ its union:

$$T_\epsilon = \bigcup_{k \in \mathbf{Z}^2} (r_\epsilon T + k\epsilon).$$

We introduce the natural functional framework for our problem by defining

$$V = \{u \in H^1(\Omega_1) \cup H^1(\Omega_2) | u = 0 \text{ on } \partial\Omega\}, \quad V^\epsilon = \{u \in V | [u] = 0 \text{ on } T_\epsilon\},$$

where $[u]$ denotes the jump on Σ defined as above. V is a Hilbert space endowed with the following scalar product:

$$\langle u, v\rangle_V = \int_{\Omega_1 \cup \Omega_2} \nabla u \nabla v$$

and V^ϵ is a subspace of V.

Let us remark that $H_0^1(\Omega)$ is a closed subspace of V^ϵ and denote by $W^\epsilon = (H_0^1(\Omega))^\perp$ its orthogonal in V^ϵ and by $W = (H_0^1(\Omega))^\perp$ its orthogonal in V. Thus $V^\epsilon = H_0^1(\Omega) \oplus W^\epsilon$ and, $V = H_0^1(\Omega) \oplus W$. Let us also define $P_{W^\epsilon} : V^\epsilon \to W^\epsilon$, the orthogonal projection onto W^ϵ.

Also, it is easy to see that the trace space of V^ϵ, TV^ϵ is identical to the trace space of W^ϵ, TW^ϵ.

In this setting, the problem (1.2) is equivalent to the following spectral problem: find $u^\epsilon \in V^\epsilon, \lambda^\epsilon \in \mathbf{R}_+$, such that

$$(\mathcal{P}_\epsilon)\begin{cases} -\Delta u^\epsilon = 0, & \text{on } \Omega_1 \cup \Omega_2 \cup T_\epsilon, \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = \lambda^\epsilon [u^\epsilon], & \text{on } \Sigma - T_\epsilon. \end{cases} \tag{2.1}$$

The corresponding equivalent variational formulation is: find $u^\epsilon \in V^\epsilon$, $\lambda^\epsilon \in \mathbf{R}_+$, such that

$$\int_{\Omega_1 \cup \Omega_2} \nabla u^\epsilon \nabla w = \lambda^\epsilon \int_{\Sigma - T_\epsilon} [u^\epsilon][w], \quad \text{for any } w \in V^\epsilon. \tag{2.2}$$

The equivalence between the problems (1.2) and (2.1) is understood in the sense that they have the same eigenvalues and related eigenvectors. Thus, if $\{z_n^\epsilon\}_{n\in\mathbf{N}}$ is an orthonormal sequence of eigenvectors for (1.2), then the sequence $\left\{\frac{1}{\sqrt{\lambda_n^\epsilon}} u_n^\epsilon\right\}_{n\in\mathbf{N}}$, where u_n^ϵ is, for any $n \in \mathbf{N}$, the solution of

$$\inf\left\{\frac{1}{2}|\nabla v|^2_{L^2(\Omega_1\cup\Omega_2)} \middle| v \in W \text{ with } [v]_\Sigma = z_n^\epsilon\right\}, \tag{2.3}$$

is an orthonormal sequence of eigenvectors for problem (2.1). Conversely, if $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ is an orthonormal sequence of eigenvectors for (2.1), then the sequence $\{z_n^\epsilon \sqrt{\lambda_n^\epsilon}\}_{n\in\mathbf{N}}$, with

$$z_n^\epsilon = [u_n^\epsilon] \quad \text{for } n \in \mathbf{N}, \tag{2.4}$$

is an orthonormal sequence for (1.2).

From the compactness of L^{-1}, we have that L has an increasing sequence of eigenvalues $\{\lambda_n^\epsilon\}_{n\in\mathbf{N}}$ and an orthonormal sequence of corresponding eigenvectors $\{z_n^\epsilon\}_{n\in\mathbf{N}}$ in $L^2(\Sigma)$. Also, from the Rayleigh's principle, we have

$$\lambda_n^\epsilon = \inf_{\substack{z\in TW^\epsilon, z\perp z_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\langle Lz, z\rangle_{L^2(\Sigma)}}{\| z \|^2_{L^2(\Sigma)}}. \tag{2.5}$$

3. Asymptotic Analysis

Because of the equivalence relations (2.3) and (2.4), we will study only the asymptotic behavior of (2.1). Similar results for the problem (1.2) will be stated as corollaries.

Now, it is easy to observe that from the equivalence relation, (2.3) and (2.4), we have

$$\inf_{\substack{z\in TW^\epsilon, z\perp z_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\langle Lz, z\rangle_{L^2(\Sigma)}}{\|z\|^2_{L^2(\Sigma)}} = \inf_{\substack{u\in W^\epsilon, u\perp u_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\|u\|_V^2}{\displaystyle\int_\Sigma [u]^2 d\sigma}. \tag{3.1}$$

From (2.5) and (3.1), we get the following representation for λ_n^ϵ, i.e.

$$\lambda_n^\epsilon = \inf_{\substack{u\in W^\epsilon, u\perp u_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\|u\|_V^2}{\displaystyle\int_\Sigma [u]^2 d\sigma}. \tag{3.2}$$

Lemma 3.1. *If* $\lim_{\epsilon\to 0} \frac{r_\epsilon}{\epsilon^2} < \infty$, *then* $C_1 \le \lambda_n^\epsilon$ *and* $\limsup_\epsilon \lambda_n^\epsilon < \infty$, *where* C_1 *is a constant with respect to* ϵ *and* n.

Proof. Using the trace continuity and (3.2), we obtain

$$\lambda_n^\epsilon \ge C_1 \quad \text{for any } n \in \mathbf{N}, \tag{3.3}$$

with C_1 not depending on ϵ, and therefore $\{\lambda_n^\epsilon\}$ is uniformly bounded from below.

We will prove next that all the limit points λ_n of $\{\lambda_n^\epsilon\}_\epsilon > 0$ are finite. We consider the following capacity potential:

$$\begin{cases} -\Delta w^\epsilon = 0, & \text{in } B_\epsilon - r_\epsilon T, \\ w^\epsilon = 1, & \text{on } r_\epsilon T, \\ w^\epsilon = 0, & \text{on } \partial B_\epsilon, \end{cases}$$

where B_ϵ is the ball of radius ϵ centered in the ϵY cube. The function w^ϵ is extended by periodicity on a layer of size ϵ around Σ and then by zero to $\mathbf{R}^3$. The sequence w^ϵ has the property that (see [1])

$$w^\epsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega).$$

Consider $u \in V \cap [C^\infty(\Omega_1) \cup C^\infty(\Omega_2)]$, with the orthogonal decomposition $u = \bar{u}_1 + \bar{u}_2$, where $\bar{u}_1 \in W$ and $\bar{u}_2 \in H_0^1(\Omega)$. We suppose that $\bar{u}_1 \neq 0$ and $\bar{u}_2 \neq 0$. Define $z^\epsilon = (1 - w^\epsilon)u$, then z^ϵ satisfies

$$z^\epsilon \rightharpoonup u \text{ weakly in } V,\ [z^\epsilon] = (1 - w^\epsilon)[\bar{u}_1] \quad \text{on } \Sigma \text{ and } z^\epsilon \in V^\epsilon.$$

We make the observation that, for ϵ small enough, $z^\epsilon \notin H_0^1(\Omega)$ and $z^\epsilon \notin W^\epsilon$. Indeed, we have $[z^\epsilon] = (1 - w^\epsilon)[\bar{u}_1] \neq 0$ on Σ, since $\bar{u}_1 \in W$. On the other hand, letting ϵ go to zero, we obtain:

$$\lim_{\epsilon \to 0} \langle z^\epsilon, \bar{u}_2 \rangle = \int_\Omega \nabla u \nabla \bar{u}_2 = \| \bar{u}_2 \|_V^2 > 0.$$

Therefore, there exists $\epsilon_0 > 0$, such that $\langle z^\epsilon, \bar{u}_2 \rangle \neq 0$ for any $\epsilon < \epsilon_0$, i.e. $z^\epsilon \notin W^\epsilon$ for any $\epsilon < \epsilon_0$.

From (3.2), we have that

$$\lambda_1^\epsilon \leq \frac{\|P_{W^\epsilon} z^\epsilon\|_V^2}{\displaystyle\int_\Sigma [P_{W^\epsilon} z^\epsilon]^2} \leq \frac{\|z^\epsilon\|_V^2}{\displaystyle\int_\Sigma [z^\epsilon]^2} \leq \frac{C_1}{\displaystyle\int_\Sigma [z^\epsilon]^2},$$

where we used the orthogonal decomposition $V^\epsilon = W^\epsilon \oplus H_0^1(\Omega)$ in order to obtain

$$\int_\Sigma [P_{W^\epsilon} z^\epsilon]^2 = \int_\Sigma [z^\epsilon]^2.$$

Since $\{z^\epsilon\}$ is weakly convergent to u and using the continuity of the trace, we get

$$\limsup_{\epsilon \to 0} \lambda_1^\epsilon \leq \frac{C_1}{\displaystyle\int_\Sigma [\bar{u}_1]^2} < \infty,$$

where C_1 is a constant independent of ϵ.

Next, we will use an induction argument to prove the statement for all $n \in \mathbf{N}$. Let us assume that

$$\limsup_{\epsilon \to 0} \lambda_k^\epsilon < \infty \quad \text{for any } k \leq n - 1. \tag{3.4}$$

We need to prove

$$\limsup_{\epsilon \to 0} \lambda_n^\epsilon < \infty.$$

Let $\{\lambda_n^\epsilon\}_{\epsilon > 0}$ be a subsequence of $\{\lambda_n^\epsilon\}_{\epsilon > 0}$ still denoted by ϵ. Then, using the induction hypothesis (3.4), the orthonormality of the associated sequence of eigenvectors and a diagonalization argument, we find a decreasing sequence $\{\epsilon_j\}_{j \in \mathbf{N}}$, such that $\epsilon_j \to 0$ and

$$u_k^{\epsilon_j} \overset{j}{\rightharpoonup} u_k \in \mathrm{W}, \tag{3.5}$$

$$\lim_{j \to \infty} \lambda_k^{\epsilon_j} \doteq \lambda_k < \infty, \tag{3.6}$$

for $k = \overline{1, n-1}$.

Let z^ϵ be as in the proof of Lemma 3.1, with

$$\bar{u}_1 \notin \operatorname{span}\{u_1, \ldots, u_{n-1}\}. \tag{3.7}$$

We can do that because W has infinite dimension.

From (3.2), we obtain

$$\lambda_n^{\epsilon_j} = \inf_{\substack{u \in W^{\epsilon_j}, u \perp u_i^{\epsilon_j} \\ i=\overline{1,n-1}}} \frac{\|u\|_V^2}{\int_\Sigma [u]^2 d\sigma}. \tag{3.8}$$

Consider now

$$\bar{z}^{\epsilon_j} = z^{\epsilon_j} - \sum_{i=1}^{n-1} u_i^{\epsilon_j} \langle z^{\epsilon_j}, u_i^{\epsilon_j} \rangle_V.$$

First, we can see that

$$\langle \bar{z}^{\epsilon_j}, u_i^{\epsilon_j} \rangle_V = 0 \quad \text{for any } i = \overline{1, n-1}. \tag{3.9}$$

Then $\bar{z}^{\epsilon_j} \in V^{\epsilon_j}$ and $\bar{z}^{\epsilon_j} \notin H_0^1(\Omega)$ for j big enough. Indeed from (2.2), we have

$$\langle z^{\epsilon_j}, u_i^{\epsilon_j} \rangle_V = \lambda_i^{\epsilon_j} \int_\Sigma [u_i^{\epsilon_j}][z^{\epsilon_j}]$$

and using the trace continuity, the definition of z^{ϵ_j}, (3.5) and (3.6) in the above relation implies

$$\bar{z}^{\epsilon_j} \rightharpoonup \bar{z} \doteq u - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i].$$

If we suppose

$$\left[u - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] \right] = 0,$$

this is equivalent to

$$\left[\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] \right] = 0,$$

which implies

$$\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] = 0, \tag{3.10}$$

because $\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] \in W$ and W is orthogonal on $H_0^1(\Omega)$. But (3.10) leads to a contradiction with (3.7).

Therefore $[\bar{z}] \neq 0$ and this implies the statement, i.e. $\bar{z}^{\epsilon_j} \notin H_0^1(\Omega)$ for j big enough. Next, using (3.9) and (3.8), we obtain

$$\lambda_n^{\epsilon_j} \leq \frac{\|P_{W^{\epsilon_j}} \bar{z}^{\epsilon_j}\|_V^2}{\int_\Sigma [P_{W^{\epsilon_j}} \bar{z}^{\epsilon_j}]^2} \leq \frac{\|z^{\epsilon_j}\|_V^2}{\int_\Sigma [\bar{z}^{\epsilon_j}]^2} \leq \frac{C_1}{\int_\Sigma [\bar{z}^{\epsilon_j}]^2},$$

where C_1 is a constant independent of j. Passing to the limit when $j \to \infty$, we obtain

$$\limsup_{j\to\infty} \lambda_n^{\epsilon_j} \le \frac{C_1}{\int_\Sigma [\bar{z}]^2} < \infty. \tag{3.11}$$

So we have proved that any subsequence of λ_n^ϵ has a subsequence $\{\lambda_n^{\epsilon_j}\}_{j\in\mathbf{N}}$, such that (3.11) is satisfied. Therefore, we have that

$$\limsup_{\epsilon\to 0} \lambda_n^\epsilon < \infty$$

for any $n \in \mathbf{N}$. □

The next corollary shows that the weak-limits u_n of the sequence $\{u_n^\epsilon\}_{\epsilon>0}$ of the normal eigenvectors associated with the eigenvalue λ_n^ϵ, cannot be zero.

Corollary 3.1. *Let $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ be the orthonormal sequence of eigenvectors associated with λ_n^ϵ for the problem $(\mathcal{P}_\epsilon)$. Then, every weak-limit u_n of $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ (i.e. u_n such that on a subsequence $u_n^\epsilon \overset{\epsilon}{\rightharpoonup} u_n$), is nonzero.*

Proof. Because $\|u_n^\epsilon\|= 1$ a subsequence, still denoted by u_n^ϵ, will weakly converge to some u_n. Using the variational form of $(\mathcal{P}_\epsilon)$, we have

$$\lambda_n^\epsilon = \frac{1}{\int_\Sigma [u_n^\epsilon]^2}.$$

Letting ϵ go to zero above, we obtain

$$\limsup \lambda_n^\epsilon = \frac{1}{\int_\Sigma [u_n]^2}.$$

Next, using Lemma (3.1), we obtain that

$$\int_\Sigma [u_n]^2 \neq 0$$

and this implies the statement. □

Remark 3.1. Similar results hold for the problem (2.1), i.e. all the strong-$L^2(\Sigma)$ limit points of the sequence $\{z_n^\epsilon\}_\epsilon$ are nonzero.

Let us now consider the duality operator $J^\epsilon : V^\epsilon \to (V^\epsilon)'$

$$\langle J^\epsilon u, w\rangle_{(V^\epsilon)',V^\epsilon} = \langle u, w\rangle_{V^\epsilon} \quad \text{for any } u, w \in V^\epsilon.$$

J^ϵ is an operator of subdifferential type

$$J^\epsilon = \partial\varphi^\epsilon, \varphi^\epsilon : V^\epsilon \to R, \tag{3.12}$$

$$\varphi^\epsilon(u) = \frac{1}{2}\|u\|^2_{V^\epsilon}. \tag{3.13}$$

By using the results in Damlamian [5] and Attouch [1], we have the following lemma:

Lemma 3.2. *The sequence of functionals* $\{\varphi^\epsilon\}$ *is* Γ*-convergent weakly in* V *to* φ *given by*

$$\varphi(u) = \frac{1}{2}\left(\| u\|_V^2 + \frac{C}{4}\int_\Sigma [u]^2\right),$$

where

$$C = \begin{cases} R \cdot \operatorname{cap} T, & \text{if } \lim_{\epsilon\to 0} \frac{r_\epsilon}{\epsilon^2} = R < \infty. \\ \infty, & \text{if } \lim_{\epsilon\to 0} \frac{r_\epsilon}{\epsilon^2} = \infty. \end{cases}$$

We have used $\operatorname{cap} T$ for the capacity of the set T in $\mathbf{R}^3$, i.e.

$$\operatorname{cap} T = \inf\left\{\int_{\mathbf{R}^3} |\nabla w|^2 dx \,|\, w \in H^1(\mathbf{R}^3), w \geq 1 \text{ a.e. on } T\right\}.$$

Corollary 3.2. *The sequence of operators* J^ϵ *is* G *convergent to* $\partial\varphi$*, with respect to the weak* $\times$ *strong topology in* $V \times V'$.

Proof. Using the G-convergence result for subdifferentials of Γ-convergent sequences (see Attouch [1, Theorem 3.67]), we have that the Γ-convergences of the sequence φ^ϵ to φ imply the G-convergence of the subdifferentials,

$$\partial\varphi^\epsilon \xrightarrow{G} \partial\varphi.$$ □

Next, we state the first homogenization result for problem (2.1).

Theorem 3.1. *There is a decreasing sequence* $\{\epsilon_j\}_j \in \mathbf{N}$ *with* $\epsilon_j \to 0$*, such that* $u_n^{\epsilon_j} \rightharpoonup u_n$, $\lambda_n^{\epsilon_j} \to \lambda_n$*, where* (λ_n, u_n) *solves the limit problem* $(\mathcal{P})$:

$$(\mathcal{P}) \quad \begin{cases} -\Delta u_n = 0 \\ \dfrac{\partial(u_n)^+}{\partial n} = -\dfrac{\partial(u_n)^-}{\partial n} = \left(\lambda_n - \dfrac{C}{4}\right)[u_n], \end{cases}$$

where $C \neq \infty$ *is as in Lemma* 3.2.

Proof. Let an arbitrary fixed $n \in \mathbf{N}$. Let $\{\lambda_n^\epsilon\}_{\epsilon>0}$ be the sequence of eigenvalues for the problem $(\mathcal{P}_\epsilon)$ and u_n^ϵ the corresponding orthonormal sequence of eigenvectors. Then there is a subsequence $\{\epsilon_j\}_j \in \mathbf{N}$, such that:

$$u_n^{\epsilon_j} \rightharpoonup u_n \quad \text{and} \quad \lambda_n^{\epsilon_j} \to \lambda_n.$$

We have proved in Lemma 3.1 that $\lambda_n < \infty$.
Let $f_n^{\epsilon_j} \in V'$ be defined as

$$f_n^{\epsilon_j}(w) = \lambda_n^{\epsilon_j} \int_\Sigma [u_n^{\epsilon_j}][w] \quad \text{for all } w \in V.$$

Using the variational formulation (2.2), we have:

$$f_n^{\epsilon_j}(w) = \langle J^{\epsilon_j} u_n^{\epsilon_j}, w\rangle_{(V^\epsilon)', V^\epsilon} \quad \text{for all } w \in V^{\epsilon_j}.$$

This implies

$$f_n^{\epsilon_j} \in \partial\varphi^{\epsilon_j}\,. \tag{3.14}$$

The next observation is that:

$$f_n^{\epsilon_j} \xrightarrow{j\to\infty} f_n \text{ strongly in } V', \tag{3.15}$$

where

$$f_n(w) = \lambda_n \int_\Sigma [u_n][w] \quad \text{for all } w \in V.$$

The proof of the above convergence is straightforward. Indeed,

$$\|f_n^{\epsilon_j} - f_n\|_{V'} = \sup_{\substack{w\in W\\ \|w\|_V \le 1}} \left(\lambda_n^{\epsilon_j} \int_\Sigma [u_n^{\epsilon_j}][w] - \lambda_n \int_\Sigma [u_n][w]\right).$$

Now from the reflexivity of the space V, we have that there exists $w_0^j \in V$ with $\|w_0^j\|_V \le 1$, such that

$$\begin{aligned}\|f_n^{\epsilon_j} - f_n\|_{V'} &= \left(\lambda_n^{\epsilon_j} \int_\Sigma [u_n^{\epsilon_j}][w_0^j] - \lambda_n \int_\Sigma [u_n][w_0^j]\right)\\ &= (\lambda_n^{\epsilon_j} - \lambda_n) \int_\Sigma [u_n^{\epsilon_j}][w_0^j] + \lambda_n \int_\Sigma [u_n^{\epsilon_j} - u_n][w_0^j].\end{aligned}$$

Thus, from the Cauchy–Schwartz inequality

$$\begin{aligned}\|f_n^{\epsilon_j} - f_n\|_{V'} \le |\lambda_n^{\epsilon_j} - \lambda_n| &\left(\int_\Sigma [u_n^{\epsilon_j}]^2\right)^{1/2} \left(\int_\Sigma [w_0^j]^2\right)^{1/2}\\ &+ \lambda_n \left(\int_\Sigma [u_n^{\epsilon_j} - u_n]^2\right)^{1/2} \left(\int_\Sigma [w_0^j]^2\right)^{1/2}.\end{aligned}$$

Next, we will use the following interpolation inequality (see [8]):

$$\|u\|^2_{L^2(\Sigma)} \le M \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)} \quad \forall u \in V \tag{3.16}$$

and the fact that $\|w_0^j\|_V \le 1$ to obtain:

$$f_n^{\epsilon_j} \xrightarrow{j\to\infty} f_n \text{ strongly in } V'.$$

Therefore, from (3.14), (3.15) and using the Corollary (3.2), we obtain that:

$$f_n \in \partial\varphi(u_n). \tag{3.17}$$

But (3.17) is exactly the problem $(\mathcal{P})$. □

The limit problem ($\mathcal{P}$) is equivalent to the following spectral problem for the DtN operator defined above. Indeed, problem ($\mathcal{P}$) is:

Find $\lambda \in \mathbf{R}$ and $z \in H^{\frac{1}{2}}(\Sigma)$ such that:

$$Lz = \left(\lambda - \frac{C}{4}\right) z. \tag{3.18}$$

Using the equivalence relations (2.3) and (2.4), the next corollary is a obvious consequence of the above discussions.

Corollary 3.3. *There is a decreasing sequence $\{\epsilon_j\}_j \in \mathbf{N}$ with $\epsilon_j \to 0$, such that $z_n^{\epsilon_j} \to z_n$ strongly in $L^2(\Sigma)$ and $\lambda_n^{\epsilon_j} \to \lambda_n$, where (λ_n, z_n) solves the limit problem:*

$$Lz_n = \left(\lambda_n - \frac{C}{4}\right) z_n,$$

where $C \neq \infty$ is as in Lemma 3.2.

The main homogenization result is:

Theorem 3.2. *If $\lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon^2} < \infty$, then:*

(i) $\lim_{\epsilon \to 0} \lambda_n^\epsilon = \lambda_n$ *on the entire sequence.*
(ii) *There is a decreasing sequence $\{\epsilon_j\}_{j \in \mathbf{N}}$ with $\epsilon_j \to 0$, such that $u_n^{\epsilon_j} \rightharpoonup u_n$ weakly in V and $z_n^{\epsilon_j} \to z_n$ strongly in $L^2(\Sigma)$.*
(iii) $\lambda_n = \left(\frac{C}{4} + \beta_n\right)$ *with $Lz_n = \beta_n z_n$, where β_n is the nth eigenvalue of the DtN operator L, and C is as in Lemma* 3.2.

Proof. Suppose there is $\lambda \neq \lambda_n$ for any $n \in \mathbf{N}$ eigenvalue for the limit problem. Let $u \in W$ be the associated normal eigenvector, i.e. $\|u\|_V = 1$ and

$$\langle u, w \rangle = \left(\lambda - \frac{C}{4}\right) \int_\Sigma [u][w] \quad \text{for all } w \in W.$$

There is $m \in \mathbf{N}$, such that

$$\lambda < \lambda_{m+1}. \tag{3.19}$$

From the Lax Milgram lemma, we have that there exists $w^\epsilon \in W^\epsilon$, such that

$$\langle J^\epsilon w^\epsilon, w \rangle_{(V^{\epsilon\prime}, V^\epsilon)} = \lambda \int_\Sigma [u][w], \quad \text{for all } w \in W^\epsilon.$$

It can be earily seen that w^ϵ is bounded in the norm of V.

Then, on a subsequence still denoted by ϵ, we have

$$w^\epsilon \rightharpoonup \bar{w} \quad \text{as } \epsilon \to 0,$$

for some $\bar{w} \in W$. But, if we consider $f_\lambda \in V'$ with $f_\lambda(w) = \lambda \int_\Sigma [u][w]$, then clearly

$$f_\lambda(w) = \langle J^\epsilon w^\epsilon, w \rangle_{(V^{\epsilon\prime}, V^\epsilon)} \Rightarrow f_\lambda \in \partial\varphi^\epsilon(w^\epsilon).$$

So, using the G-convergence result stated in (3.2), we obtain

$$f_\lambda \in \partial\varphi(\bar{w}) \Leftrightarrow \langle \bar{w}, v \rangle + \frac{C}{4} \int_\Sigma [u][\bar{w}] = \lambda \int_\Sigma [u][v]$$

for any $v \in W$.

Therefore, because of the definition of u, we have that $u = \bar{w}$. Now by Uryson's property, we can see that

$$w^\epsilon \rightharpoonup u \quad \text{when } \epsilon \to 0.$$

Let

$$v^\epsilon = w^\epsilon - \sum_{i=1}^{m} u_i^\epsilon (w^\epsilon, u_i^\epsilon)_V.$$

We can see that

$$(w^\epsilon, u_i^\epsilon)_V = \lambda_i^\epsilon \int_\Sigma [u_i^\epsilon][w^\epsilon] \xrightarrow{\epsilon} \lambda_i \int_\Sigma [u][u_i].$$

But, using the variational form of problem $(\mathcal{P})$, the last integral in the above equality is zero by the assumption $\lambda \neq \lambda_n$ for any $n \in \mathbf{N}$.

Thus, $v^\epsilon \rightharpoonup u$ weakly in V. Noticing that $v^\epsilon \in W^\epsilon$ and $v^\epsilon \perp u_i^\epsilon$ for all $i = \overline{1, m}$, from the Rayleigh's principle for (2.1), we have

$$\lambda_{m+1}^\epsilon \leq \frac{\|v^\epsilon\|_V^2}{\displaystyle\int_\Sigma [v^\epsilon]^2}. \tag{3.20}$$

Now, from the definition of w^ϵ and the inequality (3.16), we have

$$\lim_{\epsilon\to 0} \|v^\epsilon\|_V^2 = \lim_{\epsilon\to 0} \|w^\epsilon\|_V^2 = \lambda \int_\Sigma [u]^2.$$

From the last relation and Theorem 3.1, passing to the limit when $\epsilon \to 0$ in (3.20), we obtain the contradiction.

Using the equivalence relations (2.3), (2.4) and Corollary 3.3 we obtain (ii) and (iii). □

Next, following an idea in [1], we give a Mosco-convergence (see [1] for the definition of Mosco-convergence) result for the case $\lim_{\epsilon\to 0} \frac{r_\epsilon}{\epsilon^2} < \infty$.

Theorem 3.3. *Let $\lim_{\epsilon\to 0} \frac{r_\epsilon}{\epsilon^2} < \infty$ and $i \in \mathbf{N}$ be arbitrarily fixed.*

Then, if m_i is the order of multiplicity of λ_i, i.e.

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+m_i-1} < \lambda_{i+m_i},$$

the sequence of subspaces generated by $\{u_i^\epsilon, \dots, u_{i+m_i-1}^\epsilon\}$ Mosco-converge in $L^2(\Omega)$ to the eigenspace $\{u_i, \dots, u_{i+m_i-1}\}$ associated with λ_i.

Proof. We remark that the multiplicity of λ_i^ϵ might be strictly smaller than that of λ_i. So, if we denote

$$\operatorname{span}\{u_i^\epsilon, \dots, u_{i+m_i-1}^\epsilon\} \doteq S_i^\epsilon \quad \text{and} \quad \operatorname{span}\{u_i, \dots, u_{i+m_i-1}\} \doteq S_i,$$

we can see that as in the above remark S_i^ϵ may be strictly larger than the eigenspace of λ_i^ϵ. Now, from Theorem 3.2, we have that there is a subsequence $\{u_n^{\epsilon_j}\}_{j\in\mathbf{N}}$, such that

$$\lim_{\epsilon\to 0} \lambda_n^\epsilon = \lambda_n \quad \text{and} \quad u_n^{\epsilon_j} \rightharpoonup u_n,$$

where (u_n, λ_n) solves the spectral limit problem $\mathcal{P}$.

From the linearity of $\mathcal{P}^\epsilon$ and $\mathcal{P}$, we can say that

$$\limsup_{\epsilon\to 0} S_i^\epsilon \subset S_i.$$

We can easily see that for arbitrarily fixed $i, j \in \mathrm{N}$, with $i \neq j$ and

$$u_i^\epsilon \rightharpoonup u_i \quad \text{and} \quad u_j^\epsilon \rightharpoonup u_j$$

we have

$$\langle u_i, u_j \rangle_V = 0.$$

Indeed, from

$$0 = \langle u_i^\epsilon, u_j^\epsilon \rangle_V \; = \lambda_i^\epsilon \int_\Sigma [u_i^\epsilon][u_j^\epsilon]$$

passing to the limit when $\epsilon \to 0$, we have

$$\lambda_i \int_\Sigma [u_i][u_j] = 0 \Rightarrow \langle u_i, u_j \rangle_V = 0,$$

using the variational form of the limit problem. Next using the linear independence of $\{u_i, \ldots, u_{i+m_i-1}\}$ and the fact that the dimension of the eigenspace associated with λ_i is m_i, we have in fact that

$$\limsup_{\epsilon\to 0} S_i^\epsilon = S_i.$$

Because of the compact imbeding of H^1 in L^2, we have that there is a subsequence ϵ_j, such that

$$\liminf_{\epsilon\to 0} S_i^\epsilon = \limsup_{j\to\infty} S_i^{\epsilon_j}.$$

Now, if there is v, such that

$$v \notin \liminf_{\epsilon\to 0} S_i^\epsilon,$$

then from the above relation, we have

$$v \notin \limsup_{j\to\infty} S_i^{\epsilon_j} = S_i,$$

which implies

$$S_i \subset \liminf_{\epsilon\to 0} S_i^\epsilon.$$

So we have proved the statement. □

The next corollary is a consequence of the above results and states.

Corollary 3.4. *Let* $\lim_{\epsilon\to 0}\frac{r_\epsilon}{\epsilon^2} < \infty$ *and* $i \in \mathbf{N}$ *be arbitrarily fixed. Then, if* m_i *is the order of multiplicity of* λ_i*, i.e.*

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+m_i-1} < \lambda_{i+m_i},$$

the sequence of subspaces generated by $\{z_i^\epsilon, \dots, z_{i+m_i-1}^\epsilon\}$ *Mosco-converge in* $L^2(\Sigma)$ *to the eigenspace* $\{z_i, \dots, z_{i+m_i-1}\}$ *associated with* λ_i *for the problem* (3.18).

Next we will analyze the case when λ_i is a simple eigenvalue of the limit problem. We have the following result:

Theorem 3.4. *Let* $\lim_{\epsilon\to 0}\frac{r_\epsilon}{\epsilon^2} < \infty$*. If* $\lambda_n^\epsilon \to \lambda_n$ *and* λ_n *is a simple eigenvalue of the limit problem* $(\mathcal{P})$*, then the whole sequence* $\{u_n^\epsilon\}$ *is convergent,* $u_n^\epsilon \rightharpoonup u_n$*, where* u_n *is an eigenvector for* $(\mathcal{P})$ *associated with* λ_n *and* $\|u_n\|_V^2 = \left(\frac{1}{\beta_n}\frac{C}{4} + 1\right)^{-1}$*, where* β_n *is as in Theorem* 3.2 *and* C *is defined in Lemma* 3.2.

Proof. Let $\tilde{u}_n$ be the unit eigenvector associated with λ_n. Because λ_n is simple, we will have that λ_n^ϵ will be simple for ϵ small enough. We can select u_{nn}^ϵ, such that for every $\epsilon > 0$

$$\langle u_n^\epsilon, \tilde{u}_n \rangle \geq 0. \tag{3.21}$$

From the orthogonality of $(u_n^\epsilon)_{n\in\mathbf{N}}$, we have that their limits $(u_n)_{n\in\mathbf{N}}$ form an orthogonal subsequence. Indeed, using Theorem 4.2 and (3.16), we have that there is a subsequence still denoted by ϵ, such that we can pass to the limit when $\epsilon \to 0$ in the next equality

$$0 = \langle u_n^\epsilon, u_m^\epsilon \rangle_V = \lambda_n^\epsilon \int_\Sigma [u_n^\epsilon][u_m^\epsilon].$$

In the limit when $\epsilon \to 0$ in the last equality, we obtain

$$\lambda_n \int_\Sigma [u_n][u_m] = 0 \Leftrightarrow \langle u_n, u_m \rangle_V \cdot \frac{\lambda_n}{\lambda_n - \frac{C}{4}} = 0 \Leftrightarrow \langle u_n, u_m \rangle_V = 0,$$

and therefore the orthogonality of the limits eigenfunctions is proved.

On the other hand, using the orthonormality of $(u_n^\epsilon)_{n\in\mathbf{N}}$, we have that for any subsequence $(u_n^{\epsilon_j})_j$ there is a subsequence of it $(u_n^{\epsilon_{j_k}})_k$, such that $u_n^{\epsilon_{j_k}} \rightharpoonup u_n$.

Because λ_n is a simple eigenvalue, we find that there is a constant r, such that $u_n = r \cdot \tilde{u}_n$. Then, from the orthonormality, we find again

$$\|u_n^{\epsilon_{j_k}}\|_V = 1 \Leftrightarrow \left\langle u_n^{\epsilon_{j_k}}, u_n^{\epsilon_{j_k}} \right\rangle_V = 1 \Leftrightarrow \lambda_n^{\epsilon_{j_k}} \int_\Sigma [u_n^{\epsilon_{j_k}}][u_n^{\epsilon_{j_k}}] = 1.$$

Passing to the limit when $k \to \infty$ in the above equality and using $\lambda_n = \frac{C}{4} + \beta_n$, we obtain

$$\left(\frac{C}{4} + \beta_n\right) \int_\Sigma [u_n][u_n] = 1 \Leftrightarrow \langle u_n, u_n \rangle_V \cdot \left(\frac{1}{\beta_n}\frac{C}{4} + 1\right) = 1.$$

Therefore, because $u_n = r \cdot \tilde{u}_n \ \forall n \in \mathbf{N}$, we have that

$$|r| = \left(\frac{1}{\beta_n}\frac{C}{4} + 1\right)^{-1/2}.$$

But, because of (3.21), we find that $r > 0$. Thus

$$r = \left(\frac{1}{\beta_n}\frac{C}{4} + 1\right)^{-1/2}.$$

So, we have proved that every subsequence $\{u_n^{\epsilon_{j_k}}\}_{k\in\mathbf{N}}$ of $\{u_n^\epsilon\}_{\epsilon>0}$ has a subsequence of it, which converges in the weak topology of V, to $u_n = \tilde{u}_n\left(\frac{1}{\beta_n}\frac{C}{4} + 1\right)^{-1/2}$. Therefore, the conclusion follows immediately. □

The next corollary follows from the equivalence relations (2.3), (2.4) and Corollary 3.3.

Corollary 3.5. *Let* $\lim_{\epsilon\to 0}\frac{r^\epsilon}{\epsilon^2} < \infty$. *If* $\lambda_n^\epsilon \to \lambda_n$ *and* λ_n *is a simple eigenvalue of the limit problem* $(\mathcal{P})$, *then the entire sequence of eigenvectors for the problem* (2.1), $\{z_n^\epsilon\}_\epsilon$, *is convergent to* z_n *strongly in* $L^2(\Sigma)$, *where* z_n *is an eigenvector for* (3.18) *and* $\|z_n\|^2_{L^2(\Sigma)} = \frac{1}{\frac{C}{4}+\beta_n}$.

3.1. *Case* $\lim_{\epsilon\to 0}\frac{r_\epsilon}{\epsilon^2} = \infty$

In this case, we can see that the sequence $\{\varphi^\epsilon\}_{\epsilon>0}$ defined in 3.2, Γ-converge to φ and we have

$$\varphi(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx, & \text{if } u \in H_0^1(\Omega),\\ \infty, & \text{otherwise.}\end{cases}$$

Now, suppose that there is $n \in \mathbf{N}$, such that $\lambda_n^\epsilon \xrightarrow{\epsilon} \lambda_n < \infty$.

Then, using the same approach as before, we obtain from Theorem 3.1 and Corollary 3.2 that $f_n \in \partial\varphi(u_n)$. This means that

$$u_n \in \mathrm{Dom}(\varphi) = H_0^1(\Omega).$$

But, we know that $u_n^\epsilon \in W^\epsilon \subset W$, which means that

$$u_n \in W.$$

Using the fact that $W = (H_0^1(\Omega))^\perp$ in V, we obtain $u_n = 0$, which contradicts Corollary 3.1. Therefore, $\lambda_n^\epsilon \xrightarrow{\epsilon} \infty$. Now, from the variational form of (2.1), if u_n^ϵ is the normal eigenvector associated to λ_n^ϵ, we have

$$\frac{1}{\lambda_n^\epsilon} = \int_\Sigma [u_n^\epsilon]^2.$$

Consider $u_n \in W$ to be the weak limit of u_n^ϵ when $\epsilon \to 0$. Passing to the limit for $\epsilon \to 0$ in the equality above, we obtain

$$\int_\Sigma [u_n]^2 = 0.$$

This together with the fact that $u_n \in W$ and $W \perp H_0^1(\Omega)$ give us that $u_n = 0$. So, in this case, we have that all the eigenvectors of the $\mathcal{P}_\epsilon$ converges to zero and all the eigenvalues of the same problem converges to ∞.

3.2. *Case* $\lim_{\epsilon\to 0} \frac{r_\epsilon}{\epsilon^2} = 0$

Although this can be seen as a particular case for all the results stated above, we will discuss it separately due to the fact that we can obtain a stronger variant of Theorem 3.3. This case is very interesting because the holes T_ϵ "disappear" in the limit problem.

First, we can observe following [1, Theorem 1.27] that in this case, we have

$$w^\epsilon \to 0 \quad \text{strongly in } H_0^1(\Omega),$$

where w^ϵ is the capacity potential defined in Lemma 3.1.

It can be easily seen that for each $u \in W$, smooth, using w^ϵ we can construct the sequence $\tilde{u}_\epsilon = u - w^\epsilon u$, such that $\tilde{u}_\epsilon \in V^\epsilon$ and $P_{W^\epsilon}\tilde{u}_\epsilon \to u$ strongly in W.

Using the strong convergence of the capacity potential, we obtain:

$$\tilde{u}_\epsilon \to u \quad \text{strongly in } V \quad \text{when } \epsilon \to 0. \tag{3.22}$$

Now, we have

$$\tilde{u}_\epsilon = P_{W^\epsilon}\tilde{u}_\epsilon + (\tilde{u}_\epsilon - P_{W^\epsilon}\tilde{u}_\epsilon).$$

Let $a = \lim_{\epsilon\to 0} P_{W^\epsilon}\tilde{u}_\epsilon$ and $b = \lim_{\epsilon\to 0}(\tilde{u}_\epsilon - P_{W^\epsilon}\tilde{u}_\epsilon)$ be two (arbitrarily chosen) weak limit points of $\{P_{W^\epsilon}\tilde{u}_\epsilon\}_{\epsilon>0}$ and $\{\tilde{u}_\epsilon - P_{W^\epsilon}\tilde{u}_\epsilon\}_{\epsilon>0}$, respectively.

It is easy to see that $a \in (H_0^1(\Omega))^\perp$ and $b \in H_0^1(\Omega)$. Therefore, we obtain $u = a + b$. Thus $b = 0$ and $a = u$. From the arbitrary choice of a and b, and the compactness of the above sequences, we obtain that

$$\begin{aligned} P_{W^\epsilon}\tilde{u}_\epsilon &\rightharpoonup u, \\ \tilde{u}_\epsilon - P_{W^\epsilon}\tilde{u}_\epsilon &\rightharpoonup 0. \end{aligned} \tag{3.23}$$

Next, we will use the following lemma in order to get the conclusion.

Lemma 3.3. *Let $\{a_n\}_{n\in\mathbf{N}}$ and $\{b_n\}_{n\in\mathbf{N}}$ be two sequences in V (where V can be a general Hilbert space, and $(\,.\,,\,.)$ the scalar product in V) such that $a_n \perp b_n$ for every $n \in \mathbf{N}$.*

If $(a_n + b_n) \to L$ strongly in V and $a_n \rightharpoonup L$, then $a_n \to L$ and $b_n \to 0$ strongly in V as $n \to \infty$.

Proof. First we have that

$$\|a_n\|_V^2 = (a_n + b_n, a_n) \to \|L\|_V^2.$$

Therefore, we have that

$$\|a_n\|_V \to \|L\|_V .$$

Then, from reflexivity of V we obtain the result. □

Next, from the above lemma, (3.23), (3.22) and using the orthogonal decomposition of $\tilde{u}_\epsilon$, we obtain

$$\begin{aligned} P_{W^\epsilon}\tilde{u}_\epsilon &\to u \quad \text{strongly in } V,\\ \tilde{u}_\epsilon - P_{W^\epsilon}\tilde{u}_\epsilon &\to 0 \quad \text{strongly in } V. \end{aligned} \tag{3.24}$$

Noticing that W^ϵ is a closed subspace of W, using (3.24) we can easily prove that

$$P_{W^\epsilon}u \to u.$$

Indeed, we have

$$\|P_{W^\epsilon}u - u\|_V \leq \|\tilde{u}_\epsilon - u\|_V + \|P_{W^\epsilon}u - P_{W^\epsilon}\tilde{u}_\epsilon\|_V + \|P_{W^\epsilon}\tilde{u}_\epsilon - \tilde{u}_\epsilon\|_V. \tag{3.25}$$

But, for any $u \in V$, we have

$$\begin{aligned} \|P_{W^\epsilon}u - P_{W^\epsilon}\tilde{u}_\epsilon\|_V &= \|P_{W^\epsilon}(P^\epsilon u) - P_{W^\epsilon}\tilde{u}_\epsilon\|_V \leq \|P^\epsilon u - \tilde{u}_\epsilon\|_V\\ &\leq \|P^\epsilon u - u\|_V + \|\tilde{u}_\epsilon - u\|_V \leq 2 \cdot \|\tilde{u}_\epsilon - u\|_V. \end{aligned} \tag{3.26}$$

Thus, from (3.24), (3.22) and (3.26), the right-hand member in (3.25) goes to 0 when $\epsilon \to 0$. This implies, using a density argument that,

$$P_{W^\epsilon}u \to u \quad \text{for every } u \in W. \tag{3.27}$$

Let $P_{W^\epsilon} \equiv \mathcal{R}_\epsilon$.

Now, for $u \in W^\epsilon$ we define $K^\epsilon\colon W^\epsilon \to W^\epsilon$ and $K\colon W \to W$ as

$$\langle K^\epsilon u, w\rangle_V = \int_{\Sigma - T_\epsilon} [u][w], \quad \text{for any } w \in W^\epsilon, \tag{3.28}$$

and

$$\langle Ku, w\rangle_V = \int_\Sigma [u][w], \quad \text{for any } w \in W. \tag{3.29}$$

We can see that K^ϵ and K, are compact and symmetric operators and they have the eigenvalues $\{\frac{1}{\lambda_n^\epsilon}\}_{n\in\mathbf{N}}$ and $\{\frac{1}{\beta_n}\}_{n\in\mathbf{N}}$, respectively, and the associated eigenvectors sequence $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ and $\{u_n\}_{n\in\mathbf{N}}$, respectively. It is easy to check now that $\mathcal{R}_\epsilon$ verify the properties stated in [10, Sec. 11.1]. In these conditions all the results obtained in [10, Chap. 11] are valid in our case too. Define

$$N(\beta_n, K) = \left\{u \in W, Ku = \frac{1}{\beta_n}u\right\}$$

as in [10]. Now, following the results in [10], we have $\lambda_n^\epsilon \to \beta_n$ and

$$\left|\frac{1}{\lambda_n^\epsilon} - \frac{1}{\beta_n}\right| \leq 2 \cdot \sup_{u\in N(\beta_n,K),\|u\|_W=1} \|K^\epsilon\mathcal{R}_\epsilon u - \mathcal{R}_\epsilon Ku\|_W, \tag{3.30}$$

and for the eigenvectors $\{u_n^\epsilon\}_{n\in\mathbb{N}}$, the following stronger version of Theorem 3.3 holds:

Theorem 3.5. *Let $i \geq 1$ be an integer and*

$$\lambda_{i-1} < \lambda_i = \cdots = \lambda_{i+m_i-1} < \lambda_{m_i+i}, \text{ i.e.}$$

the multiplicity of the eigenvalue λ_i is equal to m_i, then for any $w \in N(\beta_i, K)$, $\|w\|_V = 1$, there exists a linear combination $\bar{u}^\epsilon$ of eigenvectors $u_i^\epsilon, \ldots, u_{i+m_i-1}^\epsilon$ of K^ϵ, such that

$$\|\bar{u}^\epsilon - w\|_V \leq M_i \|K^\epsilon \mathcal{R}_\epsilon w - \mathcal{R}_\epsilon K w\|_V + \|\mathcal{R}_\epsilon w - w\|_V, \tag{3.31}$$

where the constant M_i does not depend on ϵ.

Remark 3.2. Using (3.27) and the relation above, we can see that Theorem 3.5 states, in fact, the Mosco-convergence in the strong topology of V of the sequence of the spaces generated by $\{u_i^\epsilon, \ldots, u_{i+m_i-1}^\epsilon\}$ to the eigenspace associated with λ_i, and this is stronger than Theorem 3.3.

Remark 3.3. As a last observation, we can see that by using (2.3) and (2.4) similar results to those obtained in Secs. 3.1 and 3.2 can be stated and proved for the problem (1.2).

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