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Asymptotic analysis of a multiscale parabolic problem with a rough fast oscillating interface

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Abstract This paper is concerned with the well posedness and homogenization for a multiscale parabolic problem in a cylinder Q of $\mathbb{R}^N$. A rapidly oscillating non-smooth interface inside Q separates the cylinder in two heterogeneous connected components. The interface has a periodic microstructure, and it is situated in a small neighborhood of a hyperplane which separates the two components of Q. The problem models a time-dependent heat transfer in two heterogeneous conducting materials with an imperfect contact between them. At the interface, we suppose that the flux is continuous and that the jump of the solution is proportional to the flux. On the exterior boundary, homogeneous Dirichlet boundary conditions are prescribed. We also derive a corrector result showing the accuracy of our approximation in the energy norm.

Keywords Parabolic problem · Homogenization · Heat propagation · Rough interface · Correctors

Mathematics Subject Classification 35J75 · 35J65 · 35B27

1 Introduction

This work is devoted to the homogenization of a heat transfer problem posed on a domain separated by a non-smooth interface. The interface is modeled as a highly oscillatory Lipschitz surface (for example, the "sawtooth interface" sketched in Fig. 1) of height $O(\varepsilon^\kappa)$ (with $\kappa > 0$) and the resulting interfacial resistance gives rise to the flux of temperature proportional to a jump of the temperature, by a factor of order ε^γ, where ε is the small parameter characterizing the small scale in the problem and $\gamma \leq 1$ is a given real parameter. The complexity of the domain geometry and the imperfect contact on the interface create interesting multiscale phenomena with different macroscale behaviors depending on model parameters κ and γ.

A similar geometric setting was recently considered in the papers of Donato and Piatnitski [16] and Donato and Giachetti [15] that discussed stationary diffusion problems. Domains with rough surfaces or boundaries can be found in many applications such as flows, elastic bodies and electromagnetic waves over rough walls or

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interfaces. The roughness in the interface influences the general response of the system under consideration, which rends the model difficult to handle numerically. This motivates the use of multiscale analysis, in order to obtain a macroscopic homogenized model with a flat interface.

The use of homogenization to analyze problems in a domain with a rough or rapidly oscillating boundary can be traced back from the works of Kohler et al. [25] and of Brizzi and Chalot [6,7]. A similar approach was applied by Nevard and Keller [31] to analyze Maxwells equation and to the equations of the theory of linear elasticity. In [1], Achdou et al. studied boundary conditions or wall laws for a laminar flow over a rough wall with periodic roughness elements using homogenization.

Another related work on boundary homogenization can be found in Chechkin et al. [10] which considered the asymptotic behavior of solutions of an elliptic problem with an inhomogeneous Fourier boundary condition in domains with rapidly oscillating locally periodic boundary. An unlimited growth of the $(n-1)$-dimensional volume of the boundary as the small parameter tends to zero is assumed therein. On the other hand, for the cases studied in [4–8,20,21,32], the $(n-1)$-dimensional volume of the oscillating boundary remained uniformly bounded.

Asymptotic analysis of imperfect transmission problems on two-component composites due to interfacial resistances (as modeled by Carslaw and Jaeger [9]) was considered for different types of PDEs. First, Auriault and Ene [2] considered the elliptic case, after which [13,18,27] continued the study. For the parabolic and hyperbolic cases, one can check [14,17,19,23]. Hummel [22] showed earlier in the general case that when $\gamma \geq 1$, the solution becomes unbounded.

Homogenization problems on a prefractal layer were studied by Lancia et al. [26] (see also references therein), while the multiscale analysis for optimal control problems on domains with highly oscillating boundaries was studied by Nandakumaran et al. [28] (see also [29]), among others. For general references on homogenization, one can see [3,11,32].

In this paper, we study the well posedness and prove several homogenization results for a parabolic problem with an imperfect contact on the rough fast oscillating interface separating a domain occupied by heterogeneous materials. The peculiarity of this time-dependent problem is apparent in the lack of regularity for the time derivative of the ε solution which further complicates the homogenization procedure in general, and in particular the identification of the initial data. This is overcome by using a suitable compactness result (Theorem 4), stated and proved in Sect. 4. Another challenge appearing in the present time-domain analysis was the proof of uniform convergence with respect to time of the ε-solution to the homogenized solution needed to assess the convergence of the initial data and in the corrector analysis.

Proposing several new arguments to deal with the challenges appearing in the time-dependent problem and building up on the ideas presented in [15,16], depending on values of κ [introduced in (3)] and γ [introduced at (26)], we characterize in Theorem 5 the homogenized limit as the unique solution of a macroscale problem and prove suitable time-domain energy convergence and associated corrector results (Theorem 6). More explicitly we characterize three possible macroscale behaviors as follows:

1. If ($\kappa \geq 1$ and $\gamma = 0$) or ($0 < \kappa < 1$ and $\gamma = 1 - \kappa$), then the macroscale problem is modeled by a parabolic PDE over a domain separated by a hyperplane with the continuous flux across it given by a homogenized law.
2. If ($\kappa \geq 1$ and $\gamma < 0$) or ($0 < \kappa < 1$ and $\gamma < 1 - \kappa$), then the contribution of the microscale transmission interface disappears in the homogenized limit and the macroscale model is governed by a parabolic PDE in a smooth domain with homogeneous Dirichlet boundary conditions.
3. If ($\kappa \geq 1$ and $\gamma > 0$) or ($0 < \kappa < 1$ and $\gamma > 1 - \kappa$), then the microscale transmission interface has a very strong effect in the limit and the macroscale problem is modeled by a parabolic homogenized PDE on two disjoint domains with identical initial conditions and homogeneous mixed boundary conditions, zero flux on the flat part of the boundary and zero temperature otherwise.

This paper is organized as follows. In Sect. 2, the problem's geometric setting and the relevant functional spaces with their properties are presented. We then describe the parabolic problem and our assumptions in Sect. 3. The existence and uniqueness of the solution of our problem are also shown in Sect. 3, using a theorem based on a Galerkin type method. In Sect. 4, some uniform a priori estimates and compactness results which are important for homogenization are derived. The homogenization of our multiscale problem (limit analysis for $\varepsilon << 1$) is examined in Sect. 5, while the corrector results showing the form of the second term in the asymptotic expansion for the multiscale solution are discussed in Sect. 6. Lastly, in Sect. 7, we present a physical interpretation of the results as well as comments about possible applications.

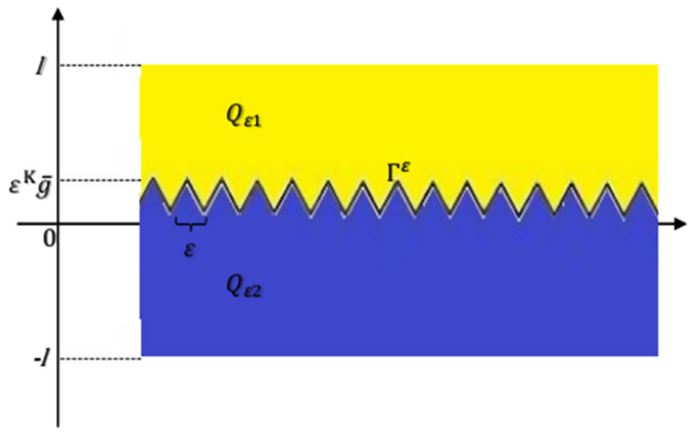

Fig. 1 An example of a possible geometry for a "sawtooth" interface

2 Preliminaries

In this work, we use the geometric framework and notations introduced in [16] (also used in [15]). Let $N \geq 2$ and suppose ω is a smooth bounded subset of $\mathbb{R}^{N-1}$ with l a positive number. We define the domain Q by

$$Q = \omega \times]-l, l[, \tag{1}$$

which is an open bounded cylinder in $\mathbb{R}^N$.

We denote by $Y =]0, 1[^N$ the volume reference cell and by $Y' =]0, 1[^{N-1}$ the surface reference cell. Furthermore, we let ε denote a positive sequence converging to zero. Assume $g : Y' \to \mathbb{R}$ to be a Y'-periodic positive Lipschitz continuous function, that is, there exists $L_g > 0$ such that,

$$|g(y') - g(y'_1)| \leq L_g |y' - y'_1|, \quad \text{for every} \quad y', y'_1 \in Y'. \tag{2}$$

Suppose $\kappa > 0$ and $x' = (x_1, \ldots, x_{N-1})$. We divide the set Q in two subdomains

$$Q_{\varepsilon 1} = \left\{ x \in Q, x_N > \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) \right\} \tag{3}$$

and

$$Q_{\varepsilon 2} = \left\{ x \in Q, x_N < \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) \right\}, \tag{4}$$

which are called the upper and the lower parts of Q, respectively.

The set

$$\Gamma_\varepsilon = \left\{ x \in Q, x_N = \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) \right\} \tag{5}$$

represents an oscillating interface which separates $Q_{\varepsilon 1}$ and $Q_{\varepsilon 2}$ (see Fig. 1).

As observed in [16], the case $\kappa = 1$ presents a self-similar geometry because the interface Γ_ε can be obtained by dilatation of the fixed function $y_N = g(y')$ in $\mathbb{R}^N$. The case $\kappa > 1$ represents the "flat" case (i.e., $\nabla_{x'} x_N \to 0$ as $\epsilon \to 0$), while the case $0 < \kappa < 1$ describes a highly oscillating interface (see [16] for details).

Setting $\overline{g} = \max g$, by construction, the set $\omega \times [0, \varepsilon^\kappa \overline{g}]$ contains the oscillating interface, and the measure of this set goes to zero as $\varepsilon \to 0$ (see Fig. 1). Consequently,

$$\chi_{Q_{\varepsilon i}} \to \chi_{Q_i} \quad \text{strongly in } L^p(Q),\ 1 \leq p < +\infty, \quad \text{and weakly * in } L^\infty(Q).$$

In the sequel, we will also make use of the decomposition of $\omega \times]0, \varepsilon^\kappa \overline{g}[$ introduced in [16], that is,

$$\omega \times]0, \varepsilon^\kappa \overline{g}[= B_{\varepsilon 1} \cup B_{\varepsilon 2} \cup \Gamma_\varepsilon,$$

where

$$B_{\varepsilon i} = \omega \times]0, \varepsilon^\kappa \overline{g}[\cap Q_{\varepsilon i}, \quad i = 1, 2. \tag{6}$$

We suppose that A is a Y-periodic matrix field satisfying, for $0 < \alpha < \beta$,

$$(A(y)\lambda, \lambda) \geq \alpha|\lambda|^2, \quad |A(y)\lambda| \leq \beta\lambda, \quad \text{a.e. in } Y \text{ and for any } \lambda \in \mathbb{R}^N. \tag{7}$$

Moreover, h will denote an Y'-periodic function such that, for some $h_0 \in \mathbb{R}_+^*$,

$$h \in L^\infty(\Gamma), \text{ and } 0 < h_0 < h(y'), \text{ a.e. on } \Gamma, \tag{8}$$

where

$$\Gamma = \{y_N = g(y'),\ y' \in Y'\}.$$

We set, for any $\varepsilon > 0$,

$$A^\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big), \qquad h^\varepsilon(x') = h\Big(\frac{x'}{\varepsilon}\Big). \tag{9}$$

For any function v defined on Q we set

$$v_{\varepsilon 1} = v_{|Q_{\varepsilon 1}} \qquad v_{\varepsilon 2} = v_{|Q_{\varepsilon 2}}. \tag{10}$$

Hence, for any $v \in L^2(Q)$ we have

$$\|v\|^2_{L^2(Q)} = \|v_{\varepsilon 1}\|^2_{L^2(Q_{\varepsilon 1})} + \|v_{\varepsilon 2}\|^2_{L^2(Q_{\varepsilon 2})}.$$

Also, we use the notations:

- $\tilde{v}$ for the zero extension to the whole of $\mathbb{R}^N$ of a function v defined on a subset of Q,
- χ_E, the characteristic function of any set $E \subset \mathbb{R}^N$,
- $m_{Y'}(v) = \dfrac{1}{|Y'|}\displaystyle\int_{Y'} f\, dy'$, the average on Y' of any function $v \in L^1(Y')$.
- C to denote any generic positive constant independent of ε.

We define the limit domains with a flat interface by

$$Q_1 = \{x \in Q : x_N > 0\},\ Q_2 = \{x \in Q : x_N < 0\},\ \Gamma_0 = \{x \in Q : x_N = 0\} \tag{11}$$

and, for any function v defined on Q,

$$v_1 = v_{|Q_1} \qquad v_2 = v_{|Q_2}.$$

Observe that from definitions (3), (4) and (6),

$$Q_1 = Q_{\varepsilon 1} \cup B_{\varepsilon 2}, \qquad Q_2 = Q_{\varepsilon 2} \setminus B_{\varepsilon 2}.$$

In the sequel, we also use the notations

$$Q_\varepsilon = Q \setminus \Gamma_\varepsilon, \qquad Q_0 = Q \setminus \Gamma_0, \qquad \Gamma_{\varepsilon,0} = \Gamma_\varepsilon \cup \Gamma_0, \qquad Q_{\varepsilon,0} = Q \setminus \Gamma_{\varepsilon,0}.$$

Now, for our functional spaces, we define the space W^ε_{i0} by

$$W^\varepsilon_{0i} := \{v_i \in H^1(Q_{\varepsilon i}) \mid v = 0 \text{ on } \partial Q \cap \partial Q_{\varepsilon i}\},$$

equipped with the norm

$$\|v_i\|_{W^\varepsilon_{0i}} = \|\nabla v_i\|_{L^2(Q_{\varepsilon i})}. \tag{12}$$

As in [16], we also introduce [under notation (10)] the space W^ε_0 defined by

$$W^\varepsilon_0 := \{v \in L^2(Q) \mid v_{\varepsilon 1} \in H^1(Q_{\varepsilon 1}),\ v_{\varepsilon 2} \in H^1(Q_{\varepsilon 2}) \text{ and } v = 0 \text{ on } \partial Q\}, \tag{13}$$

equipped with the norm

$$\|v\|_{W^\varepsilon_0} := \|\nabla v\|_{L^2(Q_\varepsilon)}, \tag{14}$$

where

$$\nabla v = \widetilde{\nabla v_{\varepsilon 1}} + \widetilde{\nabla v_{\varepsilon 2}},$$

that is, we identify ∇v with the absolutely continuous part of the gradient of v.

Let us observe that (14) is a norm, due to the following Poincaré inequality: there exists a constant C such that, for any $v \in W_0^\varepsilon$,

$$\|v\|_{L^2(Q)} \le C\|\nabla v\|_{L^2(Q_\varepsilon)}. \tag{15}$$

Then we introduce the space

$$W_0^0 := \{v \in L^2(Q) \mid v_1 \in H^1(Q_1),\ v_2 \in H^1(Q_2) \quad v = 0 \text{ on } \partial Q\},$$

equipped with the norm

$$\|v\|_{W_0^0} := \|\nabla v\|_{L_2(Q_0)}.$$

In this paper, we use the usual product norm, that is, if E_1 and E_2 are Hilbert spaces then

$$\forall (u, v) \in E_1 \times E_2, \|(u, v)\|_{E_1\times E_2} = \left(\|u\|_{E_1}^2 + \|v\|_{E_2}^2\right)^{\frac{1}{2}}.$$

Remark 1 It is straightforward from the definition and notation (10) that if $v \in W_0^\varepsilon$ then $(v_{\varepsilon 1}, v_{\varepsilon 2}) \in W_{01}^\varepsilon \times W_{02}^\varepsilon$. On the other hand, if $(v_1, v_2) \in W_{01}^\varepsilon \times W_{02}^\varepsilon$ then $v = \widetilde{v_1} + \widetilde{v_2} \in W_0^\varepsilon$.

Moreover, the map

$$\phi : v \in W_0^\varepsilon \to (v_{\varepsilon 1}, v_{\varepsilon 2}) \in W_{01}^\varepsilon \times W_{02}^\varepsilon$$

is a bijective isometry, that is,

$$\|v\|_{W_0^\varepsilon}^2 = \|(v_{\varepsilon 1}, v_{\varepsilon 2})\|_{W_{01}^\varepsilon\times W_{02}^\varepsilon}^2.$$

Indeed from (12) and (14),

$$\begin{aligned}\|v\|_{W_0^\varepsilon}^2 &= \int_{Q_{\varepsilon 1}} |\nabla v_{\varepsilon 1}|^2\, dx + \int_{Q_{\varepsilon 2}} |\nabla v_{\varepsilon 2}|^2\, dx\\ &= \|\nabla v_{\varepsilon 1}\|_{L^2(Q_{\varepsilon 1})}^2 + \|\nabla v_{\varepsilon 2}\|_{L^2(Q_{\varepsilon 2})}^2 = \|v_{\varepsilon 1}\|_{W_{01}^\varepsilon}^2 + \|v_{\varepsilon 2}\|_{W_{02}^\varepsilon}^2.\end{aligned}$$

Since ϕ is bijective, in this paper we identify $v \in W_0^\varepsilon$ with its image $\phi(v) \in W_{01}^\varepsilon \times W_{02}^\varepsilon$.

Proposition 1 *Let $v \in (W_0^\varepsilon)'$. Accordingly with Remark 1, let V be the map defined by*

$$V : (u_1, u_2) \in W_{01}^\varepsilon \times W_{02}^\varepsilon \to v(u) = v(\widetilde{u_1}) + v(\widetilde{u_2}), \quad \textit{for } u = (u_1, u_2).$$

Then $V \in (W_{01}^\varepsilon)' \times (W_{02}^\varepsilon)'$. Conversely, if $V = (V_1, V_2) \in (W_{01}^\varepsilon)' \times (W_{02}^\varepsilon)'$ then

$$v : u \in W_0^\varepsilon \to V(u_{\varepsilon 1}, u_{\varepsilon 2}) = V_1(u_{\varepsilon 1}) + V_2(u_{\varepsilon 2})$$

defines an element of $(W_0^\varepsilon)'$. Moreover,

$$\|V\|_{(W_{01}^\varepsilon)'\times(W_{02}^\varepsilon)'} = \|V_1\|_{(W_{01}^\varepsilon)'} + \|V_2\|_{(W_{02}^\varepsilon)'} = \|v\|_{(W_0^\varepsilon)'}. \tag{16}$$

Proof Suppose $v \in (W_0^\varepsilon)'$. Observe that from the preceding remark,

$$\begin{aligned}|V(u_1, u_2)| = |v(u)| &\le \|v\|_{(W_0^\varepsilon)'}\|u\|_{W_0^\varepsilon}\\ &= \|v\|_{(W_0^\varepsilon)'}\|(u_1, u_2)\|_{W_{01}^\varepsilon\times W_{02}^\varepsilon}.\end{aligned}$$

This gives

$$\|V\|_{(W_{01}^\varepsilon)'\times(W_{02}^\varepsilon)'} = \sup_{u\neq 0} \frac{|V(u_1, u_2)|}{\|(u_1, u_2)\|_{W_{01}^\varepsilon\times W_{02}^\varepsilon}} \le \|v\|_{(W_0^\varepsilon)'}. \tag{17}$$

Therefore, $V \in (W_{01}^\varepsilon)' \times (W_{02}^\varepsilon)'$.

On the other hand, let $V \in (W^\varepsilon_{01})' \times (W^\varepsilon_{02})'$. Then

$$\begin{aligned}|v(u)| = |V(u_{\varepsilon 1}, u_{\varepsilon 2})| &= |V_1(u_{\varepsilon 1}) + V_2(u_{\varepsilon 2})| \\ &\leq \|V_1\|_{(W^\varepsilon_{01})'}\|u_{\varepsilon 1}\|_{W^\varepsilon_{01}} + \|V_2\|_{(W^\varepsilon_{02})'}\|u_{\varepsilon 2}\|_{W^\varepsilon_{02}} \\ &\leq \|V\|_{(W^\varepsilon_{01})'\times(W^\varepsilon_{02})'}\|(u_{\varepsilon 1}, u_{\varepsilon 2})\|_{W^\varepsilon_{01}\times W^\varepsilon_{02}} \\ &= \|V\|_{(W^\varepsilon_{01})'\times(W^\varepsilon_{02})'}\|u\|_{W^\varepsilon_0}.\end{aligned}$$

Thus,

$$\|v\|_{(W^\varepsilon_0)'} = \sup_{u\neq 0}\frac{|v(u)|}{\|u\|_{W^\varepsilon_0}} \leq \|V\|_{(W^\varepsilon_{01})'\times(W^\varepsilon_{02})'}, \tag{18}$$

from which we have $v \in (W^\varepsilon_0)'$. Equality of the norms follows from (17) and (18). □

Proposition 2 *There exists a positive constant C such that*

$$\|v_{\varepsilon 1} - v_{\varepsilon 2}\|_{L^2(\Gamma_\varepsilon)} \leq C \max\left\{1, \varepsilon^{\frac{\kappa-1}{2}}\right\} \|v\|_{W^\varepsilon_0},$$

for every $v = (v_{\varepsilon 1}, v_{\varepsilon 2})$ in W^ε_0.

Proof In terms of the coordinates x' (see Remark 2.3 of [16]), we can write

$$\begin{aligned}\| v_{\varepsilon 1} - v_{\varepsilon 2}\|^2_{L^2(\Gamma_\varepsilon)} &= \int_{\Gamma_\varepsilon}(v_{\varepsilon 1} - v_{\varepsilon 2})^2\,\mathrm{d}s \\ &= \int_\omega \left(v_{\varepsilon 1}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right)\right) - v_{\varepsilon 2}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right)\right)\right)^2 \\ &\quad \times \left(1 + \varepsilon^{2(\kappa-1)}|\nabla_{y'}g(y')|^2\right)^{\frac{1}{2}}_{y'=\frac{x'}{\varepsilon}}\,\mathrm{d}x'.\end{aligned}$$

Clearly, since g is Lipschitz continuous,

$$\|(1 + \varepsilon^{2(\kappa-1)}|\nabla_{y'}g(y')|^2)_{y'=\frac{x'}{\varepsilon}}\|_{L^\infty(\omega)} < C\max\{1, \varepsilon^{2(\kappa-1)}\},$$

so that

$$\|v_{\varepsilon 1} - v_{\varepsilon 2}\|^2_{L^2(\Gamma_\varepsilon)} \leq C\max\{1, \varepsilon^{\kappa-1}\}\int_\omega\left(v_{\varepsilon 1}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right)\right) - v_{\varepsilon 2}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right)\right)\right)^2\mathrm{d}x. \tag{19}$$

For fixed ε, let $z_{\varepsilon 1}$ be defined as

$$z_{\varepsilon 1}(x', x_N) = \widetilde{v_{\varepsilon 1}}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) + x_N\right),$$

where $x' \in \omega$ and $0 < x_N \leq l$. Observe that

$$z_{\varepsilon 1}(x', 0) = \widetilde{v_{\varepsilon 1}}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right)\right), \quad \text{on} \quad \omega \tag{20}$$

and

$$z_{\varepsilon 1}(x', l) \equiv 0, \text{ a.e. in } \omega.$$

Then since $\omega \times \{0\} \subset \partial Q_1$, using the trace theorem and the Poincaré inequality (in the direction of x_N), we have

$$\begin{aligned}\|z_{\varepsilon 1}(x', 0)\|_{L^2(\omega)} &\leq C\left(\|z_{\varepsilon 1}(x', x_N)\|_{L^2(Q_1)} + \left\|\frac{\partial z_{\varepsilon 1}}{\partial x_N}(x', x_N)\right\|_{L^2(Q_1)}\right) \\ &\leq C\left\|\frac{\partial z_{\varepsilon 1}}{\partial x_N}(x', x_N)\right\|_{L^2(Q_1)}.\end{aligned} \tag{21}$$

In fact, the proofs of these results in Q_1 are similar to the analogous ones in the half-space (see for instance [12], Chapter 7). On the other hand, since

$$\frac{\partial z_{\varepsilon 1}}{\partial x_N}(x', x_N) = \frac{\partial \widetilde{v_{\varepsilon 1}}}{\partial x_N}(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) + x_N\Big),$$

we get

$$\left\| \frac{\partial z_{\varepsilon 1}}{\partial x_N}(x', x_N) \right\|_{L^2(Q_1)} = \left\| \frac{\partial v_{\varepsilon 1}}{\partial x_N} \right\|_{L^2(Q_{\varepsilon 1})}.$$

This together with (21) implies

$$\|z_{\varepsilon 1}(x', 0)\|_{L^2(\omega)} \leq C \left\| \frac{\partial \widetilde{v_{\varepsilon 1}}}{\partial x_N} \right\|_{L^2(Q_{\varepsilon 1})}. \tag{22}$$

Now, let $z_{\varepsilon 2}$ be defined, for fixed ε, as

$$z_{\varepsilon 2}(x', x_N) = \widetilde{v_{\varepsilon 2}}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) + x_N\right),$$

where $x' \in \omega$ and $-(l + \eta) < x_N < 0$, for some $\eta > 0$. Here, we have

$$z_{\varepsilon 2}(x', 0) = \widetilde{v_{\varepsilon 2}}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right)\right), \quad \text{on} \quad \omega \tag{23}$$

and

$$z_{\varepsilon 2}(x', (l + \eta) \equiv 0, \text{ a.e. in } \omega.$$

Then as done above, but for $Q_\eta = \omega \times] - (l + \eta), 0[$ instead of Q_1, we have

$$\begin{aligned} \|z_{\varepsilon 2}(x', 0)\|_{L^2(\omega)} &\leq C \left(\|z_{\varepsilon 2}(x', x_N)\|_{L^2(Q_\eta)} + \left\| \frac{\partial z_{\varepsilon 2}}{\partial x_N}(x', x_N) \right\|_{L^2(Q_\eta)} \right) \\ &\leq C \left\| \frac{\partial z_{\varepsilon 2}}{\partial x_N}(x', x_N) \right\|_{L^2(Q_\eta)}, \end{aligned} \tag{24}$$

where C is independent of ε. Also, since

$$\frac{\partial z_{\varepsilon 2}}{\partial x_N}(x', x_N) = \frac{\partial \widetilde{v_{\varepsilon 2}}}{\partial x_N}\left(x', \varepsilon^\kappa g\left(\frac{x'}{\varepsilon}\right) + x_N\right),$$

we have

$$\left\| \frac{\partial z_{\varepsilon 2}}{\partial x_N}(x', x_N) \right\|_{L^2(Q_\eta)} = \left\| \frac{\partial \widetilde{v_{\varepsilon 2}}}{\partial x_N} \right\|_{L^2(Q_{\varepsilon 2})}.$$

This implies together with (24) that

$$\|z_{\varepsilon 2}(x', 0)\|_{L^2(\omega)} \leq C \left\| \frac{\partial \widetilde{v_{\varepsilon 2}}}{\partial x_N} \right\|_{L^2(Q_{\varepsilon 2})}. \tag{25}$$

From (19), (20), (22), (23), (25) and Remark 1, we get

$$\begin{aligned} \|v_{\varepsilon 1} - v_{\varepsilon 2}\|_{L^2(\Gamma_\varepsilon)} &\leq C \max\left\{1, \varepsilon^{\frac{\kappa - 1}{2}}\right\} \left(\int_\omega |(z_{\varepsilon 1}(x', 0) - z_{\varepsilon 2}(x', 0)|^2 \, dx' \right)^{\frac{1}{2}} \\ &\leq C \max\left\{1, \varepsilon^{\frac{\kappa - 1}{2}}\right\}\Big\} \left(2\|z_{\varepsilon 1}(x', 0)\|^2_{L^2(\omega)} + 2\|z_{\varepsilon 2}(x', 0)\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \\ &\leq C \max\left\{1, \varepsilon^{\frac{\kappa - 1}{2}}\right\} \|v\|_{W_0^\varepsilon}. \end{aligned}$$

□

3 Statement of the problem

The goal of this paper is to prove some existence and homogenization results as $\varepsilon \to 0$, of the following problem:

$$\begin{cases} u'_\varepsilon - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f & \text{in} \quad Q_\varepsilon \times]0,T[, \\ (A^\varepsilon \nabla u_\varepsilon)_1 \cdot \nu_\varepsilon = (A^\varepsilon \nabla u_\varepsilon)_2 \cdot \nu_\varepsilon & \text{on} \quad \Gamma_\varepsilon \times]0,T[, \\ (A^\varepsilon \nabla u_\varepsilon)_1 \cdot \nu_\varepsilon = -\varepsilon^\gamma (u_{\varepsilon 1} - u_{\varepsilon 2}) h^\varepsilon & \text{on} \quad \Gamma_\varepsilon \times]0,T[, \\ u_\varepsilon = 0 & \text{on} \quad \partial Q \times]0,T[, \\ u_\varepsilon(x,0) = u^0_\varepsilon & \text{in} \quad Q_\varepsilon, \end{cases} \tag{26}$$

where $\gamma \leq 1$, ν_ε is the outward normal to $Q_{\varepsilon 1}$ and $A^\varepsilon, h^\varepsilon$ satisfy (7)–(9).

Further, we make the following assumptions on the data:

$$\begin{cases} u^0_\varepsilon \in L^2(Q), \\ f \in L^2(0,T; L^2(Q)). \end{cases} \tag{27}$$

To establish the existence of a solution of problem (26), we consider its variational formulation as follows:

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{W}^\varepsilon, \text{ such that} \\ \langle u'_{\varepsilon 1}, \ v_{\varepsilon 1} \rangle_{(W^\varepsilon_{01})', W^\varepsilon_{01}} + \langle u'_{\varepsilon 2}, v_{\varepsilon 2} \rangle_{(W^\varepsilon_{02})', W^\varepsilon_{02}} \\ + \displaystyle\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \, \mathrm{d}x + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(v_{\varepsilon 1} - v_{\varepsilon 2}) \, \mathrm{d}\sigma \\ = \displaystyle\int_Q f v \, \mathrm{d}x, \quad \text{for every } v \in L^2(0,T, W^\varepsilon_0), \\ u^\varepsilon(x,0) = u^0_\varepsilon \text{ in } Q, \end{cases} \tag{28}$$

where

$$\mathcal{W}^\varepsilon := \{ v = (v_{\varepsilon 1}, v_{\varepsilon 2}) \in L^2(0,T; W^\varepsilon_0) \quad \text{and } v' \in L^2(0,T; (W^\varepsilon_0)') \},$$

equipped with the norm [see (14)]

$$\|v\|_{\mathcal{W}^\varepsilon} = \|\nabla v\|_{L^2(0,T;L^2(Q_\varepsilon))} + \|v'\|_{L^2(0,T;(W^\varepsilon_0)'}.$$

Using a Galerkin type method (see Zeidler [34], Theorem 23.A and Corollary 23.26, pp. 424–426), we can deduce directly the following existence and uniqueness result for our problem:

Theorem 1 *Let $T > 0$ and $\varepsilon > 0$ be fixed. Suppose that W^ε_0, A^ε and h^ε are defined by (13), (7), (8) and (9), respectively. If (27) holds then problem (28) has a unique solution.*

Analogously, we define the space $\mathcal{W}^0$ given by

$$\mathcal{W}^0 := \{ v = (v_1, v_2) \in L^2(0,T; W^0_0) \quad \text{and} \quad v' \in L^2(0,T; (W^0_0)') \},$$

equipped with the norm

$$\|v\|_{\mathcal{W}^0} = \|\nabla v\|_{L^2(0,T;L^2(Q_0))} + \|v'\|_{L^2(0,T;(W^0_0)'}.$$

Remark 2 If $v \in L^2(0,T; W^\varepsilon_0)$ then in view of Proposition 2,

$$\|v_{\varepsilon 1} - v_{\varepsilon 2}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} \leq C \max\left\{1, \varepsilon^{\frac{\kappa-1}{2}}\right\} \|v\|_{L^2(0,T;W^\varepsilon_0)}.$$

Remark 3 As a consequence of Proposition 1, it is straightforward to check that if $v \in L^2(0,T; (W^\varepsilon_0)')$ and $w \in L^2(0,T; W^\varepsilon_0)$ then

$$\begin{aligned} \langle v, w \rangle_{L^2(0,T;(W^\varepsilon_0)'), L^2(0,T;W^\varepsilon_0)} &= \langle v_{\varepsilon 1}, w_{\varepsilon 1} \rangle_{L^2(0,T;(W^\varepsilon_{01})'), L^2(0,T;W^\varepsilon_{01})} \\ &\quad + \langle v_{\varepsilon 2}, w_{\varepsilon 2} \rangle_{L^2(0,T;(W^\varepsilon_{02})'), L^2(0,T;W^\varepsilon_{02})}. \end{aligned}$$

Remark 4 Observe that the properties satisfied by W^ε_0 and stated in Remarks 1, 2, 3 and Propositions 1, 2 still hold in W^0_0, with the obvious changes, simply by writing them for $\varepsilon = 0$.

4 A priori estimates and compactness result

In this section, we prove some uniform estimates (with respect to ε) as well as a compactness result which is essential for the homogenization of our problem. To do that, we assume that the initial data u_ε^0 are bounded in $L^2(Q)$ that is,

$$\|u_\varepsilon^0\|_{L^2(Q)} \leq C. \tag{29}$$

Theorem 2 *Let u_ε be the solution of problem (28) with A^ε and h^ε as in Theorem 1. Suppose (27) and (29) hold. Then,*

$$\begin{cases} (i)\ \|u_\varepsilon\|_{L^\infty(0,T;L^2(Q))} < C, \\ (ii)\ \|u_{\varepsilon 1} - u_{\varepsilon 2}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} < C\varepsilon^{-\frac{\gamma}{2}}, \\ (iii)\ \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(Q_\varepsilon))} < C, \\ (iv)\ \|u'_\varepsilon\|_{L^2(0,T;(W_0^\varepsilon)')} \leq C\left(1 + \varepsilon^{\frac{\gamma}{2}} \max\left\{1, \varepsilon^{\frac{\kappa-1}{2}}\right\}\right), \\ (v)\ \|u'_{\varepsilon 2}\|_{L^2(0,T;H^{-1}(Q_2))} \leq C. \end{cases} \tag{30}$$

Moreover, for every $\delta > 0$,

$$\|u'_{\varepsilon 1}\|_{L^2(0,T;H^{-1}(Q_1^\delta))} \leq C, \quad \textit{for every} \quad \varepsilon \leq \varepsilon_\delta, \tag{31}$$

where

$$Q_1^\delta =: \{x \in Q : x_N > \delta\} \tag{32}$$

and ε_δ is such that $Q_1^\delta \cap \Gamma_\varepsilon = \emptyset$, for every $\varepsilon \leq \varepsilon_\delta$, with C independent of δ and ε.

Proof Choose $v = (u_{\varepsilon 1}, u_{\varepsilon 2})$ in the variational formulation (28). Applying integration on $[0, T]$ and using the Hölder inequality we get for all $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \,\mathrm{d}x\,\mathrm{d}s + \varepsilon^\gamma \int_0^t \int_{\Gamma_\varepsilon} h^\varepsilon |u_{\varepsilon 1} - u_{\varepsilon 2}|^2 \,\mathrm{d}\sigma_x\,\mathrm{d}s \\ &\quad = \frac{1}{2}\|u_\varepsilon^0\|^2_{L^2(Q)} + \int_0^t \int_Q f u_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leq \frac{1}{2}\|u_\varepsilon^0\|^2_{L^2(Q)} + \int_0^t \|f(s)\|_{L^2(Q)} \|u_\varepsilon(s)\|_{L^2(Q)} \,\mathrm{d}s. \end{aligned}$$

The properties of A^ε and h^ε yield

$$\begin{aligned} &\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(Q)} + \alpha \int_0^t \int_{Q_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s + \varepsilon^\gamma h_0 \int_0^t \int_{\Gamma_\varepsilon} |u_{\varepsilon 1} - u_{\varepsilon 2}|^2 \,\mathrm{d}\sigma_x\,\mathrm{d}s \\ &\quad \leq \frac{1}{2}\|u_\varepsilon^0\|^2_{L^2(Q)} + \int_0^t \|f(s)\|_{L^2(Q)} \|u_\varepsilon(s)\|_{L^2(Q)} \,\mathrm{d}s \\ &\quad \leq \frac{1}{2}\|u_\varepsilon^0\|^2_{L^2(Q)} + \frac{1}{2}\int_0^t \left(\|f(s)\|^2_{L^2(Q)} + \|u_\varepsilon(s)\|^2_{L^2(Q)}\right) \mathrm{d}s. \end{aligned}$$

From here, we obtain for any $t \in]0, T[$,

$$\|u_\varepsilon(t)\|^2_{L^2(Q)} \leq \|u_\varepsilon^0\|^2_{L^2(Q)} + \|f\|^2_{L^2(0,T;L^2(Q))} + \int_0^t \|u_\varepsilon(s)\|^2_{L^2(Q)} \,\mathrm{d}s.$$

Using Gronwall's Lemma, (27) and (29), we conclude that

$$\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(Q))} \leq C.$$

Hence, we have (i). Using the above computations with $t = T$, we deduce further that

$$\alpha \int_0^T \|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon)} \,\mathrm{d}s + \varepsilon^\gamma h_0 \int_0^T \|u_{\varepsilon 1} - u_{\varepsilon 2}\|^2_{\Gamma_\varepsilon} \,\mathrm{d}s \leq C,$$

obtaining (ii) and (iii). To show (iv), we take $v = (v_{\varepsilon 1}, v_{\varepsilon 2}) \in W_0^\varepsilon$ as test function in the variational formulation (28). Using the Hölder inequality, the boundedness of A and h, results (ii) and (iii), in view of (16), Remark 2 and Proposition 2 we have

$$
\begin{aligned}
&|\langle u_\varepsilon', v\rangle_{L^2(0,T;(W_0^\varepsilon)'),L^2(0,T;W_0^\varepsilon)}| \\
&= |\langle u_{\varepsilon 1}', v_{\varepsilon 1}\rangle_{L^2(0,T;(W_{01}^\varepsilon)'),L^2(0,T;W_{01}^\varepsilon)} + \langle u_{\varepsilon 2}', v_{\varepsilon 2}\rangle_{L^2(0,T;(W_{02}^\varepsilon)'),L^2(0,T;W_{02}^\varepsilon)}| \\
&= \Big| \int_0^T \int_Q f v \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \varepsilon^\gamma \int_0^T \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(v_{\varepsilon 1} - v_{\varepsilon 2}) \mathrm{d}\sigma_x \,\mathrm{d}t \Big| \\
&\le \|f\|_{L^2(0,T;L^2(Q))} \|v\|_{L^2(0,T;L^2(Q))} + \beta \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(Q_\varepsilon))} \|\nabla v\|_{L^2(0,T;L^2(Q_\varepsilon))} \\
&\quad + \|h^\varepsilon\|_{L^\infty(\Gamma_\varepsilon)} \varepsilon^{\frac{\gamma}{2}} \|u_{\varepsilon 1} - u_{\varepsilon 2}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} \varepsilon^{\frac{\gamma}{2}} \|v_{\varepsilon 1} - v_{\varepsilon 2}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} \\
&\le C \|v\|_{L^2(0,T;W_0^\varepsilon)} + C \varepsilon^{\frac{\gamma}{2}} \max\left\{1, \varepsilon^{\frac{\kappa-1}{2}}\right\} \|v\|_{L^2(0,T;W_0^\varepsilon)} \\
&= C\left(1 + \varepsilon^{\frac{\gamma}{2}} \max\left\{1, \varepsilon^{\frac{\kappa-1}{2}}\right\}\right) \|v\|_{L^2(0,T;W_0^\varepsilon)}.
\end{aligned}
\tag{33}
$$

This proves (iv). To prove (v), let $v_2 \in H_0^1(Q_2)$. Then, choosing $v = \widetilde{v_2} \in W_0^\varepsilon$ in the previous computation gives

$$
\begin{aligned}
|\langle u_{\varepsilon 2}', v_2\rangle| &= \Big| \int_0^T \int_{Q_2} f v_2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{Q_2} A^\varepsilon \nabla u_\varepsilon \nabla v_2 \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \|f\|_{L^2(0,T;Q_2)} \|v_2\|_{L^2(0,T;Q_2)} + \beta \|\nabla u_\varepsilon\|_{L^2(0,T;Q_2)} \|\nabla v_2\|_{L^2(0,T;Q_2)} \\
&\le C \|v_2\|_{L^2(0,T;H_0^1(Q_2))},
\end{aligned}
$$

since here the boundary term equals zero, which gives the result. Similarly for $v \in H_0^1(Q_1^\delta)$, choosing $v = \widetilde{v_1}$ as test function in (33) gives (31). This ends the proof. □

As shown in [15], a function in W_0^0 which present a jump on Γ_0, can be approximated by functions in W_0^ε which have jumps on Γ_ε. This is important since it allows to use test functions with jumps on the oscillating interface and obtain, when passing to the limit, test functions with jumps on the flat interface. We state this property below as a lemma and rewrite the proof for clarity and convenience.

Lemma 1 [15,16] *Let $\varphi \in W_0^0$. Then, for every ε, there exists $\varphi_\varepsilon \in W_0^\varepsilon$ such that the sequence $\{\varphi_\varepsilon\}$ verifies*

$$
\begin{cases}
(i)\ \ \varphi_\varepsilon \to \varphi, \quad \text{strongly in } L^2(Q) \text{ and in } H^1(Q_1^\delta), \\
(ii)\ \ \chi_{Q_{\varepsilon i}} \nabla \varphi_\varepsilon \rightharpoonup \chi_{Q_i} \nabla \varphi, \quad \text{weakly in } (L_2(Q))^N, \quad i = 1, 2, \\
(iii)\ \ \|\nabla \varphi_\varepsilon\|_{L^2(\Pi_\delta \setminus \Gamma_\varepsilon)} \to \|\nabla \varphi\|_{L^2(\Pi_\delta \setminus \Gamma_0)},
\end{cases}
\tag{34}
$$

for every $\delta > 0$, where Q_1^δ is given by (32) and

$$
\Pi_\delta =: \{x \in Q : 0 \le x_N \le \delta\}. \tag{35}
$$

Proof Let $\varphi \in W_0^0$ be given by

$$
\varphi = (\varphi_1, \varphi_2) = (\psi_{1|Q_1}, \psi_{2|Q_2}),
$$

with ψ_1 and $\psi_2 \in H_0^1(Q)$. Then, the claimed sequence $\{\varphi_\varepsilon\}$ can be obtained by setting for every ε,

$$
\varphi_\varepsilon = (\psi_1|_{Q_{\varepsilon 1}}, \psi_2|_{Q_{\varepsilon 2}}) \in W_0^\varepsilon,
$$

observing that for any $\delta > 0$,

$$
Q_1^\delta \subset Q_{\varepsilon 1}, \quad \text{for} \quad \varepsilon \text{ small enough.}
$$

□

We state the following theorem by Simon [33] that would be useful in proving our compactness result in the succeeding theorem.

Theorem 3 [33] *Let X, B, Y be Banach spaces with $X \subset B \subset Y$ and $X \to Y$ a compact embedding. Suppose F is bounded in $L^p(0,T;X)$ where $1 \leq p < \infty$ and $\frac{\partial F}{\partial t}$ be bounded in $L^1(0,T;Y)$. Then F is relatively compact in $L^p(0,T;B)$.*

Theorem 4 *Let Γ_ε be defined by (5) and suppose that $\{v_\varepsilon\}_\varepsilon$ is a family of functions $v_\varepsilon \in \mathcal{W}^\varepsilon$ such that*

$$\begin{cases} (i)\ \|v_\varepsilon\|_{L^2(0,T;W_0^\varepsilon)} \leq C, \\ (ii)\ \|v'_{\varepsilon 1}\|_{L^2(0,T;H^{-1}(Q_1^\delta))} \leq C, \\ (iii)\ \|v'_{\varepsilon 2}\|_{L^2(0,T;H^{-1}(Q_2))} \leq C, \end{cases} \tag{36}$$

where Q_1^δ is given by (32).

Then, there exists a subsequence (still denoted $\{\varepsilon\}$) and a function $v \in L^2(0,T;W_0^0)$ such that

$$\begin{cases} (i)\ v_\varepsilon \to v, & \text{strongly in } L^2(0,T;L^2(Q)), \\ (ii)\ \chi_{Q_{\varepsilon 1}}\nabla v_\varepsilon \rightharpoonup \chi_{Q_1}\nabla v & \text{weakly in } L^2(0,T;(L^2(Q))^N), \\ (iii)\ \chi_{Q_{\varepsilon 2}}\nabla v_\varepsilon \rightharpoonup \chi_{Q_2}\nabla v & \text{weakly in } L^2(0,T;L^2(Q))^N). \end{cases} \tag{37}$$

Moreover, the following convergence holds:

$$v'_\varepsilon \rightharpoonup v', \text{ weakly in } \mathcal{D}'([0,T]\times\{Q_1 \times Q_2\}) \tag{38}$$

and for any $\varphi = (\varphi_1, \varphi_2)$ in $\mathcal{D}(Q_1)\times\mathcal{D}(Q_2)$ and $\psi \in L^2(0,T)$,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\langle v'_\varepsilon, \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1^\delta)\times H^{-1}(Q_2)),L^2(0,T;H_0^1(Q_1^\delta)\times H_0^1(Q_2))} \\ &\quad = \langle v', \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1)\times H^{-1}(Q_2)),L^2(0,T;H_0^1(Q_1)\times H_0^1(Q_2))}. \end{aligned} \tag{39}$$

Furthermore, let φ be given in W_0^0 and let $\{\varphi_\varepsilon\}$ be the corresponding sequence given by Lemma 1. If

$$\|v'_\varepsilon\|_{L^2(0,T;(W_0^\varepsilon)')} \leq C, \tag{40}$$

for some C independent of ε then $v \in \mathcal{W}^0$ and for every $\psi \in L^2(0,T)$ the following convergence holds:

$$\langle v'_\varepsilon, \psi\varphi_\varepsilon\rangle_{L^2(0,T;(W_0^\varepsilon)'),L^2(0,T;W_0^\varepsilon)} \to \langle v', \psi\varphi\rangle_{L^2(0,T;(W_0^0)'),L^2(0,T;W_0^0)}. \tag{41}$$

Proof For any fixed $\delta > 0$, let Q_1 and Q_1^δ be given by (11) and (32), respectively. We show first that

$$v_\varepsilon \to v, \quad \text{strongly in } L^2(0,T;L^2(Q_1^\delta)) \times L^2(0,T;L^2(Q_2)).$$

Applying Theorem 3 with

$$X = H^1(Q_1^\delta)\times H^1(Q_2), \quad B = L^2(Q_1^\delta)\times L^2(Q_2), \quad Y = H^{-1}(Q_1^\delta)\times H^{-1}(Q_2),$$

and using (15) and (36) (ii)–(iii), we get

$$\{v_\varepsilon\} \text{ relatively compact in } L^2(0,T;L^2(Q_1^\delta)) \times L^2(0,T;L^2(Q_2)), \tag{42}$$

since for ε sufficiently small (depending on δ), one has $Q_1^\delta \subset Q_\varepsilon$.

To prove (37) (i), we use a diagonalization argument for the sequence $\{v_\varepsilon\}$. Let $\{\delta_n\}$ be a positive sequence converging to zero.

Applying (42) for $\delta = \delta_1$, there is a subsequence $\{v_{\varepsilon_{n_1}}^{(\delta_1)}\}$ of $\{v_\varepsilon\}$ (depending on δ_1) which converges to v in $L^2(0,T;L^2(Q_1^{\delta_1}))\times L^2(0,T;L^2(Q_2))$. Similarly, applying (42) for $\delta = \delta_2$, there exists a subsequence of $\{v_{\varepsilon_{n_1}}^{(\delta_1)}\}$, denoted by $\{v_{\varepsilon_{n_k}}^{(\delta_2)}\}$ (depending on δ_2) which converges to v in $L^2(0,T;L^2(Q_1^{\delta_2}))\times L^2(0,T;L^2(Q_2))$.

Proceeding in this manner, for $\delta = \delta_j$, we get a subsequence $\{v_{\varepsilon_{n_k}}^{(\delta_j)}\}$ of $\{v_{\varepsilon_{n_k}}^{(\delta_{j-1})}\}$ (depending on δ_j) which converges to v in $L^2(0,T;L^2(Q_1^{\delta_j}))\times L^2(0,T;L^2(Q_2))$. Note that all of these sequences are subsequences of $\{v_\varepsilon\}$.

Taking the "diagonal sequence" $\{v_{\varphi(j)}\}_{j\in\mathbb{N}}$ defined by $v_{\varphi(j)} = v^{(\delta_j)}_{\varepsilon_{n_j}}$, for every $j \in \mathbb{N}$, which is still a subsequence of $\{v_\varepsilon\}$), we have

$$v_{\varphi(j)} \to v \text{ strongly in } L^2(0,T;L^2(Q_1^\delta)) \times L^2(0,T;L^2(Q_2)), \quad \text{for all} \quad \delta > 0. \tag{43}$$

What we want to prove is that

$$v_{\varphi(j)} \to v \qquad \text{strongly in } L^2(0,T;L^2(Q)).$$

It remains to show that $v_{\varphi(j)} \to v$ strongly in $L^2(0,T;L^2(Q_1))$, that is,

$$\int_0^T \int_{Q_1} (v_{\varphi(j)} - v)^2 \,\mathrm{d}x\,\mathrm{d}t \to 0.$$

By definition, for $\eta > 0$, we need to find a j_η such that if $j > j_\eta$, then

$$\int_0^T \int_{Q_1} (v_{\varphi(j)} - v)^2 \,\mathrm{d}x\,\mathrm{d}t < \eta. \tag{44}$$

Now, for $\delta_0 > 0$ let us decompose the integral above as

$$\begin{aligned}\int_0^T \int_{Q_1} (v_{\varphi(j)} - v)^2 \,\mathrm{d}x\,\mathrm{d}t = &\int_0^T \int_{Q_1^{\delta_0}} (v_{\varphi(j)} - v)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_{Q_1 \setminus Q_1^{\delta_0}} (v_{\varphi(j)} - v)^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{45}$$

Let us show that there exists a δ_0 such that the second term of the right-hand side of (45) becomes smaller than $\frac{\eta}{2}$. To do that, let us write for any δ,

$$\int_0^T \int_{Q_1 \setminus Q_1^\delta} (v_{\varphi(j)} - v)^2 \,\mathrm{d}x\,\mathrm{d}t \le 2\int_0^T \int_{Q_1 \setminus Q_1^\delta} \left(v^2_{\varphi(j)} + v^2\right) \mathrm{d}x\,\mathrm{d}t. \tag{46}$$

Observe first that there exists a δ^* such that

$$\int_0^T \int_{Q_1 \setminus Q_1^{\delta^*}} v^2 \,\mathrm{d}x\,\mathrm{d}t < \frac{\eta}{8}, \quad \text{for all} \quad \delta < \delta^*. \tag{47}$$

Now, let us consider

$$\int_0^T \int_{Q_1 \setminus Q_1^\delta} v^2_{\varphi(j)} \,\mathrm{d}x\,\mathrm{d}t.$$

We adapt to our case the same process used to prove (2.25) in [16], but integrating in time and with δ instead of $\varepsilon^\kappa \bar{g}$. Therefore, if $e_n = (0,\ldots,0,1)$ then

$$v_{\varphi(j)}(x + \delta e_n) - v_{\varphi(j)}(x) = \int_x^{x+\delta e_n} \frac{\partial v_{\varphi(j)}}{\partial x_n} \,\mathrm{d}x_n, \quad \text{a.e.} \quad x \in Q_1 \setminus Q_1^\delta.$$

It follows that

$$\int_0^T \int_{Q_1 \setminus Q_1^\delta} v^2_{\varphi(j)}(x) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{Q_1 \setminus Q_1^\delta} \left(v_{\varphi(j)}(x + \delta e_n) - \int_{x_n}^{x_n+\delta e_n} \frac{\partial v_{\varphi(j)}}{\partial s} \,\mathrm{d}s \right)^2 \mathrm{d}x\,\mathrm{d}t.$$

Following the straightforward computations in [16], using notation (1) we get

$$\int_0^T \int_{Q_1 \setminus Q_1^\delta} v^2_{\varphi(j)} \,\mathrm{d}x\,\mathrm{d}t \le 2\int_0^T \left(\int_{\omega \times]\delta, 2\delta[} v^2_{\varphi(j)} \,\mathrm{d}x + \delta^2 \|v_{\varphi(j)}\|^2_{W_0^\varepsilon} \right) \mathrm{d}t. \tag{48}$$

As shown in [16], by the Sobolev embedding theorem and (36) (i) if $N > 2$ and $2^* = \dfrac{2N}{N-2}$ we have

$$\begin{aligned}\int_0^T \int_{\omega\times]\delta,2\delta[} v^2_{\varphi(j)}(x)\, \mathrm{d}x\, \mathrm{d}t &\le \|v_{\varphi(j)}\|^{\frac{(N-2)}{N}}_{L^2(0,T;L^{2^*}(\omega\times]\delta,2\delta[))} \cdot \mathrm{meas}(\omega\times]\delta,2\delta[)^{\frac{2}{N}}\\ &\le C\|v_{\varphi(j)}\|^{\frac{(N-2)}{N}}_{L^2(0,T;L^{2^*}(\omega\times]\delta,\ell[)} \cdot (\delta)^{\frac{2}{N}}\\ &\le C(\delta)^{\frac{2}{N}},\end{aligned} \tag{49}$$

for every j, where we used the fact that the embedding operator from $H^1(\omega\times]\delta,\ell[)$ into $L^{2^*}(\omega\times]\delta,\ell[)$ is uniformly bounded with respect to δ.

If $N = 2$, the embedding $H^1(\omega\times]\delta,\ell[) \subset L^p(\omega\times]\delta,\ell[)$ is continuous for any $p < \infty$, so that we can use in the computation above any $p > 2$ instead of 2^*, and then $\delta^{\frac{2}{N}}$ is replaced by $\delta^{\frac{p-2}{p}}$ in (49).

From (49) and (48), together with assumption (36) (i), it follows that there exists δ^{**} so that

$$\int_0^T \int_{Q_1\setminus Q_1^\delta} v^2_{\varphi(j)}\, \mathrm{d}x\, \mathrm{d}t < \frac{\eta}{8}, \text{ for all } \delta < \delta^{**}. \tag{50}$$

By choosing $\delta_0 = \min\{\delta^*, \delta^{**}\}$, from (46), (47) and (50), we have that

$$\int_0^T \int_{Q_1\setminus Q_1^{\delta_0}} (v_{\varphi(j)} - v)^2\, \mathrm{d}x\, \mathrm{d}t < \frac{\eta}{2}. \tag{51}$$

On the other hand, for the above δ_0, by using (43), there exists a j_0 (depending on η and δ_0) such that

$$\int_0^T \int_{Q_1^{\delta_0}} (v_{\varphi(j)} - v)^2\, \mathrm{d}x\, \mathrm{d}t < \frac{\eta}{2}, \text{ for all } \quad j > j_0. \tag{52}$$

From (51) and (52) applied in (45), we obtain (44).

Following the same arguments in [16] adapted to our time-dependent case and using (36) (i) together with convergence (37) (i), one can show (37) (ii) and (iii).

To prove (38), first notice that from (36) (ii) and (iii) we have that

$$v'_\varepsilon = (v'_{\varepsilon 1}, v'_{\varepsilon 2}) \in \mathcal{D}'([0,T] \times \{Q_1 \times Q_2\})$$

Next, take $\varphi = (\varphi_1, \varphi_2)$ in $\mathcal{D}(Q_1) \times \mathcal{D}(Q_2)$. Observe that for ε sufficiently small there exists $\delta_\varphi > 0$ such that supp $\varphi_1 \subset Q_1^\delta$ so that $\varphi_1 \in H_0^1(Q_1^\delta)$. From (37) (i), for every $\psi \in \mathcal{D}(0,T)$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0} \langle v'_\varepsilon, \psi\varphi\rangle_{\mathcal{D}'([0,T]\times\{Q_1\times Q_2\}),\mathcal{D}([0,T]\times\{Q_1\times Q_2\})}\\ &\quad= \lim_{\varepsilon\to 0} \langle v'_\varepsilon, \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1^\delta)\times H^{-1}(Q_2)),L^2(0,T;H_0^1(Q_1^\delta)\times H_0^1(Q_2))}\\ &\quad= -\lim_{\varepsilon\to 0} \int_0^T \int_Q \psi' v_\varepsilon \varphi\, \mathrm{d}x\, \mathrm{d}t\\ &\quad= -\int_0^T \int_Q \psi' v\varphi\, \mathrm{d}x\, \mathrm{d}t\\ &\quad= \langle v', \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1)\times H^{-1}(Q_2)),L^2(0,T;H_0^1(Q_1)\times H_0^1(Q_2))}\\ &\quad= \langle v', \psi\varphi\rangle_{\mathcal{D}'([0,T]\times\{Q_1\times Q_2\}),\mathcal{D}([0,T]\times\{Q_1\times Q_2\})}\end{aligned}$$

which proves (38) and by density, we get (39).

To prove (41), under assumption (40), let φ be given in W_0^0 and $\{\varphi_\varepsilon\}$ be given by Lemma 1. From (34) (i) and (37) (i), for every $\psi \in \mathcal{D}(0,T)$, we have

$$
\begin{aligned}
&-\int_0^T \int_Q \psi' v\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\lim_{\varepsilon\to 0}\int_0^T \int_Q \psi' v_\varepsilon\varphi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&= -\lim_{\varepsilon\to 0}\int_0^T \psi' \langle v_\varepsilon, \varphi_\varepsilon\rangle_{(W_0^\varepsilon)', W_0^\varepsilon}\,\mathrm{d}t \\
&= \lim_{\varepsilon\to 0}\langle v'_\varepsilon, \psi\varphi_\varepsilon\rangle_{L^2(0,T;(W_0^\varepsilon)'),L^2(0,T;W_0^\varepsilon)} \\
&\le \lim_{\varepsilon\to 0}\|v'_\varepsilon\|_{L^2(0,T;(W_0^\varepsilon)')}\|\psi\varphi_\varepsilon\|_{L^2(0,T;W_0^\varepsilon)}.
\end{aligned}
$$

Hence, (34) (iii) and our boundedness assumption on $\{v'_\varepsilon\}$ imply that

$$
\begin{aligned}
|\langle v', \psi\varphi\rangle_{L^2(0,T;(W_0^0)'),L^2(0,T;W_0^0)}| &= \left| -\int_0^T \int_Q \psi' v\varphi \,\mathrm{d}x\,\mathrm{d}t\right| \\
\le C \lim_{\varepsilon\to 0}\|\psi\varphi_\varepsilon\|_{L^2(0,T;W_0^\varepsilon)} &= C\|\psi\varphi\|_{L^2(0,T;W_0^0)},
\end{aligned}
$$

for every φ in W_0^0 and $\psi \in \mathcal{D}(0,T)$. By a density argument, this is still true for $\psi \in L^2((0,T))$, which proves that v' belongs to $L^2(0,T;(W_0^0)')$ and convergence (41) holds true. □

The following is immediate from the preceding result and Theorem 2.

Corollary 1 *Under the assumptions of Theorem 2, there exists a subsequence (still denoted u_ε) and a function $u \in L^2(0,T;W_0^0)$ such that*

$$
\begin{cases}
(i)\ u_\varepsilon \to u, & \text{strongly in } L^2(0,T;L^2(Q)),\\
(ii)\ \chi_{Q_{\varepsilon 1}}\nabla u_\varepsilon \rightharpoonup \chi_{Q_1}\nabla u & \text{weakly in } L^2(0,T;(L^2(Q))^N),\\
(iii)\ \chi_{Q_{\varepsilon 2}}\nabla u_\varepsilon \rightharpoonup \chi_{Q_2}\nabla u & \text{weakly in } L^2(0,T;(L^2(Q))^N).
\end{cases}
\tag{53}
$$

Moreover, the following convergence holds:

$$
u'_\varepsilon \rightharpoonup u', \ \text{weakly in } \mathcal{D}'([0,T]\times\{Q_1\times Q_2\})
\tag{54}
$$

and for any $\varphi = (\varphi_1,\varphi_2)$ in $\mathcal{D}(Q_1)\times\mathcal{D}(Q_2)$ and $\psi \in L^2(0,T)$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\langle u'_\varepsilon, \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1^\delta)\times H^{-1}(Q_2)),L^2(0,T;H_0^1(Q_1^\delta)\times H_0^1(Q_2))}\\
&\quad = \langle u', \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1)\times H^{-1}(Q_2)),L^2(0,T;H_0^1(Q_1)\times H_0^1(Q_2))}.
\end{aligned}
\tag{55}
$$

Furthermore, if φ is given in W_0^0 and $\{\varphi_\varepsilon\}$ is the corresponding sequence given by Lemma 1 and if

$$
\gamma + \kappa - 1 \ge 0,
\tag{56}
$$

then for every $\psi \in L^2(0,T)$, one has $u \in \mathcal{W}^0$ and

$$
\langle u'_\varepsilon, \psi\varphi_\varepsilon\rangle_{L^2(0,T;(W_0^\varepsilon)'),L^2(0,T;W_0^\varepsilon)} \to \langle u', \psi\varphi\rangle_{L^2(0,T;(W_0^0)'),L^2(0,T;W_0^0)}.
\tag{57}
$$

Remark 5 In the case $\gamma+\kappa-1 < 0$, we are unable to prove that $\|u_\varepsilon\|_{\mathcal{W}^\varepsilon} < C$ and thus cannot prove the limit result (57). Nevertheless, the above compactness result (53), (54) and (55) still holds true for our solution u_ε.

5 Homogenization results

In this section, we describe the limit behavior of problem (28) as $\varepsilon \to 0$. To do so, first we recall the homogenized tensor A^0 (see [3]) defined by

$$A^0\lambda = m_Y(A\nabla w_\lambda) \tag{58}$$

with $w_\lambda \in H^1(Y)$ the unique solution, for any $\lambda \in \mathbb{R}^N$, of

$$\begin{cases} -\text{div}\ (A\nabla w_\lambda) = 0 & \text{in } Y, \\ w_\lambda - \lambda \cdot y & Y\text{-periodic}, \\ m_Y(w - \lambda \cdot y) = 0. \end{cases} \tag{59}$$

A crucial step in our aim of homogenizing (28) is to deal with the term in the boundary Γ^ε. We will separate the results according to the value of κ and γ as follow:

$$\begin{cases} \text{(i) } (\kappa \geq 1 \text{ and } \gamma = 0) \quad \text{or} \quad (0 < \kappa < 1 \text{ and } \gamma = 1 - \kappa); \\ \text{(ii) } (\kappa \geq 1 \text{ and } \gamma < 0) \quad \text{or} \quad (0 < \kappa < 1 \text{ and } \gamma < 1 - \kappa); \\ \text{(iii) } (\kappa \geq 1 \text{ and } \gamma > 0) \quad \text{or} \quad (0 < \kappa < 1 \text{ and } \gamma > 1 - \kappa). \end{cases} \tag{60}$$

Our main result of this section is given in the following theorem:

Theorem 5 *Under assumptions (7)–(9) and (27) let u^ε be the solution of problem (28) and A^0 be given by (58) and (59). Also, suppose that the initial condition u^0_ε satisfies:*

$$u^0_\varepsilon \rightharpoonup u^0 \qquad \textit{weakly in} \quad L^2(Q),\ i = 1, 2. \tag{61}$$

For every $\gamma \in \mathbb{R}$, there exists a function $u \in \mathcal{W}_0$ such that the following convergences hold true:

$$\begin{cases} (i)\ \ u_\varepsilon \to u, & \textit{strongly in } L^2(0, T; L^2(Q)), \\ (ii)\ \ \chi_{Q_{\varepsilon i}} \nabla u_\varepsilon \rightharpoonup \chi_{Q_i} \nabla u, & \textit{weakly in } L^2(0, T; (L^2(Q))^N), \end{cases} \tag{62}$$

and

$$\chi_{Q_{\varepsilon i}} A^\varepsilon \nabla u_\varepsilon \rightharpoonup \chi_{Q_i} A^0\ \nabla u, \qquad \textit{weakly in } L^2(0, T; (L^2(Q))^N), \tag{63}$$

for $i = 1, 2$. Moreover, in the following, we identify the limit u.

- *Suppose that (60) (i) holds. Then, the function u is the unique solution of the problem*

$$\begin{cases} u' - \text{div}(A^0\nabla u) = f & \textit{in} \quad Q_0\times]0, T[, \\ (A^0\nabla u)_2 \cdot n = (A^0\nabla u)_1 \cdot n & \textit{on} \quad \Gamma_0\times]0, T[, \\ (A^0\nabla u)_1 \cdot n = H(g, h)(u_1 - u_2) & \textit{on} \quad \Gamma_0\times]0, T[, \\ u = 0 & \textit{on} \quad \partial Q, \\ u(0) = u^0 & \textit{in} \quad Q, \end{cases} \tag{64}$$

  where $H(g, h)$ is given by

$$H(g, h) = \begin{cases} m_{Y'}\Big(h(1 + (|\nabla g|_2)^{1/2}\Big) & \textit{if} \quad \kappa = 1 \ \textit{and}\ \gamma = 0, \\ m_{Y'}(h) & \textit{if} \quad \kappa > 1 \ \textit{and}\ \gamma = 0, \\ m_{Y'}(h|\nabla g|) & \textit{if} \quad 0 < \kappa < 1 \ \textit{and}\ \gamma = 1 - \kappa, \end{cases} \tag{65}$$

- *Suppose now that (60) (ii) holds. Then, the function u belongs to $L^2(0, T; H^1_0(Q))$ with $u' \in L^2(0, T; H^{-1}(Q))$ and is the unique solution of the problem*

$$\begin{cases} u' - \text{div}(A^0\nabla u) = f & \textit{in} \quad Q\times]0, T[, \\ u = 0 & \textit{on} \quad \partial Q, \\ u(0) = u^0 & \textit{in} \quad Q. \end{cases} \tag{66}$$

– *Finally, suppose that (60) (iii) holds. Then, u_1 and u_2 are the unique solution of the following two (independent) Neumann problems:*

$$\begin{cases} u_1' - \operatorname{div}(A^0\nabla u_1) = f \quad in \quad Q_1\times]0,T[, \\ A^0\nabla u_1\cdot n = 0 \quad on \quad \Gamma_0\times]0,T[, \\ u_1 = 0 \qquad on \quad \partial Q_1\setminus\Gamma_0, \\ u(0) = u^0 \qquad in \quad Q_1, \end{cases} \tag{67}$$

$$\begin{cases} u_2' - \operatorname{div}(A^0\nabla u_2) = f \quad in \quad Q_2\times]0,T[, \\ A^0\nabla u_2\cdot n = 0 \quad on \quad \Gamma_0\times]0,T[, \\ u_2 = 0 \qquad on \quad \partial Q_2\setminus\Gamma_0, \\ u(0) = u^0 \qquad in \quad Q_2. \end{cases} \tag{68}$$

Before giving the proof of this theorem, we prove the following result:

Proposition 3 *Under the assumptions of Theorem 2, let u_ε be the solution of problem (28) and consider the (sub)-sequence given by Corollary 1 which verifies convergences (53). If φ is given in W_0^0 and $\{\varphi_\varepsilon\}$ is the corresponding sequence given by Lemma 1, then for every $\psi\in\mathcal{D}(0,T)$ we have*

$$\lim_{\varepsilon\to 0}\int_0^T\int_{Q_\varepsilon} A^\varepsilon\nabla u_\varepsilon\psi\nabla\varphi_\varepsilon\,dx\,dt = \int_0^T\int_{Q_0} A^0\nabla u\psi\nabla\varphi\,dx\,dt. \tag{69}$$

Moreover,

$$\begin{cases} (i)\ \chi_{Q_{\varepsilon 1}}A^\varepsilon\nabla u_\varepsilon \rightharpoonup \chi_{Q_1}A^0\nabla u \quad weakly\ in\ L^2(0,T;(L^2(Q))^N), \\ (ii)\ \chi_{Q_{\varepsilon 2}}A^\varepsilon\nabla u_\varepsilon \rightharpoonup \chi_{Q_2}A^0\nabla u \quad weakly\ in\ L^2(0,T;(L^2(Q))^N). \end{cases} \tag{70}$$

where A^0 is given by (58).

Proof For fixed $\delta > 0$, let Q_1^δ and Π_δ be defined by (32) and (35), respectively. This implies that for ε sufficiently small, we have (up a subsequence)

$$A^\varepsilon\nabla u_\varepsilon \rightharpoonup \xi_\delta \quad \text{weakly in} \quad L^2(0,T;L^2(Q_1^\delta)). \tag{71}$$

Then, the classical homogenization methods used with test functions in $\mathcal{D}(Q_1^\delta)$ give the following convergence of the flux, as $\varepsilon\to 0$,

$$A^\varepsilon\nabla u_\varepsilon \rightharpoonup A^0\nabla u \quad \text{weakly in} \quad L^2(0,T;L^2(Q_1^\delta)). \tag{72}$$

Hence, for any $\varphi\in W_0^0$ if $\{\varphi_\varepsilon\}$ is the corresponding sequence given by Lemma 1 we have

$$\int_0^T\int_{Q_1^\delta} A^\varepsilon\nabla u_\varepsilon\psi\nabla\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_{Q_1^\delta} A^0\nabla u\,\psi\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t, \tag{73}$$

as $\varepsilon\to 0$. Now, on the region Π_δ, from (7), Theorem 2 and convergence (34) (iii) from Lemma 1, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left|\int_0^T\int_{\Pi_\delta\setminus\Gamma_\varepsilon} A^\varepsilon\nabla u_\varepsilon\psi\nabla\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le \lim_{\varepsilon\to 0}\|A^\varepsilon\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Pi_\delta\setminus\Gamma_\varepsilon))}\|\psi\nabla\varphi_\varepsilon\|_{L^2(0,T;L^2(\Pi_\delta\setminus\Gamma_\varepsilon))} \\ &\le \lim_{\varepsilon\to 0}\beta C\|u_\varepsilon\|_{L^2(0,T;W_0^\varepsilon)}\|\nabla\varphi_\varepsilon\|_{L^2(\Pi_\delta\setminus\Gamma_\varepsilon)} \\ &\le C\lim_{\varepsilon\to 0}\|\nabla\varphi_\varepsilon\|_{L^2(\Pi_\delta\setminus\Gamma_\varepsilon)} \\ &= C\|\nabla\varphi\|_{L^2(\Pi_\delta\setminus\Gamma_0)}. \end{aligned}$$

Since the right-hand side of this inequality goes to zero as $\delta\to 0$, this together with (73) implies that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{Q_{\varepsilon 1}} A^\varepsilon\nabla u_\varepsilon\psi\nabla\varphi_{\varepsilon 1}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{Q_1} A^0\nabla u\nabla\varphi_1\,\mathrm{d}x\,\mathrm{d}t. \tag{74}$$

In a similar manner using Q_2 instead of Q_1^δ in (71)–(73) we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_{Q_{\varepsilon 2}} A^\varepsilon \nabla u_\varepsilon \psi \nabla \varphi_{\varepsilon 2}\, \mathrm{d}x\, \mathrm{d}t = \int_0^T\int_{Q_2} A^0 \nabla u \nabla \psi \varphi_2\, \mathrm{d}x\, \mathrm{d}t.$$

This together with (74) gives (69).

Now, the sequence $\{\chi_{Q_{\varepsilon 1}} A^\varepsilon \nabla u_\varepsilon\}$ is weakly compact in $L^2(0,T;L^2(Q))$ because of (7) and (30) (iii). Using (72) and the arguments to show (37) (ii), convergence (70) (i) is proved. In a similar manner, one can show (70) (ii). ◻

We present below a technical result which concerns passing to the limit in the boundary term and it will prove very useful in what follows. The result was proved for the static case in [15] (see also [16]) and the time-domain version we present here follows immediately by integrating in time.

Proposition 4 [15] *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \in \mathcal{W}^\varepsilon$ for every ε and*

$$\|w_\varepsilon\|_{L^2(0,T;W_0^\varepsilon)} \le c, \quad \|w_{\varepsilon 1} - w_{\varepsilon 2}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} \le c\, \varepsilon^{-\frac{\gamma}{2}}, \tag{75}$$

where c is a constant independent on ε. Suppose that for some $w \in L^2(0,T;W_0^0)$ one has

$$\begin{cases} (i)\ \ w_\varepsilon \to w, \quad \text{strongly in} \quad L^2(0,T;L^2(Q)),\\ (ii)\ \ \chi_{Q_{\varepsilon i}}\nabla w_\varepsilon \rightharpoonup \chi_{Q_i}\nabla w, \quad \text{weakly in} \quad L^2(0,T;(L_2(Q))^N).\end{cases}$$

– *If* (60) *(ii) holds, then*

$$w \text{ belong to } L^2(0,T;H_0^1(Q)).$$

Suppose now that $\{\psi_\varepsilon\}$ is another sequence verifying the same estimates (75) *such that for some $\psi \in L^2(0,T;W_0^0)$*

$$\begin{cases} (\ i)\ \ \psi_\varepsilon \to \psi, \quad \text{strongly in} \quad L^2(0,T;L^2(Q)),\\ (ii)\ \ \chi_{Q_{\varepsilon i}}\nabla \psi_\varepsilon \rightharpoonup \chi_{Q_i}\nabla \psi, \quad \text{weakly in} \quad L^2(0,T;(L_2(Q))^N).\end{cases}$$

– *If* (60) *(i) holds, then*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^\gamma \int_0^t\int_{\Gamma_\varepsilon} h^\varepsilon (w_{\varepsilon 1}-w_{\varepsilon 2})(\psi_{\varepsilon 1}-\psi_{\varepsilon 2})\, d\sigma\, ds\\ &\quad = H(g,h)\int_0^t\int_{\Gamma_0}(w_1-w_2)(\psi_1-\psi_2)\, d\sigma\, ds,\end{aligned} \tag{76}$$

for every $t\in[0,T]$, where $H(g,h)$ is given by (65).

– *If* (60) *(iii) holds, then*

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma \int_0^t\int_{\Gamma_\varepsilon} h^\varepsilon (w_{\varepsilon 1}-w_{\varepsilon 2})(\psi_{\varepsilon 1}-\psi_{\varepsilon 2})\, d\sigma\, ds = 0, \tag{77}$$

for every $t\in[0,T]$.

Proof of Theorem 5 Convergences (62) and (63) follow, for a subsequence, from (53) and (70), respectively. We need to identify the limit u. To this aim, we let $\psi \in \mathcal{D}(0,T)$ and φ in W_0^0, denoting by $\{\varphi_\varepsilon\}$ the corresponding sequence given by Lemma 1.

In the variational formulation (28), take $(\varphi_{\varepsilon 1}\psi, \varphi_{\varepsilon 2}\psi)$ as test function so that

$$\begin{aligned}&\langle u'_{\varepsilon 1}, \varphi_{\varepsilon 1}\psi\rangle_{(W_{01}^\varepsilon)',W_{01}^\varepsilon} + \langle u'_{\varepsilon 2}, \varphi_{\varepsilon 2}\psi\rangle_{(W_{02}^\varepsilon)',W_{02}^\varepsilon} + \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla\varphi_\varepsilon \psi\, \mathrm{d}x\\ &\quad + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1}-u_{\varepsilon 2})(\varphi_{\varepsilon 1}-\varphi_{\varepsilon 2})\psi\, \mathrm{d}\sigma = \int_Q f\varphi_\varepsilon \psi\, \mathrm{d}x.\end{aligned} \tag{78}$$

Integrating both sides with respect to t and by Remark 3, we get

$$-\int_0^T \int_{Q_\varepsilon} u_\varepsilon \varphi_\varepsilon \psi' \, dx \, dt + \int_0^T \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \varphi_\varepsilon \psi \, dx \, dt$$
$$+ \varepsilon^\gamma \int_0^T \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(\varphi_{\varepsilon 1} - \varphi_{\varepsilon 2}) \psi \, d\sigma = \int_0^T \int_Q f \varphi_\varepsilon \psi \, dx. \tag{79}$$

Using (34) (i) and (53) (i),

$$\lim_{\varepsilon \to} \int_0^T \int_{Q_\varepsilon} u_\varepsilon \varphi_\varepsilon \psi' \, dx \, dt = \int_0^T \int_Q u \varphi \psi' \, dx \, dt. \tag{80}$$

On the other hand, by Proposition 3,

$$\lim_{\varepsilon \to 0} \int_0^T \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \varphi_\varepsilon \psi \, dx \, dt = \int_0^T \int_{Q_0} A^0 \nabla u \nabla \varphi \psi \, dx \, dt. \tag{81}$$

For the limit involving the boundary term, that is, the third term on the left-hand side of (79), we distinguish the values of γ and κ according to (60).

First, suppose that (60) (i) holds. Then by Proposition 4, we have (76), that is,

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma \int_0^T \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(\varphi_{\varepsilon 1} - \varphi_{\varepsilon 2}) \psi \, d\sigma \, dt$$
$$= H(g, h) \int_0^T \int_{\Gamma_0} (u_1 - u_2)(\varphi_1 - \varphi_2) \psi \, d\sigma \, dt, \tag{82}$$

where $H(g, h)$ is given by (65).

Hence, letting $\varepsilon \to 0$ in (79) and combining (80)–(82) together with (34) (i), we have

$$-\int_0^T \int_Q u \varphi \psi' \, dx \, dt + \int_0^T \int_{Q_0} A^0 \nabla u \nabla \varphi \psi \, dx \, dt$$
$$+ H(g, h) \int_0^T \int_{\Gamma_0} (u_1 - u_2)(\varphi_1 - \varphi_2) \psi \, d\sigma \, dt = \int_0^T \int_Q f \varphi \psi \, dx \, dt.$$

Next, assume that (60) (ii) is true. By Proposition 4, u belongs to $L^2(0, T; H_0^1(Q))$. Let $\psi \in \mathcal{D}(0, T)$ and $\varphi \in \mathcal{D}(Q)$. Choosing $\varphi\psi$ as test function in the variational formulation (28), as $\varepsilon \to 0$, no boundary terms appear and we deduce that

$$u' = \operatorname{div}(A^0 \nabla u) + f \in L^2(0, T; H^{-1}(Q)).$$

Thus, u belongs to $L^2(0, T; H_0^1(Q))$ with $u' \in L^2(0, T; H^{-1}(Q))$ and is solution of the equation in problem (66).

Finally, if (60) (iii) is satisfied then by Proposition 4,

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma \int_0^T \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(\varphi_{\varepsilon 1} - \varphi_{\varepsilon 2}) \psi \, d\sigma \, dt = 0.$$

Arguing as above, this implies together with (80) and (81) that $u_i, i = 1, 2$ is a solution of the Neumann problem (66)

$$\begin{cases} u_i' - \operatorname{div}(A^0 \nabla u_1) = f & \text{in} \quad Q_i \times]0, T[, \\ A^0 \nabla u_i \cdot n = 0 & \text{on} \quad \Gamma_0 \times]0, T[, \\ u = 0 & \text{on} \quad \partial Q_i \setminus \Gamma_0. \end{cases}$$

It remains to check that in the three cases, u satisfies the initial condition. Let $\varphi = (\varphi_1, \varphi_2)$ in $\mathcal{D}(Q_1) \times \mathcal{D}(Q_2)$ and $\psi \in \mathcal{C}^\infty([0,T])$ with $\psi(T) = 0$ and $\psi(0) = 1$. For ε small enough and using $(\varphi_1\psi, \varphi_2\psi)$ as test function in (28), we have

$$\int_0^T \langle u'_{\varepsilon 1}, \varphi_1\psi\rangle_{(W^\varepsilon_{01})', W^\varepsilon_{01}}\,\mathrm{d}t + \int_0^T \langle u'_{\varepsilon 2}, \varphi_2\psi\rangle_{(W^\varepsilon_{02})', W^\varepsilon_{02}}\,\mathrm{d}t$$
$$+ \int_0^T \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_Q f\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{83}$$

Since $\psi(0) = 1$, using the initial condition in problem (28), we get

$$\begin{cases} \displaystyle\int_0^T \langle u'_{\varepsilon 1}, \varphi_1\psi\rangle_{(W^\varepsilon_{01})', W^\varepsilon_{01}}\,\mathrm{d}t = -\int_{Q_{\varepsilon 1}} u^0_{\varepsilon 1}\varphi_1\,dx - \int_0^T \int_{Q_{\varepsilon 1}} u_{\varepsilon 1}\varphi_1\psi'\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle\int_0^T \langle u'_{\varepsilon 2}, \varphi_2\psi\rangle_{(W^\varepsilon_{02})', W^\varepsilon_{02}}\,\mathrm{d}t = -\int_{Q_{\varepsilon 2}} u^0_{\varepsilon 2}\varphi_2\,\mathrm{d}x - \int_0^T \int_{Q_{\varepsilon 2}} u_{\varepsilon 2}\varphi_2\psi'\,\mathrm{d}x\,\mathrm{d}t \end{cases}$$

Substituting these identities in (83) gives

$$-\int_Q u^0_\varepsilon\varphi\,\mathrm{d}x - \int_0^T \int_{Q_{\varepsilon 1}} u_\varepsilon\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_Q f\varphi\psi\,\mathrm{d}x\,\mathrm{d}t.$$

In view of (61), (62) (i) and Proposition 3 we can pass to the limit as $\varepsilon \to 0$ in this identity to obtain

$$-\int_Q u^0\varphi\,\mathrm{d}x - \int_0^T \int_Q u\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{Q_0} A^0 \nabla u \nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_Q f\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{84}$$

On the other hand, using again $\varphi\psi$ as test function in (78) with $\psi(0) = 1$ and $\psi(T) = 0$ and passing now to the limit in the duality pairing, thanks to (16) and (54), (55), after integrating with respect to t we obtain

$$\langle u', \psi\varphi\rangle_{L^2(0,T;H^{-1}(Q_1)\times H^{-1}(Q_2)), L^2(0,T;H^1_0(Q_1)\times H^1_0(Q_2))}$$
$$+ \int_0^T \int_{Q_0} A^0 \nabla u \nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_Q f\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{85}$$

Integrating by parts in (85) we have

$$-\int_Q u(0)\varphi\,\mathrm{d}x - \int_0^T \int_Q u\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{Q_0} A^0 \nabla u \nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_Q f\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{86}$$

From (84) and (86), we conclude that

$$\int_Q (u(0) - u^0)\varphi\,\mathrm{d}x = 0$$

for every $\varphi_1 = (\varphi_1, \varphi_2)$ in $\mathcal{D}(Q_1) \times \mathcal{D}(Q_2)$, which implies that

$$u(0) = u^0. \tag{87}$$

To conclude the proof, observe that limit problems (64), (66), (67) and (68) have unique solutions since A^0 is positive definite. Hence, all the convergences involved for the three cases hold for the whole sequences. □

6 Corrector results

We complete here the convergences for the sequence of solutions $\{u_\varepsilon\}$ of problem (28) proved in Sect. 5. The following proposition provides the main tool for the corrector analysis, and it will be proved at the end of this section.

Let us first introduce $C^\varepsilon = (C^\varepsilon_{ij})_{1\le i,j\le N}$, the classical corrector matrix (see for instance [3], [11]), given by

$$\begin{cases} C^\varepsilon_{ij}(x) = C_{ij}\left(\dfrac{x}{\varepsilon}\right), & \text{a.e. on } Q \\ C_{ij}(y) = \dfrac{\partial w_j}{\partial y_i}(y), \quad i,j = 1,\dots,N & \text{a.e. on } Y. \end{cases} \tag{88}$$

where $\{e_j\}_{j=1}^N$ denotes the canonical basis of $\mathbb{R}^N$ and w_j is the solution of problem (59), written for $\lambda = e_j$.

Proposition 5 *Assume the same hypothesis as in Theorem 5. Let $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_i \in \mathcal{C}^\infty(0,T;\mathcal{D}(Q_i))$ and let $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_i = (\Phi_{i1}, \Phi_{i2}, \dots, \Phi_{iN}) \in C^\infty(0,T;(D(Q_i))^N)$ for $i = 1,2$. Let F_ε, F_0 be defined by*

$$F_\varepsilon(t) = \frac{1}{2}\|u_\varepsilon(t) - \varphi(t)\|^2_{L^2(Q_\varepsilon)} + \int_0^t \int_{Q_\varepsilon} A^\varepsilon(\nabla u_\varepsilon - C^\varepsilon\Phi)(\nabla u_\varepsilon - C^\varepsilon\Phi)dxds$$
$$+ \varepsilon^\gamma \int_0^t \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})^2 d\sigma\, ds \tag{89}$$

and

$$F_0(t) = \frac{1}{2}\|u(t) - \varphi(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_0} A^0(\nabla u - \Phi)(\nabla u - \Phi)\, dx\, ds$$
$$+ \int_0^t \int_{\Gamma_0} B(u_1 - u_2)^2 d\sigma\, ds$$

for $t \in [0,T]$, with

$$B = \begin{cases} H(g,h), & \text{if (60) } (i) \\ 0, & \text{if (60) } (ii) \text{ or (60) } (iii) \end{cases} \tag{90}$$

and where $H(g,h)$ is defined at (65). Then, if

$$u^0_\varepsilon \to u^0 \text{ strongly in } L^2(Q), \tag{91}$$

we have

$$\lim_{\varepsilon\to 0} ||F_\varepsilon - F_0||_{C^0[0,T]} = 0. \tag{92}$$

Remark 6 Assuming the same hypothesis as in Proposition 5 and consider E_ε, E_0 defined by,

$$E_\varepsilon(t) = \frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \mathrm{d}x\, \mathrm{d}s + \varepsilon^\gamma \int_0^t \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})^2 \mathrm{d}\sigma\, \mathrm{d}s$$
$$E_0(t) = \frac{1}{2}\|u(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_0} A^0 \nabla u \nabla u\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{\Gamma_0} B(u_1 - u_2)^2 \mathrm{d}\sigma\, \mathrm{d}s$$

with B defined as in (90). Then, Proposition 5 for $\varphi \equiv 0$ and $\Phi \equiv 0$ in $\mathbb{R}^N$ implies the following convergence of the energies:

$$\lim_{\varepsilon\to 0} ||E_\varepsilon - E_0||_{C^0[0,T]} = 0.$$

Before providing the proof of Proposition 5, we will present its main consequence, which is the corrector result stated in Theorem 6. Its proof makes use of the following technical lemma, which is well known in real analysis as Dini's Theorem:

Lemma 2 [24] *Let $\{g_\varepsilon\}$ be a sequence of non-decreasing real functions defined on a compact interval I of $\mathbb{R}$, which pointwise converges to a continuous function g on I. Then, g_ε converges to g uniformly on I.*

Theorem 6 *Let u_ε be the solution of problem (28). Under the assumptions of Theorem 5, we have the following convergences:*

$$\begin{cases} (i) \;\; u_\varepsilon \to u \quad \text{in } C^0([0,T]; L^2(Q)), \\ (ii) \; \lim_{\varepsilon\to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{L^2(0,T;[L^1(Q_\varepsilon)]^N)} = 0, \end{cases}$$

where $C^\varepsilon = (C^\varepsilon_{ij})_{1\le i,j\le n}$ is the classical corrector matrix described in (88).

Proof By density, for every $\delta > 0$ there exist $\varphi^\delta \in \mathcal{C}^\infty(0,T;\mathcal{D}(Q))$ and a vector function $\Phi^\delta = (\Phi^\delta_1, \Phi^\delta_2)$ with $\Phi^\delta_i = (\Phi^\delta_{i1}, \dots, \Phi^\delta_{iN})$ in $C^\infty(0,T;\mathcal{D}(Q_i)^N)$, $i = 1, 2$ satisfying

$$\begin{cases} (i) \;\; \|u - \varphi^\delta\|_{C^0([0,T];L^2(Q))} \le \delta \\ (ii) \; \|\nabla u - \Phi^\delta\|_{(L^2(0,T,L^2(Q_0)))^N} \le \delta. \end{cases} \tag{93}$$

Define

$$\begin{aligned} G_\varepsilon(t) &= F_\varepsilon(t) - \varepsilon^\gamma \int_0^t \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})^2 \mathrm{d}\sigma \, \mathrm{d}s, \\ G_0(t) &= F_0(t) - \int_0^t \int_{\Gamma_0} B(u_1 - u_2)^2 \mathrm{d}\sigma \, \mathrm{d}s, \end{aligned} \tag{94}$$

for $t \in [0,T]$, where F_ε, F_0 are defined in (89) for $\varphi = \varphi^\delta$, $\Phi_i = \Phi^\delta_i$ for $i = 1, 2$.

Let us prove that (92) and Proposition 4 imply

$$\limsup_{\varepsilon\to 0} \|G_\varepsilon\|_{C^0[0,T]} \le \|G_0\|_{C^0[0,T]} \le C\delta^2. \tag{95}$$

To do that, suppose first that (60) (i) or respectively (60) (iii) hold true. We apply Lemma 2 to the functions

$$g_\varepsilon = \varepsilon^\gamma \int_0^t \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})^2 \mathrm{d}\sigma \, \mathrm{d}s, \tag{96}$$

which are non-decreasing on $[0,T]$ to deduce that convergences (76) or (77) are uniform in $[0,T]$. This together with (92) and the definition of B at (90) imply (95).

On the other hand, if (60) (ii) holds true, then by using

$$G_\varepsilon(t) \le F_\varepsilon(t), \;\; \text{for } t \in [0,T],$$

and taking the supremum with respect to $t \in [0,T]$ above and in view of (90) and (92) we obtain (95) in this case as well. Next, the triangle inequality and (93) (i), give

$$\begin{aligned} \|u_\varepsilon - u\|^2_{C^0([0,T];L^2(Q))} &\le 2\left(\|u_\varepsilon - \varphi^\delta\|^2_{C^0([0,T];L^2(Q))} + \|\varphi^\delta - u\|^2_{C^0([0,T];L^2(Q))}\right) \\ &\le 2\|u_\varepsilon - \varphi^\delta\|^2_{C^0([0,T];L^2(Q))} + 2\delta^2. \end{aligned} \tag{97}$$

The ellipticity of A^ε implies

$$\|u_\varepsilon - \varphi^\delta\|^2_{C^0([0,T];L^2(Q))} \le 2\|G_\varepsilon\|_{C^0[0,T]}. \tag{98}$$

From (95), (97) and (98) it follows that

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - u\|^2_{C^0([0,T];L^2(Q))} \le C\delta^2. \tag{99}$$

On the other hand, by the triangle inequality and Hölder inequality,

$$\begin{aligned} \int_0^T \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{[L^1(Q_\varepsilon)]^N} \mathrm{d}t &\le 2\int_0^T \|\nabla u_\varepsilon - C^\varepsilon \Phi^\delta\|_{[L^1(Q_\varepsilon)]^N} \mathrm{d}t \\ &+ 2\|C^\varepsilon\|_{[L^2(Q)]^{N^2}} \int_0^T \|\Phi^\delta - \nabla u\|_{[L^2(Q_\varepsilon)]^N} \mathrm{d}t. \end{aligned} \tag{100}$$

The boundedness of C^ε, (93), (94) and the ellipticity of A^ε imply that

$$\int_0^T \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{[L^1(Q_\varepsilon)]^N}\, dt \le C\|G_\varepsilon\|_{C^0[0,T]}. \tag{101}$$

Thus by using (95), (100), (101) we obtain

$$\limsup_{\varepsilon\to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{L^2(0,T;[L^1(Q_\varepsilon)]^N)} \le C\delta^2. \tag{102}$$

The arbitrariness of δ in (99) and (102) implies the result. □

Remark 7 Note that the sequence $\{g_\varepsilon\}$ defined by (96) is equibounded but not equicontinuous, so that we cannot apply the classical Ascoli–Arzela theorem to prove its uniform convergence in $[0, T]$. Nevertheless, we are able to overcome this difficulty by making use of Lemma 2.

Proof of Proposition 5 Observe that (89) can be written as

$$\begin{aligned}
F_\varepsilon(t) =& \frac{1}{2}\|u_\varepsilon(t)-\varphi(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_\varepsilon} A^\varepsilon(\nabla u_\varepsilon - C^\varepsilon\Phi)(\nabla u_\varepsilon - C^\varepsilon\Phi)\mathrm{d}x\mathrm{d}s\\
&+ \varepsilon^\gamma \int_0^t\int_{\Gamma_\varepsilon} h^\varepsilon\cdot(u_{\varepsilon 1} - u_{\varepsilon 2})^2 \mathrm{d}\sigma\, \mathrm{d}s\\
=& \frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(Q)} - \int_Q u_\varepsilon(t)\varphi(t)dx + \frac{1}{2}\|\varphi(t)\|^2_{L^2(Q)} + \int_0^t\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon\, \mathrm{d}x\mathrm{d}s\\
&+\varepsilon^\gamma \int_0^t\int_{\Gamma_\varepsilon} h^\varepsilon\cdot(u_{\varepsilon 1} - u_{\varepsilon 2})^2 \mathrm{d}\sigma\, \mathrm{d}s - \int_0^t \int_{Q_\varepsilon} A^\varepsilon C^\varepsilon \Phi \nabla u_\varepsilon\, \mathrm{d}x\, \mathrm{d}s\\
&- \int_0^t \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon C^\varepsilon\Phi\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{Q_\varepsilon} A^\varepsilon C^\varepsilon \Phi C^\varepsilon \Phi\, \mathrm{d}x\, \mathrm{d}s\\
:=& \eta^1_\varepsilon(t) - \eta^2_\varepsilon(t) + \eta^3_\varepsilon(t)
\end{aligned}$$

where

$$\eta^1_\varepsilon(t) = \frac{1}{2}\|\varphi(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_\varepsilon} A^\varepsilon C^\varepsilon \Phi C^\varepsilon \Phi\, \mathrm{d}x\, \mathrm{d}s \tag{103}$$

$$\begin{aligned}
\eta^2_\varepsilon(t) =& \int_Q u_\varepsilon(t)\varphi(t)\, \mathrm{d}x + \int_0^t \int_{Q_\varepsilon} A^\varepsilon C^\varepsilon \Phi \nabla u_\varepsilon\, \mathrm{d}x\, \mathrm{d}s\\
&+ \int_0^t \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon C^\varepsilon \Phi\, \mathrm{d}x\, \mathrm{d}s
\end{aligned} \tag{104}$$

$$\begin{aligned}
\eta^3_\varepsilon(t) =& \frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(Q)} + \int_0^t \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon\, \mathrm{d}x\, \mathrm{d}s\\
&+\varepsilon^\gamma \int_0^t\int_{\Gamma_\varepsilon} h^\varepsilon\cdot(u_{\varepsilon 1} - u_{\varepsilon 2})^2 \mathrm{d}\sigma\, \mathrm{d}s
\end{aligned} \tag{105}$$

Let us study the limit of $\eta^i_\varepsilon(t)$ as $\varepsilon \to 0$ for each $i \in \{1, 2, 3\}$.
Step 1 The term $\eta^1_\varepsilon(t)$ defined in (103) is equal to

$$\frac{1}{2}\|\varphi(t)\|^2_{L^2(Q)} + \int_0^t \int_{\omega\times]\varepsilon_0^\kappa \overline{g}, l[} A^\varepsilon C^\varepsilon \Phi_1 C^\varepsilon \Phi_1\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{Q_2} A^\varepsilon C^\varepsilon \Phi_2 C^\varepsilon \Phi_2\, \mathrm{d}x\, \mathrm{d}s.$$

Observe that

$$\exists\, \varepsilon_0 \text{ such that } \operatorname{supp} \Phi_{1i} \subset \omega\times]\varepsilon_0^\kappa \overline{g}, l[, \qquad \forall i = 1, \dots, N. \tag{106}$$

This implies that

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\omega\times]\varepsilon_0^\kappa \overline{g}, l[} A^\varepsilon C^\varepsilon \Phi_1 C^\varepsilon \Phi_1\, \mathrm{d}x\, \mathrm{d}s = \lim_{\varepsilon\to 0} \int_0^t \int_{Q_1} A^\varepsilon C^\varepsilon \Phi_1 C^\varepsilon \Phi_1\, \mathrm{d}x\, \mathrm{d}s,$$

so that by (59) and standard computations,

$$\lim_{\varepsilon\to 0}\eta_\varepsilon^1(t)=\frac{1}{2}\|\varphi(t)\|^2_{L^2(Q)}+\int_0^t\int_{Q_0}A^0\Phi\Phi\ \mathrm{d}x\ \mathrm{d}s. \tag{107}$$

To show the strong convergence of $\eta_\varepsilon^1(t)$ in $C^0[0,T]$, we first observe that from (7) we have

$$\begin{aligned}||\eta_\varepsilon^1||_{C^0[0,T]}&\le\frac{1}{2}||\varphi||^2_{C^0[0,T]}\\&\quad+\beta\Big(||\Phi_1||^2_{C^0(0,T,C^0(Q_1))}+||\Phi_2||^2_{C^0(0,T,C^0(Q_2))}\Big)||C^\varepsilon||^2_{L^2(Q)}\le C.\end{aligned} \tag{108}$$

Next consider $h << 1$. By using the same ideas in (108) we obtain

$$\begin{aligned}|\eta_\varepsilon^1(t+h)-\eta_\varepsilon^1(t)|&\le||\varphi||^2_{C^0[0,T]}+\int_t^{t+h}\int_{Q_\varepsilon}A^\varepsilon C^\varepsilon\Phi C^\varepsilon\Phi\ \mathrm{d}x\ \mathrm{d}s\\&\le h\beta\left(||\Phi_1||^2_{C^0(0,T,C^0(Q_1))}+||\Phi_2||^2_{C^0(0,T,C^0(Q_2))}\right)||C^\varepsilon||^2_{L^2(Q)}\\&\le Ch.\end{aligned} \tag{109}$$

By Ascoli–Arzela theorem, (107) and estimates (108) and (109) imply

$$\eta_\varepsilon^1(t)\to\frac{1}{2}\|\varphi(t)\|^2_{L^2(Q)}+\int_0^t\int_{Q_0}A^0\Phi\Phi\ \mathrm{d}x\ \mathrm{d}s\ \text{ uniformly for } t\in[0,T]. \tag{110}$$

Step 2 Next we proceed to study the term $\eta_\varepsilon^2(t)$ defined in (104). In this regard, we rewrite $\eta_\varepsilon^2(t)$ as

$$\eta_\varepsilon^2(t)=\kappa_\varepsilon^1(t)+\kappa_\varepsilon^2(t)+\kappa_\varepsilon^3(t),$$

where

$$\kappa_\varepsilon^1(t)=\int_Q u_\varepsilon(t)\varphi(t)\ \mathrm{d}x, \tag{111}$$

$$\kappa_\varepsilon^2(t)=\int_0^t\int_{Q_\varepsilon}A^\varepsilon C^\varepsilon\Phi\nabla u_\varepsilon\ \mathrm{d}x\ \mathrm{d}s, \tag{112}$$

$$\kappa_\varepsilon^3(t)=\int_0^t\int_{Q_\varepsilon}A^\varepsilon\nabla u_\varepsilon C^\varepsilon\Phi\ \mathrm{d}x\ \mathrm{d}s. \tag{113}$$

• We perform now the limit analysis of κ_ε^1 defined in (111). Using integration by parts, taking φ as a test function in (28) and integrating in time we obtain

$$\begin{aligned}\kappa_\varepsilon^1(t)&=\int_Q u_\varepsilon(0)\varphi(0)\ \mathrm{d}x+\int_0^t\langle u_\varepsilon',\varphi\rangle_{(W_0^\varepsilon)',W_0^\varepsilon}+\int_0^t\langle u_\varepsilon,\varphi'\rangle_{(W_0^\varepsilon)',W_0^\varepsilon}\\&=\int_Q u_\varepsilon^0\varphi(0)\ \mathrm{d}x-\int_0^t\int_{Q_\varepsilon}A^\varepsilon\nabla u_\varepsilon\nabla\varphi\ \mathrm{d}x\ \mathrm{d}s+\int_0^t\int_Q f\varphi\ \mathrm{d}x\ \mathrm{d}s\\&\quad+\int_0^t\int_Q u_\varepsilon\varphi'\mathrm{d}x\ \mathrm{d}t\end{aligned}$$

From (61), (63), (62) (i) and (87) for every $t\in[0,T]$ we obtain

$$\kappa_\varepsilon^1(t)\to\int_Q u(0)\varphi(0)\ \mathrm{d}x-\int_0^t\int_{Q_0}A^0\nabla u\nabla\varphi\ \mathrm{d}x\ \mathrm{d}s+\int_0^t\int_Q f\varphi\ \mathrm{d}x\ \mathrm{d}s+\int_0^t\int_Q u\varphi'\mathrm{d}x\ \mathrm{d}t.$$

Using φ as a test function in the limit problem for u (described in Theorem 5) and integration by parts with respect to time, we obtain

$$\kappa_\varepsilon^1(t)\to\int_Q u(t)\varphi\ \mathrm{d}x\quad\text{for every}\quad t\in[0,T]. \tag{114}$$

Next, from (7), (29), (30) (iii) and Hölder inequality we obtain

$$\begin{aligned}|\kappa_\varepsilon^1(t)| &\leq C_1\left(||u_\varepsilon^0||_{L^2(Q)}+\beta||\nabla u_\varepsilon||_{L^2(0,T,W_0^\varepsilon)}+||f||_{L^2(0,T,L^2(Q))}+||u_\varepsilon||_{L^2(0,T,W_0^\varepsilon)}\right)\\ &\leq C \text{ independent of } t.\end{aligned} \tag{115}$$

For $h << 1$, by using the same ideas in (115) we deduce

$$\begin{aligned}|\kappa_\varepsilon^1(t+h)-\kappa_\varepsilon^1(t)| \leq &\left|\int_t^{t+h}\int_{Q_\varepsilon} A^\varepsilon\nabla u_\varepsilon\nabla\varphi \text{ d}x \text{ d}s\right|+\left|\int_t^{t+h}\int_Q f\varphi \text{ d}x \text{ d}s\right|\\ &+\left|\int_t^{t+h}\int_Q u_\varepsilon\varphi' \text{d}x \text{ d}t\right|\\ &\leq h^{\frac{1}{2}}\left(\beta||\nabla u_\varepsilon||_{L^2(0,T,W_0^\varepsilon)}+||f||_{L^2(0,T,L^2(Q)}+||u_\varepsilon||_{L^2(0,T,W_0^\varepsilon)}\right)\\ &\leq Ch^{\frac{1}{2}}.\end{aligned} \tag{116}$$

As above, (114)–(116) imply

$$\kappa_\varepsilon^1(t) \to \int_Q u(t)\varphi \text{ d}x \quad \text{strongly in} \quad C^0[0,T]. \tag{117}$$

• For the second term $\kappa_\varepsilon^2(t)$ defined in (112), we write,

$$\kappa_\varepsilon^2(t)=\int_0^t\int_{Q_1} A^\varepsilon C^\varepsilon \Phi_1\nabla u_{\varepsilon 1} \text{ d}x \text{ d}s+\int_0^t\int_{Q_2} A^\varepsilon C^\varepsilon \Phi_2\nabla u_{\varepsilon 2} \text{ d}x \text{ d}s.$$

From (58), if w_i is given for $\lambda = e_i$ and $w_i^\varepsilon = \varepsilon w_i(\frac{x}{\varepsilon})$ a.e. in $\mathbb{R}^N$ then

$$\begin{cases} w_i^\varepsilon \rightharpoonup x_i & \text{weakly in} \quad H^1(Q),\\ w_i^\varepsilon \to x_i & \text{strongly in} \quad L^2(Q),\\ A^\varepsilon\nabla w_i^\varepsilon \rightharpoonup A^0 e_i & \text{weakly in} \quad (L^2(Q))^N.\end{cases} \tag{118}$$

By a change of scale,

$$\int_0^t\int_\Omega A^\varepsilon\nabla w_i^\varepsilon\nabla v \text{ d}x \text{ d}s = 0 \quad \forall v\in L^2(0,T,H_0^1(\Omega)), \tag{119}$$

for every open set $\Omega\subset\mathbb{R}^N$. It follows from (118), (119) and (62) (i) that

$$\begin{aligned}\int_0^t\int_{Q_1} A^\varepsilon C^\varepsilon\Phi_1\nabla u_{\varepsilon 1} \text{ d}x \text{ d}s &= \int_0^t\int_{Q_1} A^\varepsilon\nabla w_i^\varepsilon\nabla(\Phi_{1i}u_{\varepsilon 1}) \text{ d}x \text{ d}s\\ &\quad -\int_0^t\int_{Q_1} A^\varepsilon\nabla w_i^\varepsilon\nabla\Phi_{1i}u_{\varepsilon 1} \text{ d}x \text{ d}s\\ &= -\int_0^t\int_{Q_1} A^\varepsilon\nabla w_i^\varepsilon\nabla\Phi_{1i}u_{\varepsilon 1} \text{ d}x \text{ d}s\\ &\to -\int_0^t\int_{Q_1} A^0 e_i\nabla\Phi_{1i}u_1 \text{ d}x \text{ d}s,\end{aligned} \tag{120}$$

where Einstein index summation was used in the above. In a similar manner,

$$\int_0^t\int_{Q_2} A^\varepsilon C^\varepsilon\Phi_2\nabla u_{\varepsilon 2} \text{ d}x \text{ d}s \to -\int_0^t\int_{Q_2} A^0 e_i\nabla\Phi_{2i}u_2 \text{ d}x \text{ d}s. \tag{121}$$

Therefore, applying again integration by parts in the limit integrals in (120) and (121) we obtain

$$\lim_{\varepsilon\to 0}\kappa_\varepsilon^2(t)=\int_0^t\int_{Q_0} A^0\Phi\nabla u \text{ d}x \text{ d}s. \tag{122}$$

Now, it follows from (30) (i) and the properties of Φ, A^ε and C^ε that $\kappa_\varepsilon^2(t)$ is bounded in $H^1(0,T)$. Hence, by using this and the compactness of the injection $H^1(0,T)\subset C^0(0,T)$ in (122) we obtain

$$\lim_{\varepsilon\to 0}\kappa_\varepsilon^2(t)=\int_0^t\int_{Q_0}A^0\Phi\nabla u\;\mathrm{d}x\,\mathrm{d}s,\;\text{ strongly in } C^0[0,T]. \tag{123}$$

• To handle $\kappa_\varepsilon^3(t)$ introduced in (113), we choose the test function $v=\Phi_i w_i^\varepsilon$ in the variational formulation (28). Observe that

$$\begin{aligned}\kappa_\varepsilon^3(t)&=\int_0^t\int_{Q_\varepsilon}A^\varepsilon\nabla u_\varepsilon\Phi_i\nabla w_i^\varepsilon\;\mathrm{d}x\,\mathrm{d}s\\&=\int_0^t\int_{Q_\varepsilon}A^\varepsilon\nabla u_\varepsilon\nabla(\Phi_i w_i^\varepsilon)\;\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{Q_\varepsilon}A^\varepsilon\nabla u_\varepsilon\nabla\Phi_i w_i^\varepsilon\;\mathrm{d}x\,\mathrm{d}s\\&=\int_0^t\int_{Q_\varepsilon}f\Phi_i w_i^\varepsilon\;\mathrm{d}x\,\mathrm{d}s-\int_0^t\langle u'_{\varepsilon 1},\Phi_i w_i^\varepsilon\rangle\,\mathrm{d}s-\int_0^t\langle u'_{\varepsilon 2},\Phi_i w_i^\varepsilon\rangle\,\mathrm{d}s\\&\quad-\varepsilon^\gamma\int_0^t\int_{\Gamma_\varepsilon}h^\varepsilon(u_{\varepsilon 1}-u_{\varepsilon 2})((\Phi w_i^\varepsilon)_1-(\Phi w_i^\varepsilon)_2)\;\mathrm{d}\sigma\,\mathrm{d}s\\&\quad-\int_0^t\int_{Q_\varepsilon}A^\varepsilon\nabla u_\varepsilon\nabla\Phi_i w_i^\varepsilon\;\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Now, since (106) holds and for $\varepsilon\le\varepsilon_0$, $\mathrm{supp}(\Phi_2)\subset Q_2$, we have

$$\varepsilon^\gamma\int_0^t\int_{\Gamma_\varepsilon}h^\varepsilon(u_{\varepsilon 1}-u_{\varepsilon 2})((\Phi w_i^\varepsilon)_1-(\Phi w_i^\varepsilon)_2)\;\mathrm{d}\sigma\,\mathrm{d}s\to 0, \tag{124}$$

as $\varepsilon\to 0$. On the other hand, by Remark 3 and considering (55) (extended to $L^2(0,T,H_0^1(Q_0)$ by a classical density argument), using (118) we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(\int_0^t\langle u'_{\varepsilon 1},\Phi_i w_i^\varepsilon\rangle\,\mathrm{d}s+\int_0^t\langle u'_{\varepsilon 2},\Phi_i w_i^\varepsilon\rangle\,\mathrm{d}s\right)&=\lim_{\varepsilon\to 0}\int_0^t\langle u'_\varepsilon,\Phi_i w_i^\varepsilon\rangle\,\mathrm{d}s\\&=\int_0^t\langle u',\Phi_i x_i\rangle\,\mathrm{d}s.\end{aligned} \tag{125}$$

Using (124) and (125) and together with (118) and (63), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\kappa_\varepsilon^3(t)&=\int_0^t\int_Q f\Phi_i x_i\;\mathrm{d}x\,\mathrm{d}s-\int_0^t\langle u',\Phi_i x_i\rangle\,\mathrm{d}s-\int_0^t\int_{Q_0}A^0\nabla u\nabla\Phi_i x_i\;\mathrm{d}x\,\mathrm{d}s\\&=\int_0^t\int_Q f\Phi_i x_i\;\mathrm{d}x\,\mathrm{d}s-\int_0^t\langle u',\Phi_i x_i\rangle\,\mathrm{d}s-\int_0^t\int_{Q_0}A^0\nabla u\nabla(\Phi_i x_i)\;\mathrm{d}x\,\mathrm{d}s\\&\quad+\int_0^t\int_{Q_0}A^0\nabla u\Phi\;\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{126}$$

Using the fact that (106) holds and $\mathrm{supp}(\Phi_i)\subset Q_i$, $i=1,2$, considering the limit problems satisfied by u (see Theorem 5), we obtain

$$\int_0^t\int_{Q_0}A^0\nabla u\nabla(\Phi_i x_i)\;\mathrm{d}x\,\mathrm{d}s=\int_0^t\int_Q f\Phi_i x_i\;\mathrm{d}x\,\mathrm{d}s-\int_0^t\langle u',\Phi_i x_i\rangle\,\mathrm{d}s. \tag{127}$$

Combining (126) and (127), it follows that

$$\lim_{\varepsilon\to 0}\kappa_\varepsilon^3(t)=\int_0^t\int_{Q_0}A^0\nabla u\Phi\;\mathrm{d}x\,\mathrm{d}s.$$

Now, observe that by (63), the definition of C^ε, assumption on Φ and the Hölder's inequality,

$$|\kappa_\varepsilon^3(t)| \leq \|A^\varepsilon \nabla u_\varepsilon\|_{L^2(0,T;[L^2(Q)]^n} \|C^\varepsilon\|_{[L^2(Q)]^{n^2}} \|\Phi\|_{L^\infty(0,T;[L^2(Q)]^n} \leq c,$$

where c is independent of t. Moreover, for any $h > 0$ small enough,

$$\begin{aligned}|\kappa_\varepsilon^3(t+h) - \kappa_\varepsilon^3(t)| &\leq \|A^\varepsilon \nabla u_\varepsilon\|_{L^2(0,T;[L^2(Q)]^n} \|C^\varepsilon\|_{[L^2(Q)]^{n^2}} h^{\frac{1}{2}} \|\Phi\|_{L^\infty(0,T;[L^2(Q)]^n} \\ &\leq ch^{\frac{1}{2}} \to 0, \quad \text{as } h \to 0, \text{ uniformly in } \varepsilon.\end{aligned}$$

Thus, by Ascoli–Arzela theorem,

$$\kappa_\varepsilon^3(t) \to \int_0^t \int_{Q_0} A^0 \nabla u \Phi \, \mathrm{d}x \, \mathrm{d}s \quad \text{strongly in} \quad C^0([0,T]). \tag{128}$$

Combining (117), (123) and (128), we have,

$$\lim_{\varepsilon\to 0} \eta_\varepsilon^2(t) = \int_Q u(t)\varphi \, \mathrm{d}x + \int_0^t \int_{Q_0} A^0 \Phi \nabla u \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_{Q_0} A^0 \nabla u \Phi \, \mathrm{d}x \, \mathrm{d}s \tag{129}$$

strongly in $C^0([0,T])$.
Step 3 Finally we discuss the limit behavior of $\eta_\varepsilon^3(t)$ introduced in (105). Thus, taking u_ε as test function in (28) and integrating with respect to time we obtain

$$\eta_\varepsilon^3(t) = \int_0^t \int_Q f u_\varepsilon \, \mathrm{d}\, \mathrm{d}s + \frac{1}{2}\|u_\varepsilon^0\|_{L^2(Q)}^2.$$

Using (91) and (62) (i) we obtain that

$$\lim_{\varepsilon\to 0} \eta_\varepsilon^3(t) \to \int_0^t \int_Q f u \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2}\|u^0\|_{L^2(Q)}^2, \text{ for all } t \in [0,T]. \tag{130}$$

Next, it easy to observe that (27) and the fact that $u \in C^0(0,T,L^2(Q))$ imply that the sequence of functions η_ε^3 is equibounded and equicontinuous in $C^0[0,T]$. Then (130) and Ascoli–Arzela theorem imply,

$$\eta_\varepsilon^3(t) \to \int_0^t \int_Q f u \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2}\|u^0\|_{L^2(Q)}^2, \text{ strongly in } C^0[0,T]. \tag{131}$$

Finally, using u as a test function in the limit problem (see Theorem 5) and integrating with respect to time, convergence (131) becomes

$$\begin{aligned}\eta_\varepsilon^3(t) \to \frac{1}{2}\|u(t)\|_{L^2(Q)}^2 &+ \int_0^t \int_{Q_0} A^0 \nabla u \nabla u \, \mathrm{d}x \, \mathrm{d}s \\ &+ \int_0^t \int_{\Gamma_0} B(u_1 - u_2)^2 \mathrm{d}\sigma \, \mathrm{d}s\end{aligned} \tag{132}$$

strongly in $C^0[0,T]$. Finally, from (110), (129) and (132) we obtain (92). □

7 Physical interpretation of results and applications

In this section, we will first offer a discussion on the physical interpretation of the homogenization and corrector results presented in Theorems 5 and 6 followed by a brief account of possible applications of our study and future work.

Let us assume a three-dimensional space. The main question addressed in the paper is mathematically described in (26) (see also Fig. 1), and it is our aim to characterize a computationally feasible macroscale model to describe the multiscale problem of heat flow through two different microstructures (of characteristic length $\varepsilon << 1$) separated by a rough interface, the geometry of which depends in a prescribed fashion when $\varepsilon << 1$. For simplicity, we will further assume $h = 1$ in (26) and $|\omega| = 1$ in (11) where here and in what follows $|S|$ denotes the area of the set $S \subset \mathbb{R}^2$.

First, we note that in the formulation of our main problem (26), the condition on the interface Γ_ε relating the fluxes and the jump of the temperature is written per unit area. Hence, the actual microscale physical heat transfer coefficient (the proportionality constant), K_ε, between the heat flux through Γ_ε and the temperature over Γ_ε is given by

$$K_\varepsilon = \varepsilon^\gamma |\Gamma_\varepsilon|. \tag{133}$$

Next, recall that the area of a surface S described by $z = f(\xi') = f(\xi_1, \xi_2)$ for $\xi' \in \Sigma$ and some arbitrary smooth function f is given by

$$|S| = \int_\Sigma \sqrt{1 + f_{\xi_1}^2(\xi') + f_{\xi_2}^2(\xi')}\mathrm{d}\xi'. \tag{134}$$

where f_{ξ_i} denotes the partial derivative of the function f with respect to its ith variable. Assuming that the function g introduced in (2) is smooth enough, from (134) and the definition of Γ_ε in (5), we have

$$|\Gamma_\varepsilon| = \int_\omega \sqrt{1 + \varepsilon^{2\kappa-2} g_{x_1}^2\left(\frac{x'}{\varepsilon}\right) + \varepsilon^{2\kappa-2} g_{x_2}^2\left(\frac{x'}{\varepsilon}\right)}\mathrm{d}x'. \tag{135}$$

From (135) it easily follows that:

$$\begin{aligned}
&\frac{K_\varepsilon}{\varepsilon^\gamma} \to 1, \text{ if } \kappa > 1 \\
&\frac{K_\varepsilon}{\varepsilon^{\gamma+\kappa-1}} \to m_{Y'}(|\nabla g|), \text{ if } 0 < \kappa < 1 \\
&\frac{K_\varepsilon}{\varepsilon^\gamma} \to m_{Y'}\left(\sqrt{1 + |\nabla g|^2}\right), \text{ if } \kappa = 1,
\end{aligned} \tag{136}$$

where $m_{Y'}(f)$ denotes the average on Y' (the surface reference cell) of the function f.

The results in (136) explain then the behavior obtained in Theorem 5. Indeed,

1. If ($\kappa \geq 1$ and $\gamma = 0$) or ($0 < \kappa < 1$ and $\gamma = 1 - \kappa$), then the heat transfer coefficient on the interface K_ε approaches a constant for $\varepsilon << 1$ (ε infinitely small) and thus the homogenous macroscale problem is modeled by a parabolic PDE over a domain separated by a hyperplane Γ_0 with the continuous flux proportional to the temperature jump across it with proportionality constant given by the constant limit of K_ε.
2. If ($\kappa \geq 1$ and $\gamma < 0$), then the heat transfer coefficient on the interface satisfies $K_\varepsilon \approx \varepsilon^\gamma$ for $\varepsilon << 1$. Similarly, if ($0 < \kappa < 1$ and $\gamma < 1 - \kappa$) then $K_\varepsilon \approx \varepsilon^{\gamma+\kappa-1}$ for $\varepsilon << 1$. In both of these situations, the heat transfer coefficient on the interface, i.e., K_ε, becomes infinitely large for $\varepsilon \to 0$ and so, a realistic finite flux across the interface implicitly implies that the temperature becomes continuous across the interface in the limit when $\varepsilon \to 0$. Hence, as a consequence, the contribution of the microscale transmission interface disappears in the homogenized limit and the macroscale model is governed by a parabolic PDE in the whole domain with homogeneous Dirichlet boundary conditions.
3. As above, if ($\kappa \geq 1$ and $\gamma > 0$) or respectively ($0 < \kappa < 1$ and $\gamma > 1 - \kappa$), the heat transfer coefficient on the interface satisfies $K_\varepsilon \approx \varepsilon^\gamma$ and respectively $K_\varepsilon \approx \varepsilon^{\gamma+\kappa-1}$ for $\varepsilon << 1$. In both of these situations, the heat transfer coefficient on the interface, i.e., K_ε, approaches zero as $\varepsilon \to 0$ and so the microscale transmission interface has a very strong effect in the limit and the macroscale problem is modeled by a parabolic homogenized PDE on two disjoint domains with identical initial conditions and homogeneous mixed boundary conditions, zero flux on the flat part of the boundary and zero temperature otherwise.

In Sect. 6, we present the corrector analysis for our homogenization result. First, the result of Remark 6 states the fact that the energy associated with the multiscale model (26) approaches (as $\varepsilon \to 0$) the energy of the limit problem presented in Theorem 5. Thus, as expected from the physical point of view, for a given microscale ε, the energy of the proposed macroscale model will be close to the energy of the multiscale problem. Then, the corrector results of Theorem 6 show that the macroscale problem given in Theorem 5 can indeed be used for an approximation for the evolution of both the microscale temperature and its gradient.

A direct application of our results could be their use as part of alternative strategies for the design of efficient material interfaces between given microstructures with the purpose of controlling the overall heat transfer. Another important application of our results will be for the associated multiscale approximate controllability problem where prescribed controls described as interior heat sources [mathematically appearing as additional additive terms in the right-hand side of the PDE (26)] will be employed to satisfy certain global optimality constraints. Because these controls will also depend on the microscale $\varepsilon << 1$, the overall multiscale solution will be very difficult to compute and so a homogenized macroscale model (PDE + control) will be desired instead. This problem will be studied, and the results will be reported in a forthcoming paper.

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