Asymptotic Analysis 54 (2007) 103–123
IOS Press

Error estimates for periodic homogenization with non-smooth coefficients

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Abstract. In this paper we present new results regarding the H_0^1-norm error estimate for the classical problem in homogenization using suitable boundary layer correctors. Compared with all the existing results on the subject, which assume either smooth enough coefficients or smooth data, we use the periodic unfolding method and propose a new asymptotic series to approximate the solution u_ε with an error estimate which holds true for nonsmooth coefficients and general data.

1. Introduction

This paper is dedicated to the study of the error estimates for the classical problem in homogenization using suitable boundary layer correctors.

Let $\Omega \in \mathbb{R}^N$, denote a bounded convex polyhedron or a convex bounded domain with a sufficiently smooth boundary. Consider also the unit cube $Y = (0,1)^N$. It is well known that for $A \in L^\infty(Y)^{N\times N}$, Y-periodic with $m|\xi|^2 \leqslant A_{ij}(y)\xi_i\xi_j \leqslant M|\xi|^2$, $\forall \xi \in \mathbb{R}^N$ the solutions of

$$\begin{cases} -\nabla\cdot\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

have the property that (see [18,12,3,4]),

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega),$$

where u_0 verifies

$$\begin{cases} -\nabla\cdot(\mathcal{A}^{\text{hom}}\nabla u_0(x)) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

with $\mathcal{A}_{ij}^{\text{hom}} = M_Y(A_{ij}(y) + A_{ik}(y)\frac{\partial\chi_j}{\partial y_k})$, $M_Y(\cdot) = \frac{1}{|Y|}\int_Y \cdot\, \mathrm{d}y$ and where $\chi_j \in W_{\text{per}}(Y) = \{\chi \in H^1_{\text{per}}(Y) \mid M_Y(\chi) = 0\}$ are the solutions of the local problem

$$-\nabla_y\cdot(A(y)(\nabla\chi_j + e_j)) = 0 \tag{1.3}$$

and where e_j denote the vectors of the canonical basis in $\mathbb{R}^N$.

We mention that, throughout this paper, ∇ and $(\nabla\cdot)$ will denote the full gradient and divergence operators respectively, and with $\nabla_x, (\nabla_x\cdot)$ and $\nabla_y, (\nabla_y\cdot)$ we will denote the gradient and the divergence in the slow and fast variable respectively.

We will also denote, throughout the paper, by Φ the continuous extension of any arbitrary function $\Phi \in W^{p,m}(\Omega)$ with $p, m \in \mathbb{Z}$, to the space $W^{p,m}(\mathbb{R}^N)$. With minimal assumption on the smoothness of Ω a stable extension operator can be constructed (see [20], Ch. VI, 3.1).

The formal asymptotic expansion corresponding to the above results can be written as

$$u_\varepsilon(x) = u_0(x) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

where

$$w_1\left(x, \frac{x}{\varepsilon}\right) = \chi_j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_j}. \tag{1.4}$$

We make the observation that the summation convention over repeated indices will be used in the remaining of the chapter and that the letter C will denote a constant independent of any other parameter, otherwise specified.

A classical result (see [18,12,15,3]), states that with additional regularity assumption on χ_j, the solutions of the local problems, one has

$$\left\| u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon w_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{H^1(\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}. \tag{1.5}$$

Without any additional assumptions a similar result has been recently proved by G. Griso in [10], using the Periodic Unfolding method developed in [7], i.e.,

$$\left\| u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon\chi_j\left(\frac{\cdot}{\varepsilon}\right)Q_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right)\right\|_{H^1(\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^2(\Omega)} \tag{1.6}$$

with

$$x \in \tilde{\Omega}_\varepsilon, \quad Q_\varepsilon(\phi)(x) = \sum_{i_1,\dots,i_N} M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon i)\bar{x}_{1,\xi}^{i_1} \cdot \cdots \cdot \bar{x}_{N,\xi}^{i_N}, \quad \xi = \left[\frac{x}{\varepsilon}\right]$$

for $\phi \in L^2(\Omega)$, $i = (i_1, \dots, i_N) \in \{0,1\}^N$ and

$$\bar{x}_{k,\xi}^{i_k} = \begin{cases} \dfrac{x_k - \varepsilon\xi_k}{\varepsilon} & \text{if } i_k = 1, \\ 1 - \dfrac{x_k - \varepsilon\xi_k}{\varepsilon} & \text{if } i_k = 0, \end{cases} \qquad x \in \varepsilon(\xi + Y),$$

where $M_Y^\xi(\phi)(x) = \frac{1}{\varepsilon^N}\int_{\varepsilon\xi+\varepsilon Y}\phi(y)\,dy$ and $\tilde{\Omega}_\varepsilon = \bigcup_\xi\{\varepsilon\xi + \varepsilon Y,\ \text{with } (\varepsilon\xi + \varepsilon Y) \cap \Omega \neq \emptyset\}$.

In order to improve the error estimates in (1.5) boundary layer terms have been introduced as solutions to

$$-\nabla\cdot\left(A\left(\frac{x}{\varepsilon}\right)\nabla\theta_\varepsilon\right)=0\quad\text{in }\Omega,\qquad \theta_\varepsilon=w_1\left(x,\frac{x}{\varepsilon}\right)\quad\text{on }\partial\Omega. \tag{1.7}$$

With the assumptions that $A\in C^\infty(Y)$ and is a Y-periodic matrix and that the homogenized solution u_0 is sufficiently smooth, it has been proved in [4] (see also [15]) that

$$\left\|u_\varepsilon(\cdot)-u_0(\cdot)-\varepsilon w_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)+\varepsilon\theta_\varepsilon(\cdot)\right\|_{H_0^1(\Omega)}\leqslant C\varepsilon, \tag{1.8}$$

$$\left\|u_\varepsilon(\cdot)-u_0(\cdot)-\varepsilon w_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)+\varepsilon\theta_\varepsilon(\cdot)\right\|_{L^2(\Omega)}\leqslant C\varepsilon^2. \tag{1.9}$$

Moskow and Vogelius [16], proved the above estimates assuming $A\in C^\infty(Y)$, Y-periodic matrix and $u_0\in H^2(\Omega)$ or $u_0\in H^3(\Omega)$ for (1.8) or (1.9) respectively. Inequality (1.8) is proved in Allaire and Amar [1] for the case when $A\in L^\infty(Y)$ and $u_0\in W^{2,\infty}(\Omega)$. J. Casado-Diaz proved in [5] that (1.8) holds true for $\Omega\in C^{1,1}$ and $f\in L^{N+\tau}(\Omega)$ for some $\tau>0$.

In [21], Sarkis and Versieux showed that estimates (1.8) and (1.9) respectively still hold in a more general setting, when one has $u_0\in W^{2,p}(\Omega)$, $\chi_j\in W^{1,q}_{\text{per}}(Y)$ for (1.8), and $u_0\in W^{3,p}(\Omega)$, $\chi_j\in W^{1,q}_{\text{per}}(Y)$ for (1.9), where, in both cases, $p>N$ and $q>N$ satisfy $\frac{1}{p}+\frac{1}{q}\leqslant\frac{1}{2}$. In [21] the constants in the right hand side of (1.8) and (1.9) are proportional to $\|u_0\|_{W^{2,p}(\Omega)}$ and $\|u_0\|_{W^{3,p}(\Omega)}$ respectively.

In order to improve the error estimate in (1.9) one needs to consider the second order boundary layer corrector φ_ε defined as the solution of,

$$-\nabla\cdot\left(A\left(\frac{x}{\varepsilon}\right)\nabla\varphi_\varepsilon\right)=0\quad\text{in }\Omega,\qquad \varphi_\varepsilon(x)=\chi_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2u_0}{\partial x_i\partial x_j}\quad\text{on }\partial\Omega, \tag{1.10}$$

where $\chi_{ij}\in W_{\text{per}}(Y)$ are solution of the following local problems,

$$\nabla_y\cdot(A\nabla_y\chi_{ij})=b_{ij}+\mathcal{A}^{\text{hom}}_{ij} \tag{1.11}$$

with $\mathcal{A}^{\text{hom}}$ defined by (1.2), $M_Y(b_{ij}(y))=-\mathcal{A}^{\text{hom}}_{ij}$, and $b_{ij}=-A_{ij}-A_{ik}\frac{\partial\chi_j}{\partial y_k}-\frac{\partial}{\partial y_k}(A_{ik}\chi_j)$.

In this paper, we answer the open question about error estimates for (1.1) in general convex domains and coefficients $A\in L^\infty(Y)$. Inspired by an idea of Griso presented in [10], we use the periodic unfolding method developed by Cioranescu, Damlamian and Griso [7] and a general smoothing argument to replace in (1.8) and (1.9) $w_1(x,\frac{x}{\varepsilon})$ defined in (1.4), by

$$u_1\left(x,\frac{x}{\varepsilon}\right)=\chi_j\left(\frac{x}{\varepsilon}\right)Q_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right). \tag{1.12}$$

For $u_0\in H^2(\Omega)$ we prove that

$$\left\|u_\varepsilon(\cdot)-u_0(\cdot)-\varepsilon\chi_j\left(\frac{\cdot}{\varepsilon}\right)Q_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right)+\varepsilon\beta_\varepsilon(\cdot)\right\|_{H_0^1(\Omega)}\leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{1.13}$$

where β_ε is defined by

$$-\nabla\cdot\left(A\left(\frac{x}{\varepsilon}\right)\nabla\beta_\varepsilon\right)=0 \quad \text{in } \Omega, \qquad \beta_\varepsilon = u_1\left(x,\frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \tag{1.14}$$

Assuming $u_0 \in W^{3,p}(\Omega)$ with $p > N$ we obtain

$$\left\|u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon\chi_j\left(\frac{\cdot}{\varepsilon}\right)\frac{\partial u_0}{\partial x_j} + \varepsilon\theta_\varepsilon(\cdot)\right\|_{L^2(\Omega)} \leqslant C\varepsilon^2\|u_0\|_{W^{3,p}(\Omega)}. \tag{1.15}$$

For completeness, in the Appendix, we give some of the definitions and properties related to the unfolding operator (for which we refer to [7]), and we give the proofs for the convergence results related to our smoothing argument.

As an application of (1.14) we mention that one can follow similar arguments as in [11] to prove the convergence of the Multiscale Finite Element method proposed by T. Hou and X. Wu for the general case of nonsmooth coefficients.

For a complete asymptotic analysis of the second order boundary layer corrector φ_ε defined in (1.10), together with new second order error estimates and their applications, we refer the reader to [17].

2. First order error estimates

The main result of this section is

Theorem 2.1. *Let u_ε, u_0, u_1, and β_ε be defined as in Section 1. Then we have*

$$\left\|u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right) + \varepsilon\beta_\varepsilon(\cdot)\right\|_{H_0^1(\Omega)} \leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}.$$

The proof of the theorem will be done in several steps:

Step 1. The first step is to consider the mollified coefficient matrix $(A_{ij}^n)_{i,j=1}^N$, defined in the Appendix, with the properties $\|A_{ij}^n\|_{L^\infty} \leqslant \|A_{ij}\|_{L^\infty}$, (A_{ij}^n) is a Y-periodic matrix, and

$$A_{ij}^n \to A_{ij} \quad \text{in } L^p(Y) \text{ for } 1 \leqslant p < \infty. \tag{2.1}$$

For these coefficients the corresponding functions u_ε^n, χ_j^n, u_1^n, and β_ε^n defined similarly as in Section 1, (1.1), (1.3), (1.12), and (1.14), respectively, satisfy (see Appendix):

$$\begin{aligned} &\chi_j^n \rightharpoonup \chi_j && \text{in } H^1_{\mathrm{per}}(Y),\\ &u_\varepsilon^n \overset{n}{\rightharpoonup} u_\varepsilon && \text{in } H_0^1(\Omega),\\ &u_1^n \rightharpoonup u_1 && \text{in } H^1(\Omega),\\ &\beta_\varepsilon^n \overset{n}{\rightharpoonup} \beta_\varepsilon && \text{in } H^1(\Omega). \end{aligned} \tag{2.2}$$

Step 2. Next we define

$$v_0^n(x,y) = A^n(y)Q_\varepsilon(\nabla_x u_0) + A^n(y)\nabla_y u_1^n(x,y) \tag{2.3}$$

or equivalently

$$\left(v_0^n(x,y)\right)_i = \left(A_{ij}^n(y) + A_{ik}^n(y)\frac{\partial \chi_j^n}{\partial y_k}\right)Q_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right). \tag{2.4}$$

By using the definition of χ_j^n we have $\nabla_y \cdot v_0^n = 0$. Let us denote by

$$\left(C^n(y)\right)_{ij} = A_{ij}^n(y) + A_{ik}^n(y)\frac{\partial \chi_j^n}{\partial y_k}$$

and $\mathcal{A}_n^{\text{hom}} = M_Y(C^n(y))$. It can be seen that

$$\nabla_y \cdot \left(v_0^n - \mathcal{A}_n^{\text{hom}} Q_\varepsilon(\nabla_x u_0)\right) = 0. \tag{2.5}$$

Lemma 2.2. *There exists $q^n(x,\cdot) \in [W_{\text{per}}(Y)]^N$ such that $curl_y q^n = v_0^n - \mathcal{A}_n^{\text{hom}} Q_\varepsilon(\nabla_x u_0)$.*

Proof. Let $B^n(y) = C^n(y) - \mathcal{A}_n^{\text{hom}}$. We then have

$$v_0^n - \mathcal{A}_n^{\text{hom}} Q_\varepsilon(\nabla_x u_0) = B^n(y) Q_\varepsilon(\nabla_x u_0). \tag{2.6}$$

We look for q^n of the form

$$q^n(x,y) = \phi^n(y) Q_\varepsilon(\nabla_x u_0),$$

where $\phi^n(y) = (\phi_{ij}^n(y))_{ij}$ with $\phi_{ij}^n(y) \in W_{\text{per}}(Y)$.

If we denote by B_l^n the vector $B_l^n = (B_{il}^n)_i \in [L_{\text{per}}^2(Y)]^N$ we observe that $\nabla_y \cdot B_l^n = 0$. Hence from the Theorem 3.4 in Girault and Raviart [9] adapted to the periodic case, the vectors $\phi_l^n = (\phi_{il}^n)_i \in [W_{\text{per}}(Y)]^N$ are determined as the solutions to

$$\text{curl}_y\, \phi_l^n = B_l^n \quad \text{and} \quad \text{div}_y\, \phi_l^n = 0. \tag{2.7}$$

Obviously we have

$$\text{curl}_y\, q^n(x,y) = v_0^n - \mathcal{A}_n^{\text{hom}} Q_\varepsilon(\nabla_x u_0). \qquad \square \tag{2.8}$$

Remark 2.3. From (2.2) it can be immediately seen that B^n is bounded independently of n in $[L^2(Y)]^{N\times N}$ and using the Appendix we have

$$B^n \rightharpoonup B \quad \text{in } \left[L^2(Y)\right]^{N\times N},$$

where B has an identical form as B^n and it can be easily determined from the above limit. This together with (2.7) and Theorem 3.9 in [9] adapted for the periodic case implies that ϕ^n is bounded independently of n in $(W_{\text{per}}(Y))^{N\times N}$ and we have

$$\phi_l^n \rightharpoonup \phi_l \quad \text{in } \left[W_{\text{per}}(Y)\right]^N, \quad \text{where}$$

$$\text{curl}_y\, \phi_l = B_l \quad \text{and} \quad \text{div}_y \phi_l = 0; \quad \text{for every } l \in \{1,\dots,N\}. \tag{2.9}$$

Step 3. Next we define

$$v_1^n(x,y) = \operatorname{curl}_x q^n(x,y)$$

and using Lemma 2.2 we have

$$\nabla_y \cdot v_1^n = -\nabla_x \cdot \operatorname{curl}_y q^n = -\nabla_x \cdot v_0^n - f_\varepsilon^n, \tag{2.10}$$

where

$$f_\varepsilon^n = -\nabla_x \cdot \left(\mathcal{A}_n^{\text{hom}} Q_\varepsilon(\nabla_x u_0)\right).$$

We define

$$z_\varepsilon^n(x) = u_\varepsilon^n(x) - u_0(x) - \varepsilon u_1^n\left(x, \frac{x}{\varepsilon}\right), \tag{2.11}$$

$$\mu_\varepsilon^n(x) = A^n\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^n(x) - v_0^n\left(x, \frac{x}{\varepsilon}\right) - \varepsilon v_1^n\left(x, \frac{x}{\varepsilon}\right). \tag{2.12}$$

From the above definitions, similarly as in [16] we obtain

$$\begin{aligned} A^n\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon^n(x) - \mu_\varepsilon^n(x) &= \varepsilon\left(v_1^n\left(x, \frac{x}{\varepsilon}\right) - A^n\left(\frac{x}{\varepsilon}\right)\nabla_x u_1^n\left(x, \frac{x}{\varepsilon}\right)\right) \\ &\quad + A^n\left(\frac{x}{\varepsilon}\right)\left(Q_\varepsilon(\nabla_x u_0) - \nabla_x u_0\right). \end{aligned} \tag{2.13}$$

Next, we will prove that the L^2 norm of (2.13) is of order ε. In order to do this we will show that $v_1^n(x, \frac{x}{\varepsilon})$ and $A^n(\frac{x}{\varepsilon})\nabla_x u_1(x, \frac{x}{\varepsilon})$ are bounded in L^2 independently of n and ε. We have the following estimate

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ as before. For any $\psi \in L^2(Y)$, Y-periodic, we have*

$$\left\| \nabla_x Q_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right)\psi\left(\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leqslant C\|u_0\|_{H^2(\Omega)}\|\psi\|_{L^2(Y)}.$$

Proof. We recall the definition of Q_ε

$$Q_\varepsilon(\phi)(x) = \sum_{i_1,\dots,i_N} M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon i)\bar{x}_{1,\xi}^{i_1} \cdot \dots \cdot \bar{x}_{N,\xi}^{i_N}, \quad \xi = \left[\frac{x}{\varepsilon}\right]$$

for any $x \in \tilde{\Omega}_\varepsilon$, with $\tilde{\Omega}_\varepsilon$ defined in the Appendix, and any $\phi \in L^2(\tilde{\Omega}_{\varepsilon,2})$ with $\tilde{\Omega}_{\varepsilon,2} = \{x \in \Omega; \operatorname{dist}(x,\Omega) < 2\varepsilon\}$, where $i = (i_1, \dots, i_N) \in \{0,1\}^N$ and

$$\bar{x}_{k,\xi}^{i_k} = \begin{cases} \dfrac{x_k - \varepsilon\xi_k}{\varepsilon} & \text{if } i_k = 1, \\ 1 - \dfrac{x_k - \varepsilon\xi_k}{\varepsilon} & \text{if } i_k = 0, \end{cases} \qquad x \in \varepsilon(\xi + Y).$$

Note that, on each cube $\varepsilon\xi + \varepsilon Y$ the first order derivative Q_ε takes the form

$$\frac{\partial}{\partial x_1} Q_\varepsilon(\phi)(x) = \sum_{i_2,\ldots,i_N} \frac{M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon(1, i_2, \ldots, i_n)) - M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon(0, i_2, \ldots, i_n))}{\varepsilon} \times \bar{x}_{2,\xi}^{i_2} \cdot \cdots \cdot \bar{x}_{N,\xi}^{i_N},$$

where $\xi = [\frac{x}{\varepsilon}]$. Therefore we have

$$\begin{aligned}
&\int_{\varepsilon\xi+\varepsilon Y} \left|\frac{\partial}{\partial x_1} Q_\varepsilon(\phi)(x)\right|^2 \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x \\
&\leqslant 2^{N-1} \sum_{i_2,\ldots,i_N} \left|\frac{M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon(1, i_2, \ldots, i_n)) - M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon(0, i_2, \ldots, i_n))}{\varepsilon}\right|^2 \\
&\quad \times \int_{\varepsilon\xi+\varepsilon Y} \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x \\
&= 2^{N-1} \sum_{i_2,\ldots,i_N} \left|\frac{M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon(1, i_2, \ldots, i_n)) - M_Y^\xi(\phi)(\varepsilon\xi + \varepsilon(0, i_2, \ldots, i_n))}{\varepsilon}\right|^2 \varepsilon^N \|\psi\|^2_{L^2(Y)}.
\end{aligned} \tag{2.14}$$

Using the Schwartz inequality in (2.14) together with the definition of the mean M_Y^ξ we obtain

$$\begin{aligned}
&\int_{\varepsilon\xi+\varepsilon Y} \left|\frac{\partial}{\partial x_1} Q_\varepsilon(\phi)(x)\right|^2 \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x \\
&\leqslant C\|\psi\|^2_{L^2(Y)} \sum_{i_2,\ldots,i_N} \int_{\varepsilon\xi+\varepsilon Y} \left|\frac{\phi(x + \varepsilon(1, i_2, \ldots, i_n)) - \phi(x + \varepsilon(0, i_2, \ldots, i_n))}{\varepsilon}\right|^2 \mathrm{d}x \\
&\leqslant C\|\psi\|^2_{L^2(Y)} \sum_{i_2,\ldots,i_N} \int_{\varepsilon\xi+\varepsilon Y} \left(\left|\frac{\phi(x + \varepsilon(1, i_2, \ldots, i_n)) - \phi(x)}{\varepsilon}\right|^2 \right. \\
&\quad \left. + \left|\frac{\phi(x + \varepsilon(0, i_2, \ldots, i_n)) - \phi(x)}{\varepsilon}\right|^2\right) \mathrm{d}x.
\end{aligned}$$

After summing the above inequalities over $\xi \in \{\xi \in \mathbb{Z}^N; (\varepsilon\xi + \varepsilon Y) \cap \Omega \neq \emptyset\}$, and using the inequality between the differential quotients and the gradient we obtain

$$\int_\Omega \left|\frac{\partial}{\partial x_1} Q_\varepsilon(\phi)(x)\right|^2 \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 \leqslant C\|\psi\|^2_{L^2(Y)} \|\nabla\phi\|^2_{L^2(\tilde{\Omega}_{\varepsilon,2})}.$$

This yields

$$\int_\Omega |\nabla_x Q_\varepsilon(\phi)|^2 \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 \leqslant C\|\psi\|^2_{L^2(Y)} \|\nabla\phi\|^2_{L^2(\tilde{\Omega}_{\varepsilon,2})}.$$

Recall that, u_0 denotes the stable extension of u_0 to the whole space. Therefore, choosing ϕ to be the partial derivative of u_0 the conclusion of the lemma follows.

Proof of Theorem 2.1. By applying Lemma 2.4 we can see that

$$\left\|A^n\left(\frac{x}{\varepsilon}\right)\nabla_x u_1^n\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leqslant C\|\chi_j^n\|^2_{L^2(Y)}\|u_0\|_{H^2(\Omega)} \leqslant C\|u_0\|_{H^2(\Omega)} \tag{2.15}$$

and using Remark 2.3 we obtain

$$\left\|v_1^n\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leqslant C\left(\sum_{i,j}\|\phi_{ij}^n\|^2_{L^2(Y)}\right)^{\frac{1}{2}}\|u_0\|_{H^2(\Omega)} \leqslant C\|u_0\|_{H^2(\Omega)}. \tag{2.16}$$

Using (2.15), (2.16) and the properties of Q_ε we obtain the following estimate for the left hand side of (2.13):

$$\left\|A^n\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon^n(x) - \mu_\varepsilon^n(x)\right\|_{L^2(\Omega)} \leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}. \tag{2.17}$$

For $g \in L^2(\Omega)$ we define $w_\varepsilon^n \in H_0^1(\Omega)$ solution of the following problem

$$-\nabla\cdot\left(A^n\left(\frac{x}{\varepsilon}\right)\nabla w_\varepsilon^n\right) = g \quad \text{in } \Omega, \qquad w_\varepsilon^n = 0 \quad \text{on } \partial\Omega. \tag{2.18}$$

Obviously we have

$$\|w_\varepsilon^n\|_{H_0^1(\Omega)} \leqslant \|g\|_{H^{-1}(\Omega)}. \tag{2.19}$$

Using $z_\varepsilon^n + \varepsilon\beta_\varepsilon^n$ as a test function in (2.18), with β_ε defined by (1.14) we obtain

$$\int_\Omega (z_\varepsilon^n + \varepsilon\beta_\varepsilon^n) g\,\mathrm{d}x = \int_\Omega A^n\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon^n\cdot\nabla w_\varepsilon^n\,\mathrm{d}x. \tag{2.20}$$

The right hand side can be estimated as follows

$$\begin{aligned}\int_\Omega A^n\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon^n\cdot\nabla w_\varepsilon^n\,\mathrm{d}x &= \int_\Omega\left(A^n\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon^n - \mu_\varepsilon^n\right)\cdot\nabla w_\varepsilon^n\,\mathrm{d}x - \int_\Omega(\nabla\cdot\mu_\varepsilon^n)w_\varepsilon^n\,\mathrm{d}x\\ &\leqslant \left\|A^n\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon^n - \mu_\varepsilon^n\right\|_{L^2(\Omega)}\|w_\varepsilon^n\|_{H_0^1(\Omega)} + \|\nabla\cdot\mu_\varepsilon^n\|_{H^{-1}(\Omega)}\|w_\varepsilon^n\|_{H_0^1(\Omega)}.\end{aligned} \tag{2.21}$$

We note here that $\nabla\cdot\mu_\varepsilon^n \in L^2(\Omega)$; indeed:

$$\nabla\cdot\mu_\varepsilon^n(x) = \nabla\cdot\left(A^n\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^n(x)\right) - \nabla_x\cdot v_0^n\left(x,\frac{x}{\varepsilon}\right) - \frac{1}{\varepsilon}\nabla_y\cdot v_0^n\left(x,\frac{x}{\varepsilon}\right)$$

$$- \nabla_x \cdot v_1^n\left(x, \frac{x}{\varepsilon}\right) - \nabla_y \cdot v_1^n\left(x, \frac{x}{\varepsilon}\right)$$
$$= -f(x) - \nabla_x \cdot \left(\mathcal{A}_n^{\text{hom}} Q_\varepsilon(\nabla_x u_0)\right).$$

To estimate the H^{-1} norm of $\nabla \cdot \mu_\varepsilon^n$ we consider $\phi \in H_0^1(\Omega)$ and

$$\begin{aligned}\int_\Omega (\nabla \cdot \mu_\varepsilon^n)\phi(x)\,\mathrm{d}x &= \int_\Omega (\mathcal{A}^{\text{hom}} Q_\varepsilon(\nabla u_0) - \mathcal{A}^{\text{hom}}\nabla u_0)\nabla\phi\,\mathrm{d}x + \int_\Omega (\mathcal{A}_n^{\text{hom}} - \mathcal{A}^{\text{hom}})Q_\varepsilon(\nabla u_0)\nabla\phi\,\mathrm{d}x \\ &\leqslant C\|\nabla u_0 - Q_\varepsilon(\nabla u_0)\|_{L^2(\Omega)}\|\phi\|_{H_0^1(\Omega)} \\ &\quad + \|\phi\|_{H_0^1(\Omega)}\|(\mathcal{A}_n^{\text{hom}} - \mathcal{A}^{\text{hom}})Q_\varepsilon(\nabla u_0)\|_{L^2(\Omega)} \\ &\leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}\|\phi\|_{H_0^1(\Omega)} + K_n\|\phi\|_{H_0^1(\Omega)}\|u_0\|_{H^1(\Omega)},\end{aligned} \tag{2.22}$$

where we used the properties of Q_ε and $K_n \doteq |\mathcal{A}^{\text{hom}} - \mathcal{A}_n^{\text{hom}}|$.

Therefore we proved that

$$\|\nabla \cdot \mu_\varepsilon^n\|_{H^{-1}(\Omega)} \leqslant C\varepsilon\|u_0\|_{H^2(\Omega)} + K_n\|u_0\|_{H^1(\Omega)}. \tag{2.23}$$

Thus (2.17) and (2.23) used in (2.22) imply

$$\begin{aligned}\left|\int_\Omega (z_\varepsilon^n + \varepsilon\beta_\varepsilon^n)g\,\mathrm{d}x\right| &\leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}\|w_\varepsilon^n\|_{H_0^1(\Omega)} + CK_n\|w_\varepsilon^n\|_{H_0^1(\Omega)} \\ &\leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}\|g\|_{H^{-1}(\Omega)} + CK_n\|g\|_{H^{-1}(\Omega)},\end{aligned}$$

where we used (2.19). From the above inequality we have

$$\|z_\varepsilon^n + \varepsilon\beta_\varepsilon^n\|_{H_0^1(\Omega)} \leqslant C\varepsilon\|u_0\|_{H^2(\Omega)} + CK_n. \tag{2.24}$$

From (2.1) and (2.2) we have that $K_n \to 0$ as $n \to \infty$. Using the Appendix we can pass to the limit when $n \to \infty$ in (2.24) and from (2.2) we get

$$\|z_\varepsilon + \varepsilon\beta_\varepsilon\|_{H_0^1(\Omega)} \leqslant C\varepsilon\|u_0\|_{H^2(\Omega)}$$

which is exactly what needs to be proved. □

3. The error estimate in the L^2 norm

In this section we will look for minimal assumptions on u_0 needed to prove the classical error estimates (1.9) in the case of non-smooth coefficients.

The L^2-norm of

$$u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon w_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \varepsilon\theta_\varepsilon(\cdot) \tag{3.1}$$

with $w_1(\cdot, \frac{\cdot}{\varepsilon})$ as defined in (1.4), can be estimated with additional assumptions. In [16], Moscow and Vogelius proved the estimate (1.9) assuming that $u_0 \in H^3(\Omega)$ and $A \in C^\infty(Y)$. In [21], Sarkis and Versieux prove an estimate of order ε^2, under the assumptions that $u_0 \in W^{3,p}(\Omega)$ and $\chi_j, \chi_{ij} \in W^{1,q}_{\text{per}}(Y)$ for $p, q > N$ where $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{2}$.

We extend in this section the previous results, by only requiring that $u_0 \in W^{3,p}(\Omega)$ for $p > N$, to prove (1.9) in the case of nonsmooth coefficients. In order to do this we need to introduce the second order cell problems. Therefore, let $\chi^n_{ij} \in W_{\text{per}}(Y)$ denote the solutions of

$$\nabla_y \cdot (A^n \nabla_y \chi^n_{ij}) = b^n_{ij} - M_Y(b^n_{ij}), \tag{3.2}$$

where

$$b^n_{ij} = -A^n_{ij} - A^n_{ik}\frac{\partial \chi^n_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A^n_{ik}\chi^n_j)$$

and $M_Y(\cdot)$ is the average on Y. From Corollary A.9, in the Appendix

$$|\nabla_y \chi^n_{ij}|_{L^2(Y)} < C \quad \text{and} \quad \chi^n_{ij} \rightharpoonup \chi_{ij} \quad \text{in } W_{\text{per}}(Y), \quad \forall i, j \in \{1, \ldots, N\},$$

where

$$\int_Y A(y)\nabla_y \chi_{ij} \nabla_y \psi \, dy = (b_{ij} - M_Y(b_{ij}), \psi)_{(W_{\text{per}}(Y),(W_{\text{per}}(Y))')}$$

for any $\psi \in W_{\text{per}}(Y)$ and with

$$b_{ij} = -A_{ij} - A_{ik}\frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j).$$

Next, we will only assume that $u_0 \in W^{3,p}(\Omega)$ with $N < p \leqslant \infty$ to prove the estimate of order ε^2 for (3.1). Indeed we have,

Theorem 3.1. *Let u_ε, u_0, u_1 and θ_ε defined as in Section 2. If $u_0 \in W^{3,p}(\Omega)$, $N < p \leqslant \infty$ we have*

$$\left\| u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon w_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \varepsilon\theta_\varepsilon(\cdot) \right\|_{L^2(\Omega)} \leqslant C\varepsilon^2 \|u_0\|_{W^{3,p}(\Omega)}. \tag{3.3}$$

Proof. For the sake of simplicity we will consider only the case when $N = 3$, the two dimensional case being similar. As in the previous section we can assume the smooth coefficients A^n (see (A.2)), and follow the same ideas as in [16] to define

$$u^n_2(x, y) = \chi^n_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x).$$

For $p > N$ we have that

$$\left\| \nabla_x u^n_2\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leqslant \left\| \chi^n_{ij}\left(\frac{\cdot}{\varepsilon}\right) \right\|_{L^{\frac{2p}{p-2}}(\Omega)} \left\| \nabla_x \frac{\partial^2 u_0}{\partial x_j \partial x_i} \right\|_{L^p(\Omega)}$$

and using a change in variables and the inequality (A.11) in the Appendix, we obtain

$$\left\|\nabla_x u_2^n\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\|^2_{L^2(\Omega)} \leqslant C\sum_{i,j}\left\|\nabla_x \frac{\partial^2 u_0}{\partial x_j \partial x_i}\right\|^2_{L^p(\Omega)} \leqslant C\|u_0\|^2_{W^{3,p}(\Omega)}. \tag{3.4}$$

As in [16] we will define

$$\left(v_*^n(x,y)\right)_k = A_{ki}^n(y)\chi_j^n(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A_{kl}^n(y)\frac{\partial \chi_{ij}^n}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j \partial x_i}. \tag{3.5}$$

Following similar arguments we can observe that $\nabla_x \cdot M_Y(v_*^n) = 0$. By introducing

$$R_{ki}^j = M_Y\left(A_{ki}^n \chi_j^n + A_{kl}^n \frac{\partial \chi_{ij}^n}{\partial y_l}\right).$$

Consider $\alpha_{ij}^n \in [L^2(Y)]^3$ defined by,

$$\alpha_{ij}^n = \begin{pmatrix} A_{1i}^n\chi_j^n + A_{1l}^n\frac{\partial\chi_{ij}^n}{\partial y_l} - R_{1i}^j \\ A_{2i}^n\chi_j^n + A_{2l}^n\frac{\partial\chi_{ij}^n}{\partial y_l} - R_{2i}^j \\ A_{3i}^n\chi_j^n + A_{3l}^n\frac{\partial\chi_{ij}^n}{\partial y_l} - R_{3i}^j \end{pmatrix} + \beta_{ij}^n$$

with

$$\begin{aligned}
\beta_{1j}^n &= \left(0, -\phi_{3j}^n, \phi_{2j}^n\right)^{\mathrm{T}}, \\
\beta_{2j}^n &= \left(\phi_{3j}^n, 0, -\phi_{1j}^n\right)^{\mathrm{T}}, \quad \text{for } j \in \{1,2,3\}, \\
\beta_{3j}^n &= \left(-\phi_{2j}^n, \phi_{1j}^n, 0\right)^{\mathrm{T}},
\end{aligned}$$

where T denotes the transpose and ϕ_{ij}^n are defined at (2.7). Using the symmetry of the matrix A we observe that the vectors α_{ij}^n defined above are divergence free with zero average over Y. This implies that there exists $\psi_{ij}^n \in [W_{\mathrm{per}}(Y)]^3$, (see Theorem 3.4, [9] adapted to the periodic case) so that

$$\operatorname{curl}_y \psi_{ij}^n = \alpha_{ij}^n \quad \text{and} \quad \nabla \cdot \psi_{ij}^n = 0 \quad \text{for any } i,j \in \{1,2,3\}. \tag{3.6}$$

From Corollaries A.5 and A.9 in Appendix and we observe that

$$\alpha_{ij}^n \rightharpoonup \alpha_{ij} \quad \text{in } \left[L^2(Y)\right]^3, \tag{3.7}$$

where the form of α_{ij} is identical with that of α_{ij}^n and can be obviously obtain from (3.7). Using the above convergence result and Theorem 3.9 from [9] adapted to the periodic case, we obtain that

$$\psi_{ij}^n \rightharpoonup \psi_{ij}, \quad \text{in } W_{\mathrm{per}}(Y) \quad \text{for any } i,j \in \{1,2,3\}$$

and ψ_{ij} satisfy

$$\operatorname{curl}_y \psi_{ij} = \alpha_{ij} \quad \text{and} \quad \nabla_y \cdot \psi_{ij} = 0 \quad \text{for } i, j \in \{1, 2, 3\}. \tag{3.8}$$

Next define $p^n(x, y) = \psi^n_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$ and $v^n_2(x, y) = \operatorname{curl}_x p^n(x, y)$. Obviously we have that $\nabla_x \cdot v^n_2 = 0$. It is also easy to check, that $\nabla_y \cdot v^n_2 = -\nabla_x \cdot v^n_*$, (see [16] for example). We set

$$w^n_1(x, y) = \chi^n_j(y)\frac{\partial u_0}{\partial x_j}(x),$$

$$r^n_0(x, y) = A^n(y)\nabla_x u_0 + A^n(y)\nabla_y w^n_1(x, y),$$

$$\psi^n_\varepsilon(x) = u^n_\varepsilon(x) - u_0(x) - \varepsilon w^n_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon^2 u^n_2\left(x, \frac{x}{\varepsilon}\right), \tag{3.9}$$

$$\xi^n_\varepsilon(x) = A^n\left(\frac{x}{\varepsilon}\right)\nabla u^n_\varepsilon - r^n_0\left(x, \frac{x}{\varepsilon}\right) - \varepsilon v^n_*\left(x, \frac{x}{\varepsilon}\right) - \varepsilon^2 v^n_2\left(x, \frac{x}{\varepsilon}\right). \tag{3.10}$$

As in [16] we can write

$$A^n\left(\frac{x}{\varepsilon}\right)\nabla\psi^n_\varepsilon(x) - \xi^n_\varepsilon(x) = \varepsilon^2\left(v^n_2\left(x, \frac{x}{\varepsilon}\right) - A^n\left(\frac{x}{\varepsilon}\right)\nabla_x u^n_2\left(x, \frac{x}{\varepsilon}\right)\right). \tag{3.11}$$

We use next, as in (3.4), the inequality (A.11) to obtain

$$\left\| v^n_2\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leqslant C\|u_0\|_{W^{3,p}(\Omega)}. \tag{3.12}$$

Using (3.4), (3.11) and (3.12) we get

$$\left\| A^n\left(\frac{x}{\varepsilon}\right)\nabla\psi^n_\varepsilon(x) - \xi^n_\varepsilon(x)\right\|_{L^2(\Omega)} \leqslant C\varepsilon^2\|u_0\|_{W^{3,p}(\Omega)}. \tag{3.13}$$

Similarly as in [16] we have that $\nabla \cdot \xi^n_\varepsilon(x) = 0$. Let us define φ^n_ε as solution of

$$\nabla \cdot \left(A^n\left(\frac{x}{\varepsilon}\right)\nabla\varphi^n_\varepsilon\right) = 0 \quad \text{in } \Omega, \qquad \varphi^n_\varepsilon = u^n_2\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \tag{3.14}$$

Using again Corollary A.11 in Appendix, we have that $\varphi^n_\varepsilon \rightharpoonup \varphi_\varepsilon$ in $H^1(\Omega)$ where φ_ε is the solution of

$$\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla\varphi_\varepsilon\right) = 0 \quad \text{in } \Omega, \qquad \varphi_\varepsilon = u_2\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \tag{3.15}$$

Then,

$$\|\varphi_\varepsilon\|_{L^2(\Omega)} \leqslant C\left\| u_2\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^\infty(\partial\Omega)} \leqslant C\|\chi_{ij}\|_{L^\infty(Y)}\|u_0\|_{W^{3,p}(\Omega)} \leqslant C\|u_0\|_{W^{3,p}(\Omega)}, \tag{3.16}$$

where we used [13] for the L^∞ bound on χ_{ij}. Next, similarly as in [16] we have

$$\left\|u_\varepsilon^n(\cdot) - u_0(\cdot) - \varepsilon w_1^n\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \varepsilon\theta_\varepsilon^n(\cdot) - \varepsilon^2 u_2^n\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \varepsilon^2\varphi_\varepsilon^n\right\|_{L^2(\Omega)} \leqslant C\varepsilon^2\|u_0\|_{W^{3,p}(\Omega)}$$

and passing to the limit when $n \to \infty$ using triangle inequality, (3.4) and (3.16) we get (3.3). □

Remark that the assumption that $u_0 \in W^{3,p}$, with $p > N$ was necessary for the estimate (3.16).

Appendix A

In this section we will present the proofs for some of the results used in the previous sections and which were not included in the main body of the chapter for the sake of clarity of the exposition.

A.1. Definition and properties of the unfolding operator

Let $\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N; (\varepsilon\xi + \varepsilon Y) \cap \Omega \neq \emptyset\}$ and define

$$\tilde{\Omega}_\varepsilon = \bigcup_{\xi\in\Xi_\varepsilon} (\varepsilon\xi + \varepsilon Y). \tag{A.1}$$

Let us also consider $H^1_{\text{per}}(Y)$ to be the closure of $C^\infty_{\text{per}}(Y)$ in the H^1 norm, where $C^\infty_{\text{per}}(Y)$ is the subset of $C^\infty(\mathbb{R}^N)$ of Y-periodic functions, and

$$W_{\text{per}}(Y) \doteq \left\{v \in H^1_{\text{per}}(Y)/\mathbb{R}, \frac{1}{|Y|}\int_Y v\,\mathrm{d}y = 0\right\}$$

(see [6] for properties).

Next we present several very useful technical inequalities obtained in [10].

Proposition A.1. *We have*:

(1) $\left\|\psi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}_\varepsilon)} + \left\|\nabla_y\psi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}_\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}\|\psi\|_{H^1(Y)}$ *for every* $\psi \in H^1_{\text{per}}(Y)$.

(2) $\|M_Y^\varepsilon(v)\|_{L^2(\Omega)} \leqslant \|v\|_{L^2(\tilde{\Omega}_\varepsilon)}$ *for any* $v \in L^2(\tilde{\Omega}_\varepsilon)$.

(3) $$\begin{cases} \|v - M_Y^\varepsilon(v)\|_{L^2(\Omega)} \leqslant C\varepsilon\|\nabla v\|_{[L^2(\Omega)]^N}, \\ \|v - \mathcal{T}_\varepsilon(v)\|_{L^2(\Omega\times Y)} \leqslant C\varepsilon\|\nabla v\|_{[L^2(\Omega)]^N}, \\ \|Q_\varepsilon(v) - M_Y^\varepsilon(v)\|_{L^2(\Omega)} \leqslant C\varepsilon\|\nabla v\|_{[L^2(\Omega)]^N} \quad \textit{for any } v \in H^1(\Omega). \end{cases}$$

(4) $\left\|Q_\varepsilon(v)\psi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\tilde{\Omega}_{\varepsilon,2})}\|\psi\|_{L^2(Y)}$ *for any* $v \in L^2(\tilde{\Omega}_{\varepsilon,2})$ *and* $\psi \in L^2(Y)$.

A.2. Convergence results and the smoothing argument

Let $m_n \in C^\infty$ be the standard mollifying sequence, i.e., $0 < m_n \leqslant 1$, $\int_{\mathbb{R}^N} m_n \, \mathrm{d}z = 1$, $\mathrm{sppt}(m_n) \subset B(0, \frac{1}{n})$. Define $A^n(y) = (m_n * A)(y)$, where a has been defined in the Introduction (see (1.1)). We have:

(1) $A^n - Y$-periodic matrix,
(2) $|A^n|_{L^\infty} < |A|_{L^\infty}$,
(3) $A^n \to A \quad \text{in } L^p \quad \text{for any } p \in (1, \infty)$. (A.2)

From (A.2) we have that $c|\xi|^2 \leqslant A^n_{ij}(y)\xi_i\xi_j \leqslant C|\xi|^2 \; \forall \xi \in \mathbb{R}^N$. Define

$$(A_n^{\mathrm{hom}})_{ij} = M_Y\left(A^n_{ij}(y) + A^n_{ik}(y)\frac{\partial \chi^n_j}{\partial y_k}\right), \tag{A.3}$$

where $M_Y(\cdot) = \frac{1}{|Y|}\int_Y \cdot \, \mathrm{d}y$ and $\chi^n_j \in W_{\mathrm{per}}(Y)$ are the solutions of the local problem

$$-\nabla_y \cdot \left(A(y)(\nabla \chi^n_j + e_j)\right) = 0. \tag{A.4}$$

Next we present a few important convergence results needed in the smoothing argument developed in the previous sections.

Lemma A.2. *Let $f_n, f \in H^{-1}(\Omega)$ with $f_n \rightharpoonup f$ in $H^{-1}(\Omega)$ and let $b^n, b \in L^\infty(\Omega)$, with*

$$c|\xi|^2 \leqslant b^n_{ij}(y)\xi_i\xi_j \leqslant C|\xi|^2,$$
$$c|\xi|^2 \leqslant b_{ij}(y)\xi_i\xi_j \leqslant C|\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and

$$b^n \to b \quad \text{in } L^2(\Omega).$$

Consider $\zeta_n \in H^1_0(\Omega)$ the solution of

$$\int_\Omega b^n(x)\nabla\zeta_n\nabla\psi \, \mathrm{d}x = \int_\Omega f_n\psi \, \mathrm{d}x$$

for any $\psi \in H^1_0(\Omega)$. Then we have

$$\zeta_n \rightharpoonup \zeta \quad \text{in } H^1_0(\Omega)$$

and ζ verifies

$$\int_\Omega b(x)\nabla\zeta\nabla\psi \, \mathrm{d}x = \int_\Omega f\psi \, \mathrm{d}x \quad \text{for any } \psi \in H^1_0(\Omega).$$

Proof. Immediately can be observed that

$$\|\zeta_n\|_{H_0^1(\Omega)} \leqslant C$$

and therefore there exists ζ such that for a subsequence still denoted by n we have

$$\zeta_n \rightharpoonup \zeta \quad \text{in } H_0^1(\Omega). \tag{A.5}$$

For any smooth $\psi \in H_0^1(\Omega)$ easily it can be seen that

$$\int_\Omega b^n(x)\nabla\zeta_n\nabla\psi\,\mathrm{d}x \to \int_\Omega b(x)\nabla\zeta\nabla\psi\,\mathrm{d}x$$

and this implies the statement of the lemma. Due to the uniqueness of φ one can see that the limit (A.5) holds on the entire sequence. □

Remark A.3. Using similar arguments it can be proved that the results of Lemma A.2 hold true if we replace the Dirichlet boundary conditions with periodic boundary conditions.

Corollary A.4. *Let $u_\varepsilon^n \in H_0^1(\Omega)$ be the solution of*

$$\begin{cases} -\nabla\cdot\left(A^n\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon^n\right) = f & \text{in } \Omega,\\ u_\varepsilon^n = 0 & \text{on } \partial\Omega. \end{cases}$$

We then have

$$u_\varepsilon^n \overset{n}{\rightharpoonup} u_\varepsilon \quad \text{in } H_0^1(\Omega),$$

where u_ε verifies

$$\begin{cases} -\nabla\cdot\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f & \text{in } \Omega,\\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Using (A.2) we have that

$$A^n\left(\frac{x}{\varepsilon}\right) \overset{n}{\rightharpoonup} A\left(\frac{x}{\varepsilon}\right) \quad \text{in } L^2(\Omega)$$

and the statement follows immediately from Remark A.3. □

Corollary A.5. *For $j \in \{1,\ldots,N\}$, let $\chi_j^n \in W_{\mathrm{per}}(Y)$ be the solution of*

$$-\nabla_y\cdot\left(A^n(y)(\nabla\chi_j^n + e_j)\right) = 0, \tag{A.6}$$

where $\{e_j\}_j$ denotes the canonical basis of $\mathbb{R}^N$. Then we have

$$\chi_j^n \rightharpoonup \chi_j \quad \text{in } W_{\text{per}}(Y),$$

where $\chi_j \in W_{\text{per}}(Y)$ verifies

$$-\nabla_y \cdot \big(A(y)(\nabla \chi_j + e_j)\big) = 0.$$

Proof. From (A.2) we obtain

$$\frac{\partial}{\partial y_i} A_{ij}^n(y) \rightharpoonup \frac{\partial}{\partial y_i} A_{ij}(y) \quad \text{in } \big(W_{\text{per}}(Y)\big)'.$$

The statement of the remark follows then immediately from Remark A.3. □

Proposition A.6. *Let $v \in [H^1(\Omega)]^N$ be arbitrarily fixed and for every $j \in \{1, \dots, N\}$, let $\chi_j \in W_{\text{per}}(Y)$ be defined as in* (A.4), *and $\chi_j^n \in W_{\text{per}}(Y)$, for $j \in \{1, \dots, N\}$, to be the solutions of* (A.6).
Define

$$h_\varepsilon^n(x) = h^n\left(x, \frac{x}{\varepsilon}\right) = \chi_j^n\left(\frac{x}{\varepsilon}\right) v_j, \qquad h_\varepsilon(x) = h\left(x, \frac{x}{\varepsilon}\right) = \chi_j\left(\frac{x}{\varepsilon}\right) v_j,$$
$$g_\varepsilon^n(x) = g^n\left(x, \frac{x}{\varepsilon}\right) = \chi_j^n\left(\frac{x}{\varepsilon}\right) Q_\varepsilon(v_j), \qquad g_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) = \chi_j\left(\frac{x}{\varepsilon}\right) Q_\varepsilon(v_j).$$

We have that

(1) $g_\varepsilon^n \overset{n}{\rightharpoonup} g_\varepsilon$ *in* $H^1(\Omega)$.
(2) *If* $v \in [W^{1,p}(\Omega)]^N, p > N$, *then* $h_\varepsilon^n \overset{n}{\rightharpoonup} h_\varepsilon$ *in* $H^1(\Omega)$.

Proof. First note that applying Corollary A.5 to the sequence $\{\chi_j^n\}_n$ we have

$$\chi_j^n \overset{n}{\rightharpoonup} \chi_j \quad \text{in } W_{\text{per}}(Y). \tag{A.7}$$

Next we have

$$\left\| g^n\left(x, \frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega)}^2 = \int_\Omega \left(\chi_j^n\left(\frac{x}{\varepsilon}\right) Q_\varepsilon(v_j)\right)^2 \mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \left(\nabla_y \chi_j^n\left(\frac{x}{\varepsilon}\right) Q_\varepsilon(v_j)\right)^2 \mathrm{d}x$$
$$+ \int_\Omega \left(\chi_j^n\left(\frac{x}{\varepsilon}\right) \nabla_x Q_\varepsilon(v_j)\right)^2 \mathrm{d}x \tag{A.8}$$

and

$$\left\| h^n\left(x, \frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega)}^2 = \int_\Omega \left(\chi_j^n\left(\frac{x}{\varepsilon}\right) v_j\right)^2 \mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \left(\nabla_y \chi_j^n\left(\frac{x}{\varepsilon}\right) v_j\right)^2 \mathrm{d}x$$
$$+ \int_\Omega \left(\chi_j^n\left(\frac{x}{\varepsilon}\right) \nabla_x v_j\right)^2 \mathrm{d}x. \tag{A.9}$$

For the first convergence in Theorem A.6 we use that

$$\left\|\chi_j^n\left(\frac{x}{\varepsilon}\right)Q_\varepsilon(v_j)\right\|_{H^1(\Omega)} \leqslant C\|\chi_j^n\|_{W_{\mathrm{per}}(Y)}. \tag{A.10}$$

Next we can see that (A.8) imply that

$$\left\|g^n\left(x,\frac{x}{\varepsilon}\right)-g\left(x,\frac{x}{\varepsilon}\right)\right\|^2_{L^2(\Omega)}=\int_\Omega\left(\chi_j^n\left(\frac{x}{\varepsilon}\right)-\chi_j\left(\frac{x}{\varepsilon}\right)\right)^2\left(Q_\varepsilon(v_j)\right)^2\,\mathrm{d}x$$

and using (A.10) we obtain the desired result.

For the second convergence result in Proposition A.6 we will recall now a very important inequality (see [13], Chp. 2) to be used for our estimates.

For any $p>N$ we have

$$\|\phi\|_{L^{\frac{2p}{p-2}}(\Omega)} \leqslant c(p)\left(\|\phi\|_{L^2(\Omega)}+\|\nabla\phi\|^{\frac{N}{p}}_{L^2(\Omega)}\|\phi\|^{1-\frac{N}{p}}_{L^2(\Omega)}\right) \tag{A.11}$$

for any $\phi\in H^1(\Omega)$ and where $c(p)$ is a constant which depends only on q, N, Ω.

For $v\in[W^{1,p}(\Omega)]^N$ with $p>N$, using (A.7), the Sobolev embedding $W^{1,p}(\Omega)\subset L^\infty(\Omega)$ and (A.11) in (A.9) we obtain

$$\left\|h^n\left(x,\frac{x}{\varepsilon}\right)\right\|^2_{H^1(\Omega)}<C,$$

where the constant C above does not depend on n.

Next we can easily observe that

$$\left\|h^n\left(x,\frac{x}{\varepsilon}\right)-h\left(x,\frac{x}{\varepsilon}\right)\right\|^2_{L^2(\Omega)}=\int_\Omega\left(\chi_j^n\left(\frac{x}{\varepsilon}\right)-\chi_j\left(\frac{x}{\varepsilon}\right)\right)^2(v_j)^2\,\mathrm{d}x$$

and in either of the above cases, (A.7) and a few simple manipulations imply that

$$h^n\left(x,\frac{x}{\varepsilon}\right)\xrightarrow{n}h\left(x,\frac{x}{\varepsilon}\right)\quad\text{in } L^2(\Omega).$$

This together with the bound on the sequence $\{h^n(x,\frac{x}{\varepsilon})\}_n$ implies the statement of the proposition. □

The two convergence results in the next corollary will follow immediately from Proposition A.5.

Corollary A.7. *Let* $w^n_{1\varepsilon}(x)=w^n_1(x,\frac{x}{\varepsilon})=\chi^n_j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}$ *and* $u^n_{1\varepsilon}=u^n_1(x,\frac{x}{\varepsilon})=\chi^n_j(\frac{x}{\varepsilon})Q_\varepsilon(\frac{\partial u_0}{\partial x_j})$. *Then we have*

(1) *If* $u_0\in W^{3,p}(\Omega)$ *for* $p>N$,

$$w^n_{1\varepsilon}\xrightarrow{n}w_1\quad\textit{in } H^1(\Omega).$$

(2) *If* $u_0 \in H^2(\Omega)$,

$$u_{1\varepsilon}^n \overset{n}{\rightharpoonup} u_1 \quad in\ H^1(\Omega),$$

where w_1 *and* u_1 *were defined in* (1.4) *and* (1.12) *respectively.*

Corollary A.8. *Let* θ_ε^n *be the solution of*

$$-\nabla \cdot \left(A^n\left(\frac{x}{\varepsilon}\right)\nabla\theta_\varepsilon^n\right) = 0 \quad in\ \Omega, \qquad \theta_\varepsilon^n = w_1^n\left(x, \frac{x}{\varepsilon}\right) \quad on\ \partial\Omega \tag{A.12}$$

and β_ε^n *be the solution of*

$$-\nabla \cdot \left(A^n\left(\frac{x}{\varepsilon}\right)\nabla\beta_\varepsilon^n\right) = 0 \quad in\ \Omega, \qquad \beta_\varepsilon^n = u_1^n\left(x, \frac{x}{\varepsilon}\right) \quad on\ \partial\Omega. \tag{A.13}$$

We have that

(i) *if* $u_0 \in W^{3,p}(\Omega)$, $p > N$, *then*

$$\theta_\varepsilon^n \overset{n}{\rightharpoonup} \theta_\varepsilon \quad in\ H^1(\Omega).$$

(ii) *if* $u_0 \in H^2(\Omega)$, *then*

$$\beta_\varepsilon^n \overset{n}{\rightharpoonup} \beta_\varepsilon \quad in\ H^1(\Omega),$$

where θ_ε *and* β_ε *satisfies*

$$-\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla\theta_\varepsilon\right) = 0 \quad in\ \Omega, \qquad \theta_\varepsilon = w_1\left(x, \frac{x}{\varepsilon}\right) \quad on\ \partial\Omega \tag{A.14}$$

and

$$-\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla\beta_\varepsilon\right) = 0 \quad in\ \Omega, \qquad \beta_\varepsilon = u_1\left(x, \frac{x}{\varepsilon}\right) \quad on\ \partial\Omega. \tag{A.15}$$

Proof. Using Corollary A.7 and a few simple arguments one can simply show that

$$-\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla w_1^n\left(x, \frac{x}{\varepsilon}\right)\right) \overset{n}{\rightharpoonup} -\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla w_1\left(x, \frac{x}{\varepsilon}\right)\right) \quad \text{in } H^{-1}(\Omega)$$

and

$$-\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla w_1^n\left(x, \frac{x}{\varepsilon}\right)\right) \overset{n}{\rightharpoonup} -\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla w_1\left(x, \frac{x}{\varepsilon}\right)\right) \quad \text{in } H^{-1}(\Omega).$$

Homogenizing the data in the problems (A.12) and (A.13) and using Corollary A.7 and Lemma A.2 the statement follows immediately. □

Corollary A.9. *For any $i, j \in \{1, \dots, N\}$ let $\chi_{ij}^n \in W_{\text{per}}(Y)$ be the solutions of*:

$$\nabla_y \cdot (A^n \nabla_y \chi_{ij}^n) = b_{ij}^n - M_Y(b_{ij}^n), \tag{A.16}$$

where

$$b_{ij}^n = -A_{ij}^n - A_{ik}^n \frac{\partial \chi_j^n}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}^n \chi_j^n)$$

and $M_Y(\cdot)$ is the average on Y.

Then we have

$$\chi_{ij}^n \rightharpoonup \chi_{ij} \quad \text{in } W_{\text{per}}(Y) \text{ for any } i, j \in \{1, \dots, N\},$$

where χ_{ij} satisfies

$$\int_Y A(y) \nabla_y \chi_{ij} \nabla_y \psi \, dy = (b_{ij} - M_Y(b_{ij}), \psi)_{((W_{\text{per}}(Y))', W_{\text{per}}(Y))} \tag{A.17}$$

for any $\psi \in W_{\text{per}}(Y)$ and with

$$b_{ij} = -A_{ij} - A_{ik} \frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik} \chi_j).$$

Proof. For any $\psi \in W_{\text{per}}(Y)$, we have that,

$$\int_Y (b_{ij}^n - M_Y(b_{ij}^n)) \psi \, dy = \int_Y \left(-A_{ij}^n - A_{ik}^n \frac{\partial \chi_j^n}{\partial y_k}\right) \psi \, dy + (\mathcal{A}_n^{\text{hom}})_{ij} \int_Y \psi \, dy + \int_Y A_{ki}^n \chi_j^n \frac{\partial \psi}{\partial y_k} \, dy, \tag{A.18}$$

where we used $M_Y(b_{ij}^n) = -(\mathcal{A}_n^{\text{hom}})_{ij}$ (see [16]).

Using (A.2), (A.7), and simple manipulations we can prove that

$$A_{ik}^n \frac{\partial \chi_j^n}{\partial y_k} \rightharpoonup A_{ik} \frac{\partial \chi_j}{\partial y_k} \quad \text{in } L^2(Y) \tag{A.19}$$

and

$$A_{ik}^n \chi_j^n \rightharpoonup A_{ik} \chi_j \quad \text{in } L^2(Y). \tag{A.20}$$

From (A.19), (A.2) and (A.3) we have that

$$(\mathcal{A}_n^{\text{hom}})_{ij} \to \mathcal{A}_{ij}^{\text{hom}}. \tag{A.21}$$

Finally using (A.2), (A.7), (A.19) and (A.20) in (A.18) we obtain that

$$b_{ij}^n - M_Y(b_{ij}^n) \rightharpoonup b_{ij} - M_Y(b_{ij}) \quad \text{in } (W_{\text{per}}(Y))'.$$

This and Remark A.3 complete the proof of the statement. □

Remark A.10. We can easily observe that we have

$$A^n_{ij}\chi^n_{ij} \overset{n}{\rightharpoonup} A_{ij}\chi_{ij}, \qquad A^n_{ij}\frac{\partial \chi^n_{ij}}{\partial y_k} \overset{n}{\rightharpoonup} A_{ij}\frac{\partial \chi_{ij}}{\partial y_k} \quad \text{weakly in } W_{\text{per}}(Y).$$

Corollary A.11. *Let $u_0 \in H^2(\Omega)$ be the solution of the homogenized problem* (1.2) *and $\chi^n_{ij}, \chi_{ij} \in W_{\text{per}}(Y)$ be defined by* (A.16) *and* (A.17). *Suppose that there exists $p > N$ such that $u_0 \in W^{3,p}(\Omega)$. Define $u^n_2(x,y) = \chi^n_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$ and $u_2(x,y) = \chi_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$. Consider φ^n_ε the solution of*

$$\nabla \cdot \left(A^n\left(\frac{x}{\varepsilon}\right)\nabla\varphi^n_\varepsilon\right) = 0 \quad \text{in } \Omega, \qquad \varphi^n_\varepsilon = u^n_2\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \tag{A.22}$$

Then we have that

$$u^n_2\left(x, \frac{x}{\varepsilon}\right) \overset{n}{\rightharpoonup} u_2\left(x, \frac{x}{\varepsilon}\right) \quad \text{and} \quad \varphi^n_\varepsilon \overset{n}{\rightharpoonup} \varphi_\varepsilon \quad \text{in } H^1(\Omega),$$

where φ_ε satisfies

$$\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla\varphi_\varepsilon\right) = 0 \quad \text{in } \Omega, \qquad \varphi_\varepsilon = u_2\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \tag{A.23}$$

Proof. Following similar arguments as those used in Corollary A.6 we can prove that

$$u^n_2\left(x, \frac{x}{\varepsilon}\right) \overset{n}{\rightharpoonup} \chi_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) \quad \text{in } H^1(\Omega).$$

Using the above convergence result and similar ideas as in Corollary A.8 we complete the proof of the statement. □

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