See theorem 3.3 of Fleming [3] for a proof. Sections 3.5 and 3.6 of [3] also provide a different version of the following material.

Exercises.

Exercise 13.1 Suppose $r : \mathbb{R}^3 \to [0, \infty)$ is the Euclidean radius function $r(x) := \|x\|_2$. Let $\psi : [0, \infty) \to \mathbb{R}$ be a continuous function that is C^2 on $(0, \infty)$ and let $f(x) := \psi(r(x))$.
(a) Evaluate $\nabla r(x)$ and $D^2 r(x)$ for $x \in \mathbb{R}^3 \setminus \{0\}$.
(b) Find formulae for $\nabla f(x)$, $D^2 f(x)$ in terms of derivatives of ψ for $x \in \mathbb{R}^3 \setminus \{0\}$.
(c) Prove that $f(x) := r(x)^p$ is convex on $\mathbb{R}^3$ for $1 \leq p < \infty$ and evaluate this $\nabla f(x)$.

Exercise 13.2 (a) Suppose $f(x) := \|x\|_p^p$ with $p \in (1, \infty)$. Find a formula for the gradient of this function. When is this function C^1- on $\mathbb{R}^n$?
(b) Suppose $f(x) := \|x\|_1$ with $x \in \mathbb{R}^3$. Describe the set of points where this function is not G-differentiable. Find an explicit expression for the gradient of this function at points where it is differentiable.

Exercise 13.3 A real valued function f on $\mathbb{R}^n$ is said to be homogeneous of degree p if $f(cx) = |c|^p f(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Suppose f is also G-differentiable, prove Euler's rule that

$$\langle \nabla f(x), x \rangle = p\, f(x) \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Exercises.

Exercise 14.1 Define a function $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x) := \|x\|_2{}^4 - 2\,\langle Ax, x\rangle$$

where A is an $n \times n$ symmetric matrix.

(i) Prove that this function is bounded below and has minimizers on $\mathbb{R}^n$.

(ii) Find the equation satisfied by the critical points of G on $\mathbb{R}^n$.

(iii) What mathematical properties can you say about the critical points and/or minimizers of G? What can you say about the value of this problem?

Exercises.

Exercise 15.1 Suppose $S := \{a^{(1)}, \ldots, a^{(m)}\}$ is a finite set of distinct points in space and $F : \mathbb{R}^3 \to [0, \infty)$ is defined by

$$F(x) := \sum_{j=1}^{m} c_j \, \|x - a^{(j)}\|_2^2$$

with each $c_j > 0$.

(a) Show that F is convex and coercive and that there is a unique minimizer of this function.

(b) What equations hold at the minimizer of this problem?

(c) Suppose $S = \{0, e^{(1)}, e^{(2)}, e^{(3)}\}$ and each $c_j = 1$. Find the solution of this problem.

Exercise 15.2 Suppose $\psi : [0, \infty)^n \to \mathbb{R}_+$ is a continuous, convex function with $\psi(0) = 0$ and $\psi(x) > 0$ for $x \neq 0$.

(a) Show that there are constants $c_2 \geq c_1 > 0$ such that

$$\psi(x) \geq c_1 \|x\|_1 \quad \text{and} \quad \psi(x) \geq c_2 \|x\|_\infty \quad \text{when} \quad \|x\|_\infty \geq 1. \tag{15.3}$$

and that ψ is weakly coercive on $[0, \infty)^n$.

(b) Let $A : \mathbb{R}^n \to [0, \infty)^n$ be the map defined by $A(x) := (|x_1|, |x_2|, \ldots, |x_n|)$. Define $p : \mathbb{R}^n \to [0, \infty)$ by

$$p(x) := \inf\{s > 0 : \psi(A(x)/s) \leq 1\}$$

16. Exercises.

Exercise 16.1: Given $p \in [1, \infty]$, prove that the function $f_p : M_{mn} \to [0, \infty)$ defined by $f_p(A) := \|A\|_p$ is both a norm and a convex function on M_{mn}.

Exercise 16.2: (a) Show that if $A \in M_{mn}$, then

$$\|A\|_1 = \sup_{\|x\|_1 \leq 1} \sup_{\|y\|_\infty \leq 1} \langle Ax, y \rangle.$$

(b) Find an explicit formula for the 1-norm of the matrix A.
(c) Find conditions on the entries in A for $\|A^T\|_1 = \|A\|_1$.

Exercise 16.3: (a) Suppose $A \in M_{mn}$ show that

$$\|A\|_2 = \sup_{\|x\|_2 \leq 1} \sup_{\|y\|_2 \leq 1} \langle Ax, y \rangle, \qquad \text{and}$$

(b) $\|A^T\|_2 = \|A\|_2$.

Exercise 16.4: Suppose A, B are $n \times n$ matrices and $p \in [1, \infty]$, prove that

$$\|AB\|_p \leq \|A\|_p \, \|B\|_p.$$

Exercise 16.5: Let A be an $n \times n$ skew-symmetric matrix. Show that the quadratic form $q(x) := \langle Ax, x \rangle$ is identically zero.

Exercises.

Exercise 18.1 Let A be the 3×3 matrix

$$A := \begin{pmatrix} 1 & 0 & 1 \\ 0 & a_1 & 0 \\ 1 & 0 & a_2 \end{pmatrix} \tag{18.8}$$

Evaluate the associated quadratic form on $\mathbb{R}^3$ and find conditions on a_1, a_2 for A to be positive semi-definite. When is it positive definite? When A is p.s.d under what conditions on $b \in \mathbb{R}^3$ are there solutions of $Ax = b$? Find all solutions of the equation in this case.

Exercise 18.2 Let A be an $m \times n$ matrix with rank A = m. Let $V := N(A)$ be the null space of A. What is the dimension of V? In Euclidean geometry, the projection of a vector $b \in \mathbb{R}^n$ onto V is the point $\hat{x} \in V$ that is closest to b. That is $\hat{x} := P_V b$ minimizes

$$d(x) := \|x - b\|_2^2 \quad \text{subject to} \quad Ax = 0.$$

Find the equations obeyed by $P_V b$. What is the distance of b from V?

Exercise 18.3 Let $S := \{a^{(j)} : 1 \leq j \leq m\}$ be a finite set of points in $\mathbb{R}^3$ with $m \geq 4$. Consider the problem of finding a plane in space that provides the "best approximation" to this data. Suppose this plane Σ has the equation

$$\langle c, x \rangle = \sum_{k=1}^{3} c_k x_k = \gamma.$$

Find the expression for the Euclidean distance d_j of the point $a^{(j)}$ from a point in this plane. This distance is a function of c, γ. Define the function $F : \mathbb{R}^4 \to \mathbb{R}$ by

$$F(c, \gamma) := \sum_{j=1}^{m} d_j^2 + \|c\|_2^2 - 1.$$

This is a quadratic function of c, γ. The above analysis shows that there are minimizers of this function. Find the linear equations satisfied by the minimizers. How many equations are there? When does your matrix have a non-trivial null space?