Please work only on the problems with blue bullets

Then the minimization of f is reduced to a minimization with respect to t, e_x separately. Since the unit vectors lie on a compact subset of $\mathbb{R}^n$, it is the minimization of g on $(0, \infty)$ that often is more difficult.

Exercises.

Exercise 9.1 Suppose $f : [0, L] \to \mathbb{R}$ is a continuous function with $D_+f(0) < 0,\ D_-f(L) > 0$. Show that the global minimizers of f on $[0, L]$ are attained at points in $(0, L)$.

Exercise 9.2 Prove the following inequalities for the function $f_p(x) := x^p$ with $p > 1$.
(i) Given $s > 0$, find a positive constant C(s,p) such that $x^p \geq sx - C(s,p)$ for all $x > 0$.
(ii) Find the smallest positive constant $C(p) \in (0,1)$ such that $x^{p-1} \leq x^p + C(p)$ for all $x > 0$.
(iii) Suppose $x, y > 0$, prove that

$$x^p + p\,x^{p-1}y \leq [x + y]^p \leq x^p + p\,[x + y]^{p-1}y \qquad \text{and}$$
$$[x + y]^p \leq 2^{p-1}(x^p + y^p).$$

Exercise 9.3: Suppose that $0 < p < q < r$ and $\epsilon > 0$. Show that there is a $C_\epsilon > 0$ such that

$$x^q \leq \epsilon\, x^r + C_\epsilon\, x^p \quad \text{for all} \quad x \geq 0.$$

Exercise 9.4 Suppose that $p > 1$. Show that for any $\epsilon > 0$ there is a $C_\epsilon > 0$ such that for all real x,y

$$|x\,y| \leq \epsilon\,|x|^p + C_\epsilon\,|y|^{p^*}$$

Exercise 9.5 Given x, y in $(0, \infty)$, their *harmonic mean* is $h(x,y) := \frac{2xy}{x+y}$. Prove the following
(i) If $0 < x < y$, then $x < h(x,y) < y$.
(ii) $h(x,y) \leq \sqrt{xy} \leq (x+y)/2$

Exercise 9.6 Prove that $x^{1-t}y^t \leq (1-t)x + ty$ for all $x, y > 0$ and $0 \leq t \leq 1$. When $t = 1/2$, this is the arithmetic-geometric mean inequality.

Exercise 9.7 Prove that, when $t \in [0,1]$, there is a $C_t > 0$ such that

$$x^{1-t}y^t \leq C_t\,(x + y) \quad \text{for all} \quad x, y > 0.$$

Show that $C_t := t^t(1-t)^{1-t}$ is the optimal value here. Hence prove that, when $0 < q < p$,

$$x^q y^{p-q} \leq C_{q/p}\,(x^p + y^p) \quad \text{for all} \quad x, y > 0.$$

Exercise 9.8 Prove that the following inequalities hold when $p \geq 2,\ x, y > 0$.
(i) (general parallelogram inequality) $|x+y|^p + |x-y|^p \geq 2\,[x^p + y^p]$.
(ii) (Clarkson's first inequality) $\left|\frac{x+y}{2}\right|^p + \left|\frac{x-y}{2}\right|^p \leq \frac{1}{2}[x^p + y^p]$.

Exercise 9.9 Show that the function $f(x) := |x|\,\ln(1+|x|)$ is an even, positive, coercive, strictly convex function on $\mathbb{R}$ and f is strictly increasing on $[0, \infty)$. Prove that for each $\epsilon > 0, p > 0$, there is a constant $C > 0$ such that

$$\ln(1+x) \leq \epsilon\, x^p + C. \qquad \text{for } x \geq 0.$$

Please work only on the problems with blue bullets

In undergraduate calculus classes a function is defined to be convex, (or concave up), on an open interval I provided the assumptions of the following corollary hold. Note that of the examples given above only example 8.3 always satisfies the "undergraduate" definition.

Corollary 8.9. *Let I be an open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is such that $f''(x) \geq 0$ for each $x \in I$, then f is C^1 and convex on I.*

Proof. When $f''(x)$ exists then f' is continuous at x from a standard result in calculus, so f' will be continuous on I. Moreover f' is an increasing function on I as $f''(x) \geq 0$. Hence the result follows from the preceding theorem 8.8. □

Exercises.

Exercise 8.1: (a) Complete the proof of theorem 3.1 by evaluating the derivative of the function f(p) and showing that $f'(p) \leq 0$ for all $p > 0$.
(b) Evaluate $f''(p)$ and prove that f is a convex function on $(0, \infty)$.

Exercise 8.2: Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex with $f(0) = 0$. Show that $f(y) \geq f(1)y$ for $y \geq 1$ and $f(y) \geq f(-1)|y|$ for $y \leq -1$. Hence show that f is weakly coercive on $\mathbb{R}$ provided $m := \min\{f(-1), f(1)\} > 0$.

Exercise 8.3: Suppose I is a non-trivial interval in $\mathbb{R}$ and $f, g : I \to [0, \infty)$ are convex, increasing functions on I. Show that $h(x) := f(x)\,g(x)$ is a convex increasing function on I. Give a counterexample to this result if g is allowed to be negative on a subset of I.

Exercise 8.4 Suppose $a : [0, 1] \to \mathbb{R}$ is continuous and C^2- on $(0, 1)$. Describe conditions on the function a that imply that $f(x) := \frac{a(x)}{1-x}$ is convex on $[0, 1)$.

Exercise 8.5 Show that $f(x) := x^m(1-x)^p$ is concave on $[0, 1]$ for all integers m, p. Find the maximum value of f on $[0, 1]$. Show that $({}^nC_m)\, x^m\,(1-x)^{n-m} \leq 1$ for all $m \in I_n$. Hence prove that $({}^nC_m) \leq (m = 1)^{n-m}$ for $0 \leq m \leq n$.

Exercise 8.6 Suppose that $f : [0, 1] \to \mathbb{R}$ is convex and increasing. Prove that f must be continuous at 0 and u.s.c. at 1. State and prove the analogous result when f is convex and decreasing on $[0, 1]$.

Exercise 8.7 (a) Show that the function $f : (0, \infty) \to [0, \infty)$ defined by $f(x) := x\,\ln(1+x)$ is strictly increasing, C^2 and strictly convex.
(b) Let g be the inverse function of f. Specify the domain and range of g, Show that g is strictly increasing and strictly concave.

Exercise 8.8 Suppose that $f(x) := a\,x + b x^{-\beta}$ for $x > 0$, with a, b, β all strictly positive. Show that f is convex and has a minimizer on $(0, \infty)$. Find the minimizer of this function and find the numbers $C > 0, \gamma$ such that

$$\inf_{x>0} f(x) = C\,a^{\gamma}\,b^{1-\gamma}.$$

Verify that this C is independent of a and b. When $b, \beta > 0$ and $a = 0$ show that f is convex and bounded below but does not have a minimizer on $(0, \infty)$.

Please work only on the problems with blue bullets

In particular the values of these two problems are the same and, provided the value is not ∞, the minimizers will also be the same. Thus the existence theorem above also applies to problems defined on proper closed subsets of $\mathbb{R}^n$.

Exercises.

Exercise 5.1. If $p(x) = x^{2m} + a_1 x^{2m-1} + \ldots + a_{2m-1}x + a_{2m}$ is a polynomial of even degree on $\mathbb{R}$, then $\alpha(p) := \inf_{x\in\mathbb{R}}(p(x))$ is finite and there is minimizer $\hat{x}$ of p on $\mathbb{R}$. Give an example of a polynomial of degree 4 with nonunique minimizers.

Exercise 5.2. Suppose $p(x) = x^4 + ax^2 + bx$ with $a \geq 0$. Show that $\alpha(p) \geq \frac{-b^2}{4a}$ when $a > 0$. Show that this polynomial has a unique minimizer when $a = 0$ and find the minimal value.

Exercise 5.3. Suppose that $f(x) := x^2 + 2a\cos x - 2bx$ with a, b constants. Show that this function has minimizers on $\mathbb{R}$ and find the equations that they satisfy. Find bounds on this minimizer and show that the minimum value is less than or equal to $(a+1)^2 - b^2 - 1$.

Exercise 5.4. Suppose y is a real number and $x^2 \leq \epsilon^2 y^2 + C$ with $\epsilon, C > 0$. Prove that $|x| \leq \epsilon\,|y| + \sqrt{C}$. More generally show that $|x|^p \leq \epsilon^p\,|y|^p + C$ for some $C > 0, p > 1$ implies that there is a $C_1 > 0$ such that $|x| \leq \epsilon\,|y| + C_1$.

Exercise 5.5. If $p(x) = x^{2m+1} + a_1 x^{2m} + \ldots + a_{2m}x + a_{2m+1}$ is a polynomial of odd degree, then $\alpha(p) = -\infty$ and there is no minimizer, or maximizer, of p in $\mathbb{R}$.

Exercise 5.6: Suppose E is a non-empty closed subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is continuous. Let f_e be the extension of f to $\mathbb{R}$ defined by equation (5.5) above. Prove that f_e is l.s.c. on $\mathbb{R}$. Give a counterexample that shows this result need not hold when E is an open interval.

Exercise 5.7. Suppose K_1, K_2 are compact sets in $\mathbb{R}^m, \mathbb{R}^n$ respectively and $K := K_1 \times K_2$. When $F : K \to \mathbb{R}$ is a continuous function, prove that
(i) $G_1 : K_1 \to \mathbb{R}$ defined by $G_1(x) := \inf_{y\in K_2} F(x,y)$ is a continuous function on K_1. (The same proof shows that $G_2 : K_2 \to \mathbb{R}$ defined by $G_2(y) := \inf_{x\in K_2} F(x,y)$. is continuous.)
(ii) Prove that

$$\inf_{x\in K_1} G_1(x) = \inf_{y\in K_2} G_2(y) = \inf_{(x,y)\in K} F(x,y)$$

(iii) Do the preceding equalities hold when K_1, K_2 are arbitrary non-empty subsets of $\mathbb{R}^m, \mathbb{R}^n$ respectively? If so, prove it. If not find a counterexample.

6. 1-Dimensional Local Minimizers and Critical Points

In your first calculus course, you were taught some elementary results about finding the local minimizers, and maximizers of differentiable functions of a real variable. This section will summarize this material - and point out some issues with what you were probably told.