Definitions 1D optimization

Level set (contour), X - metric spaces.

$f: X \to \mathbb{R}$, $L_c(f) = \{x \in X, f(x) = c\}$, $c \in \mathbb{R}$.

Synoptic (or sublevel) sets of f.

$S_c(f) = \{x \in X ; \ f(x) \le c\}$

- f continuous, $L_c(f)$, $S_c(f)$ are closed sets.

$\mathrm{dom}(f) = \{x \in X, f(x) \in \mathbb{R}\}$.

graph of f: $G(f) = \{(x, f(x)) ; x \in X\}$.

f is proper if $\mathrm{dom}(f) \neq \phi$ and $f(x) \neq -\infty$ for any $x \in X$.

Ex. $f: \mathbb{R}^3 \longrightarrow [0, \infty]$, $f(x) = \begin{cases} \frac{1}{|x|} & x \neq 0 \\ \infty & x = 0 \end{cases}$

$\hookrightarrow$ continuous, proper.

$S_c(f) = \{x \,/\, |x| \geq \frac{1}{c}\}$, $L_c(f) = \{|x| = \frac{1}{c}\}$.

Ex $\quad f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$

$$\ln(x) = \begin{cases} \ln|x|. & x \neq 0. \\ -\infty & x = 0. \end{cases}$$

f - continuous, on $\mathbb{R}^n$.
not proper.

~~$S_c(f) = \{ \ln|x| \leq c$~~

$S_c(f) = \{ |x| \leq e^c \}$.

$L_c(f) = \{ |x| = e^c \}$.

Ex. $\quad f : \mathbb{R}^n \longrightarrow [0, \infty)$.

$f(x) = \ln(1 + |x|) \geq 0$

Continuous, proper.

$f : E \longrightarrow \mathbb{R}, \quad E \subset X$
nonempty subset.

$$f_k(x) = \inf \{ f(y) : y \in E, \ 0 < d(x, y) < \tfrac{1}{k} \}$$

$\forall x \in E, \quad \{ f_k(x) \}_{k \geq 1}$ is increasing sequence

$$\liminf_{y \to x} f(y) = \lim_{k \to \infty} f_k(x) = \sup_{k \geq 1} f_k(x)$$

Similarly.

$$g_k(x) := \sup \{ f(y) : y \in E, \ 0 < d(x,y) < \tfrac{1}{k} \}.$$

$$\limsup_{y \to x} f(y) = \lim_{k \to \infty} g_k(x) = \inf_{k \geq 1} g_k(x)$$

$f : E \to \mathbb{R}$, l.s.c. at $x \in E$

$$f(x) \leq \liminf_{y \to x} f(y).$$

f u.s.c at $x \in E$ if.

$$f(x) \leq \liminf_{y \to x} f(y).$$

Theorem $f : X \to \overline{\mathbb{R}}$, f is l.s.c iff $S_c(f)$ are closed. for each $c \in \mathbb{R}$.

Ex. $f(x) = \begin{cases} \sin \frac{1}{x} & , x \neq 0 \\ c & x = 0 \end{cases}$

limit points at 0, $[-1, 1]$.

Envelopes of family of functions

$\mathcal{F} = \{ f_j, \ j \in J \}$. function on X.

$$F(x) = \inf_{j \in J} f_j(x) \ , \quad G(x) = \sup_{j \in J} f_j(x).$$

If ~~if~~ f_j continuous and J is finite

then F, G continuous on X.

For J infinite

* If f_j is l.s.c fn each j on X

then G is l.s.c on X.

If each f_j is u.s.c ~~on~~ X then

F is u.s.c on X.

Proof Use characterisation of l.s.c in terms of synoptic sets.

$$S_c(G) = \bigcap_{j \in J} S_c(f_j)$$

Minimization problems

$$f : X \to \overline{\mathbb{R}}$$

$$\alpha(f, X) = \inf_{x \in X} f(x)$$

value of problem.

$$M(f) = \{ x \in X, \ f(x) = \alpha(f, X) \}.$$

↳ minimisers of f on X.

Maximization problem. similar

Ex. $f : \mathbb{R} \to \overline{\mathbb{R}}$, $f(x) = e^x$

$\alpha(f, \mathbb{R}) = \inf_{x \in \mathbb{R}} e^x = 0$ but there is no minimiser of f on $\mathbb{R}$.

Ex. $f = ax^2 + bx + c$, $x \in \mathbb{R}$, $a > 0$.

$$\inf f(x) = c - \frac{b^2}{4a} \quad \left(\frac{-\Delta}{4a} \right)$$

Unique minimiser at $\hat{x} = -\frac{b}{2a}$.

$\overline{\mathbb{R}}$ - metric space with respect to

$$d_e(x, y) = \int_x^y \frac{dt}{1+t^2} = | \arctan y - \arctan x |$$

↳ Compact metric space

$\circ$ $x_k \to x$ in $\mathbb{R}$ then

$x_k \to \hat{x}$ in $(\overline{\mathbb{R}}, d_e)$.

<u>Notations</u>

$\Gamma = \{x_k\}_{k \geq 1} \subset X$ descent sequence for f in X if $f(x_{k+1}) \leq f(x_k)$, $k \geq 1$.

Γ is strict descent sequence if
$f(x_{k+1}) < f(x_k)$, $k \geq 1$

Γ is a minimizing sequence for f on X if it is a descent sequence and also
$\lim_{k \to \infty} f(x_k) = \inf_{k \geq 1} f(x_k) = \alpha(f, X)$

<u>Thm</u> (Weierstrass). $K \neq \phi$, K compact
$K \subset X$, $f : K \to \mathbb{R}$ continuous
$\alpha(f, K) = \inf_{x \in K} f(x)$, $\beta(f, K) = \sup_{x \in K} f(x)$ are finite and attained in K

The existence theorem is not constructive

* $K \neq \phi$, compact subset of (X, d).

  $f : K \to \overline{\mathbb{R}}$ l.s.c.

  $\exists\ \hat{x} \in K$ minimising f on K.

- $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ l.s.c and coercive.

  Then f has a minimiser on $\mathbb{R}^n$

  ($f(x) \to \infty$, $|x| \to \infty$).

- $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ l.s.c. $\exists\, c \in \mathbb{R}$ s.t.

  $S_c(f)$ is non-empty and bounded.

  $\exists\ \hat{x} \in \mathbb{R}^n$ minimiser for f on $\mathbb{R}^n$.

Show $S_c(f)$ are compact for c non empty for c.

* minimising f on K (non-empty, closed.)

  equivalent to minimising

$$f_e = \begin{cases} f & \text{on } K \\ \infty & \text{on } \mathbb{R}^n \setminus K \end{cases}$$

The min value of f_1, f_2 are the same and if it is not ∞. the minimisers are the same.

1D minimisers and critical points

- $f : \mathbb{R} \to \overline{\mathbb{R}}$, $\tilde{x} \in \mathbb{R}$.

$\tilde{x}$ is a local minimiser if $\exists\, \delta > 0$ s.t. $f(x) \geq f(\tilde{x})$ for $|x - \tilde{x}| < \delta$.

$\tilde{x}$ is a strict local minimiser of f if it is a local minimiser and $f(x) > f(\tilde{x})$ for $0 < |x - \tilde{x}| < \delta$.

$\tilde{x}$ is an isolated local minimiser of f and if $\hat{x}$ is a local minimiser of f then is a $\delta_1 > 0$ s.t. f has no other local minimiser in $|x - \tilde{x}| < \delta_1$.

Ex.
$$f(x) = \begin{cases} x^2\,[2 + \cos(x^{-1})] & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Study (i) local minimisers, global minimiser

- $f : [a,b] \to \mathbb{R}$ continuous.

$\exists\, x_1, x_2, x_3$, $a \le x_1 < x_3 < x_2 \le b$.

s.t. $f(x_3) < \min(f(x_1), f(x_2))$. $(*)$

then there is a local minimizer of f in (x_1, x_2).

Proof $\alpha = \inf_{[x_1,x_2]} f$ ~~finite~~ finite.

$\exists\, \hat{x} \in [x_1, x_2]$ s.t. $f(\hat{x}) = \alpha$.

$(*)$ implies that $\hat{x} \ne x_1$, $\hat{x} \ne x_2$.

$I = (a,b)$. $f : I \to \mathbb{R}$. continuous.

$$D_- f(x) = \lim_{t \to 0_+} \frac{f(x) - f(x-t)}{t} \in \overline{\mathbb{R}}$$

$$D_+ f = \lim_{t \to 0_+} \frac{f(x+t) - f(x)}{t} \in \overline{\mathbb{R}}$$

If $\overbrace{D_- f(x) = D_+ f(x)}^{\text{finite}}$ - f diff $x \in (a,b)$

if $D_+ f(a)$ exist and finite then f diff at a

if $D_- f(b)$ exist and finite then f diff at b

*. $f : (a,b) \longrightarrow \mathbb{R}$ Continuous ~~at~~ and $\hat{x}$ local minimizer of f on (a,b).

Then

1. $\liminf_{|h| \to 0} \dfrac{f(\hat{x}+h) - f(\hat{x})}{h} \geq 0.$

2. If f has left and right derivative at $\hat{x}$ then
$D_- f(\hat{x}) \leq 0$ and $D_+ f(\hat{x}) \geq 0$.

3. f is diff at $\hat{x}$, $f'(\hat{x}) = 0$

Ex. $f(x) = x^3$, $f'(0) = 0$ but 0 is not a local minimizer.

*. $f : (a,b) \longrightarrow \mathbb{R}$ is continuous and f has left and right derivatives at $\hat{x}$ with $D_- f(\hat{x}) < 0$, $D_+ f(\hat{x}) > 0$.

Then $\hat{x}$ is a strict local minimizer.

Ex. $f: \mathbb{R} \to [0,\infty)$
$f(x) = \sqrt{|x|}$. $\in C^\infty$ on any interval

not containing zero.

Not diff at 0. but

$$D_- f(0) = \lim_{h\to 0_+} \frac{f(0) - f(-h)}{h}$$

$$= \lim_{h\to 0_+} \frac{-\sqrt{h}}{h} = -\infty.$$

$$D_+ f(0) = +\infty.$$

0 is the unique minimiser

• $f(x) = |x|$ — Unique minimum at $\hat{x} = 0$.

$\tilde{x} \in (a,b)$ critical point of f if f is diff at $\tilde{x}$ and $f'(\tilde{x}) = 0$.

$\tilde{x}$ isolated critical point if $\exists\, \delta > 0$ s.t $f'(x) \neq 0$ for all $x \in B_\delta(\tilde{x}) \setminus \{\tilde{x}\}$.

~~A critic.~~

A critical point $\tilde{x} \in (a,b)$ of f is called a saddle point of f if.

$\forall \delta > 0$, $\exists$ $x_1, x_2 \in B_\delta(\tilde{x})$ s.t.

$$\left(f(x_1) - f(\tilde{x})\right) \cdot \left(f(x_2) - f(\tilde{x})\right) < 0$$

- $f: (a,b) \to \mathbb{R}$ continuous and C^1 on (a,b).

and $\exists$ $x_1, x_2 \in I$ s.t $a < x_1 < x_2 < b$

and $f'(x_1) < 0 < f'(x_2)$.

Then $\exists$ at least a local minimizer $\hat{x}$ of f in (x_1, x_2)

<u>Proof.</u> $\exists$ δ s.t $f(x_1+\delta) < f(x_1)$

$f(x_2-\delta) < f(x_2)$.

x_3 s.t

$f(x_3) = \min\{f(x_1+\delta), f(x_2-\delta)\}$

$f(x_3) < \min\{f(x_1), f(x_2)\}$.