Critical points

- $\tilde{x} \in (a,b)$ — Critical point of f, provided f is differentiable at $\tilde{x}$ and $f'(\tilde{x}) = 0$!

  $\tilde{x}$ is called isolated critical point if $\exists\, \delta > 0$ s.t. $f'(x) \neq 0 \;\; \forall x \in B_\delta(\tilde{x}) \setminus \{\tilde{x}\}$

  ($B_\delta(\tilde{x})$ - ball centered in $\tilde{x}$ and radius δ).
  $B_\delta(\tilde{x}) = (\tilde{x}-\delta, \tilde{x}+\delta)$.

- A critical point $\tilde{x} \in (a,b)$ of f is said to be a saddle point of f if for any $\delta > 0$. there are points $x_1, x_2 \in B_\delta(\tilde{x})$ s.t.

$$\left(f(x_1) - f(\tilde{x})\right) \cdot \left(f(x_2) - f(\tilde{x})\right) < 0.$$

Saddle Point is a critical point which is not a local extremum. !

Error estimates and second order conditions.

I - $f : (a,b) \to \mathbb{R}$ continuous and C^1 in (a,b). Assume there exists $x_1, x_2 \in I$ such that

$$a < x_1 < x_2 < b \text{ and } f'(x_1) < 0 < f'(x_2).$$

Then there is at least one local minimizer $\hat{x}$ of f in (x_1, x_2).

Proof. $\exists\, \delta > 0$ s.t. $f(x_1+\delta) < f(x_1)$ and

$f(x_2-\delta) < f(x_2)$. Take x_3 s.t.

$$f(x_3) = \min\{f(x_1+\delta), f(x_2-\delta)\}.$$

Thus from the above result. I. we have.
$\exists\ \hat{x}$ local minimizer of f in (c_1, c_2) and.
$f'(\hat{x}) = 0$ because f is C^1.

If $a < x < \hat{x}$ then i) imply.

$$f'(\hat{x}) - f'(x) \geq c_0(\hat{x} - x) > 0$$

$$\Downarrow$$

$$f'(x) < 0 \quad \text{and} \quad |\hat{x} - x| < c_0^{-1}|f'(x)|.$$

Similary if $\hat{x} < x < b \Rightarrow f'(x) > 0$ and

$$|\hat{x} - x| < c_0^{-1}|f'(x)|.$$

$\hat{x}$ is the unique critical point and thus the unique minimiser.

Note that condition ii) above in II. holds if $f \in C^2(a,b)$ and $f''(x) \geq c_0 > 0$. $\forall x \in (a,b)$.

Then $f(x_3) < \min(f(x_1), f(x_2))$. (*)

Weierstrass theorem $\alpha = \inf_{[x_1,x_2]} f(x)$ is finite and attained in $\hat{x} \in [x_1, x_2]$.

(*) implies that $\hat{x} \neq x_1$, $\hat{x} \neq x_2$.

II • $f: (a,b) \to \mathbb{R}$, C^1 and there exists points c_1, c_2, $c_1 < c_2$ in (a,b) and $c_0 > 0$ such that:

i) $f'(x_2) - f'(x_1) \geq c_0(x_2 - x_1)$ for all

$a < x_1 < x_2 < b$.

ii) $f'(c_1)\, f'(c_2) < 0$.

Then there is a unique minimizer $\hat{x}$ of f in (a,b) with:

$$\hat{x} \in [c_1, c_2] \text{ and } |x - \hat{x}| \leq c_0^{-1} |f'(x)| \quad \forall x \in (a,b).$$

Proof. $f'(x_2) \geq f'(x_1) + c_0(x_2 - x_1) > f'(x_1)$ ($\square$) $\forall\ a < x_1 < x_2 < b$.

So for $x_1 = c_1$, $x_2 = c_2$ we have from ($\square$) and ii) that $f'(c_1) < 0$, $f'(c_2) > 0$

Second order necessary condition

III. $f : (a,b) \longrightarrow \mathbb{R}$ $C^1(a,b)$, and let $\hat{x}$ be a local minimum (Assuming there exists one).

i) If $\hat{x}$ is an isolated critical point of f then

$$\liminf_{h \to 0} \frac{f'(\hat{x}+h)}{h} = c \geq 0. \quad (**)$$

ii) If $f''(\hat{x})$ exists then $f''(\hat{x}) \geq 0$.

Proof. ~~Assume c < 0.~~

$\hat{x}$ isolated critical point implies that $\exists\, \delta > 0$, s.t. $f'(\hat{x}+h) \neq 0$ for $0 < |h| < \delta$.

Assume $c < 0$. Then $(**)$ implies $\exists\, h_1$ with $0 < |h_1| < \delta$ such that

$$h_1^{-1} f'(\hat{x}+h_1) \leq \frac{c}{2} < 0.$$

If $h_1 > 0 \Rightarrow f'(\hat{x}+h) < 0$ for all $0 < h < h_1$

$$\Rightarrow f(\hat{x}+h) - f(\hat{x}) = \int_0^h f'(\hat{x}+s)\,ds < 0.$$

$\forall h < h_1$

Contradiction with $\hat{x}$ being local minimizer.

If $h_1 \cancel{<} 0 \Rightarrow f'(x+h_1) > 0.$ $\forall$ $-h_1 < h < 0$

$\cancel{\forall}$ Then

$$f(\hat{x}) - f(\hat{x}+h). = \int_h^0 f'(\hat{x}+s)\,ds > 0.$$

Again contradiction with the minimizer character of $\hat{x}$.

Thus $c \geq 0$.

(ii) If f'' exists then

$$f''(\hat{x}) = \lim_{h\to 0} \frac{f'(\hat{x}+h)}{h} = c$$

for some real c. because $f'(\hat{x}) = 0$.

Repeat the argument at i).

Second order sufficient condition

Suppose f is C^1 on (a,b) and $\hat{x} \in (a,b)$ is a critical point of f. Also assume

$$\liminf_{h \to 0} h^{-1} f'(\hat{x}+h) = c > 0. \quad (**)$$

Then $\hat{x}$ is an isolated strict local minimiser of f on (a,b).

Proof $(**) \implies \exists\, \delta > 0$ s.t.

$$|h| < \delta \implies \frac{f'(\hat{x}+h)}{h} \geq \frac{c}{2}.$$

Then $f(\hat{x}+h) - f(\hat{x}) = \int_0^h f'(x+s)\,ds \geq \frac{c}{2}\int_0^h s\,ds = \frac{ch^2}{4}$

for $0 \leq h < \delta$

$$f(\hat{x}) - f(\hat{x}+h) = \int_h^0 f'(x+s)\,ds \leq -\frac{c}{4}h^2.$$

for $-\delta < h < 0$.

$\hat{x}$ ~~[illegible]~~ isolated (strict) local minimiser