Note that the above second order sufficient condition holds if $f''(x) > 0$.

Univariate convex Functions

$I \neq \phi$, interval of $\mathbb{R}$.

I connected, contains at lest two points. $(|I| > 0)$.

$f : I \to \mathbb{R}$ is convex if.

$$f((1-t)x + ty) \leq (1-t)f(x) + t f(y) \quad (C)$$

$$0 \leq t \leq 1, \quad x, y \in I$$

f is strictly convex if (C) holds with $<$. whenever $x \neq y$ and $0 < t < 1$.

f is (strictly) concave if $-f$ is (strictly) convex.

- $f : I \to \mathbb{R}$, is such that for $x \in I$ there is a $\psi \in \mathbb{R}$ s.t.

$$(\square\square) \quad f(y) \geq f(x) + \psi(y - x) \quad \forall y \in I.$$

Then f is convex in I

Proof Let $x \in I$ and $y > x$.

Let $z = (1-t)x + ty.$, $0 < t < 1$.

Then ($\square\square$) implies.

$$f(x) \geq f(z) + \psi(x-z) = f(z) + \psi t(x-y)$$

$$f(y) \geq f(z) + \psi(y-z) = f(z) + \psi(1-t)(y-x).$$

Multiply the first inequality with $(1-t)$ and the second with t and add to obtain

$$f(z) \leq t f(y) + (1-t) f(x).$$

Similar argument for $y < x$.

$f : I \to \mathbb{R}$ is such that for each $x \in I$:

$\exists\, \psi \in \mathbb{R}$ s.t.

$$f(y) > f(x) + \psi(y-x). \quad \forall y \in I,\ y \neq x.$$

then f is strictly convex on I

• $f : I \to \mathbb{R}$ is differentiable at each point $x \in I$. The f is convex on I iff.

(o) $f(y) \geq f(x) + f'(x)(y-x) \quad \forall\, x, y \in I$.

It is strictly convex on I iff. (o) holds with $>$ in place of $\geq$ and $y \neq x$.

<u>Proof</u>. (o) $\Rightarrow$. f convex from the above result. Conversely if f is convex on I, $0 < t \leq 1$ then.

$$f(y) \geq t^{-1}\left[f((1-t)x + ty) - (1-t)f(x)\right] =$$
$$= f(x) + t^{-1}\left[f(x + t(y-x)) - f(x)\right]$$
$$\forall\, x, y \in I.$$

Let $t \to 0_+$ and (o) is obtained.

Similarly for strict convexity.

The above result says that the graph of a differentiable convex function lies on or above its tangent line at each point in I.

Ex. Show $|x|^p$ is convex when $p \geq 1$.
Sketch the graph of the function for a number of values of p.

Ex. $f(x) = e^{\alpha x}$ is convex on $\mathbb{R}$ for any choice of $\alpha \in \mathbb{R}$.

Ex. $h(x) = \begin{cases} x \ln x & x > 0 \\ 0 & x = 0 \end{cases}$

h is convex and continuous on $[0, \infty)$.

- $f: I \to \mathbb{R}$. The first-order Newton divided difference is the slope of the straight line joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (x_1 \neq x_0).$$

- Suppose $f: I \to \mathbb{R}$ convex and $x_1 < x_2 < x_3$ in I.

$$f[x_1, x_2] \leq f[x_1, x_3] \leq f[x_2, x_3].$$

Proof. $x_2 \in (x_1, x_3) \to x_2 = (1-t)x_1 + t x_3$ for some $t \in (0,1)$

Convexity.

$$f(x_2) \le (1-t) f(x_1) + t f(x_3)$$

$$\Downarrow$$

$$f(x_2) - f(x_1) \le t\big(f(x_3) - f(x_1)\big)$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{t\big(f(x_3) - f(x_1)\big)}{t(x_3 - x_1)} \qquad (\square\circ)$$

because $x_2 - x_1 = t(x_3 - x_1)$.

$(\square\circ) \iff f[x_1, x_2] \le f[x_3, x_1]$.

$$f(x_3) - f(x_2) \ge (1-t)\big(f(x_3) - f(x_1)\big)$$

$x_3 - x_2 = (1-t)(x_3 - x_1)$.

$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} \ge \frac{(1-t)\big(f(x_3) - f(x_1)\big)}{(1-t)(x_3 - x_1)}.$$

- $f : I \to \mathbb{R}$ convex and differentiable at $x_1 \in I$. Then

$$f[x, x_1] \ge f'(x_1) \quad \forall\, x \ge x_1$$

- Note that $f[x-\sigma, x]$ is decreasing as a function of $\sigma \to 0$ and as well $f[x, x+s]$ is decreasing as a function of s

Diff of convex functions

$I = [a,b]$. $f : I \to \mathbb{R}$ convex.

i). $D_+ f(a)$ exists $(\neq +\infty)$.

ii) $x \in (a,b)$, $D_- f(x)$, $D_+ f(x)$ exists and are finite, $D_- f(x) \le D_+ f(x)$

$D_- f(x)$, $D_+ f(x)$ are increasing fct. on (a,b).

iii) $D_- f(b)$ exists. $(\neq -\infty)$.

iv. $J = [c,d] \subset (a,b)$, the f is Lipschitz continuous on J with. with

$|f(y) - f(x)| \le L|y-x|$

$$L = \max\{D_+ f(c), D_- f(d)\}.$$

i). Observe that $f[a, a+s]$ ~~increasing~~ decreasing fct of s.

$D_+ f(a)$ exists in $[-\infty, \infty)$.

ii) $x \in (a,b)$, $x_1 = x - \sigma$, $x_3 = x + s$.

Take $x_2 = x$ in. $\Rightarrow$ $f[x_1,x_2] \le f[x_1,x_3] \le f[x_2,x_3]$

$x_1 < x_2 < x_3$

Then $f[x-\sigma, x] \le f[x, x+s]$

$\sigma \to 0^+$ the then $f[x-\sigma, x]$ is increasing and

is bounded above by $f[x, x+s]$.

$D_- f(x)$ exist and is finite.

Similarly show $f[x, x+s] \xrightarrow[s \to 0_+]{} D_+ f(x)$

Let.

$x_1 < x_2 \le b$ $\quad$ ($x_1 - s < x_1 < x_2 - s < x_2$.

$$f[x_1 - s, x_1] \le f[x_2 - s, x_2].$$

$$\Downarrow$$

$$D_- f(x_1) \le D_- f(x_2)$$

Similar for $D_+ f(x)$.

iii). Similar to i

iv). If $f \in C^1$ then

$$f(y) - f(x) = \int_x^y f' \le (y-x) f'(\xi).$$

$$|f(y) - f(x)| \le |y-x| \max_{\xi \in [x,y]} \{ f' \} =$$

- $I = (a,b)$, $f : I \to \mathbb{R}$ convex. Then f is locally Lipschitz continuous on I.

- $f : I \to \mathbb{R}$ diff at each $x \in I$.
  Then f is convex on I if and only if f' is an increasing fct. If f'' exists $f'' \geq 0$.

$\Rightarrow$ f' increasing is implied by the above

$\Leftarrow$ $f(y) = f(x) + f'(\xi)(y-x) \geq f(x) + f'(x)(y-x)$
$x < \xi < y$

Similarly if $y < x$. Then f is convex!