1-D Optimization and inequalities

$\alpha(f,I) = \inf_{x\in I} f(x) \in \mathbb{R}$ $\quad f: I \to \mathbb{R}$ continuous

I - compact

$\beta(f,I) = \sup_{x\in I} f(x) \in \mathbb{R}.$

and there exists minimizers and maximizers.

- Let $\hat{x}$, $f(\hat{x}) = \alpha(f,I)$.

i) If $\hat{x} = a$ then $f[x,a] \geq 0 \quad \forall x \in I$.

ii) If $\hat{x} \in (a,b)$ then.

$f[x,\hat{x}] \leq 0$ for $a < x < \hat{x}$ and

$f[\hat{x},x] \geq 0$ for $\hat{x} < x < b$. $\quad (*)$

iii) if $\hat{x} = b$ then $f[x,b] \leq 0$ for all $x \in I$.

- $f: I \to \mathbb{R}$ continuous and convex.

If f i) $D_+ f(a) \geq 0$ then a minimizes f on I.

ii) $\hat{x} \in (a,b)$ satisfies $(*)$. then $\hat{x}$ minimizes f on I

iii) $D_- f(b) \leq 0$ then b minimizes f on I.

The set of all the minimizers is a closed subinterval of $[a,b]$.

<u>Proof</u>

ii). $f[x,\hat{x}] \leq 0$ for $a < x < \hat{x}$
$f[\hat{x},x] \geq 0$ for $\hat{x} < x < b$. $\Big\}$ $(*)$

$\hat{x} \in (a,b)$.

Assume $\hat{x}$ is not a minimizer. Then $\exists\, x_0$ s.t.

$f(x_0) < f(\hat{x})$ $(**)$

If $x_0 < \hat{x}$. Then take x s.t.

$x_0 < \hat{x} < x$.

Since f convex we have.

$f[x_0,\hat{x}] \leq f[x_0,x] \leq f[\hat{x},x]$.

But $(*)$ implies $f[x_0,\hat{x}] \leq 0$.

On the other hand. $(**)$ implies.

$f[x_0,\hat{x}] > 0$ Contradiction!

If $x_0 > \hat{x}$.

$(*) \Rightarrow f[x_0, \hat{x}] \geq 0$.

$(**) \Rightarrow f[x_0, \hat{x}] < 0$

Contradiction.

Similar proof for i), iii).

The set of all minimisers is

$$S = \{ x \in [a,b],\ f(x) = \alpha \} \text{ closed}$$

if f is continuous.

Also. $x_1, x_2 \in S$, $x = x_1 t + (1-t) x_2$

$$f(x) \leq (1-t) f(x_2) + t f(x_1) = \alpha.$$

$$\Downarrow$$

$$x \in S.$$

S is a subinterval. closed of I.

- If $f : [a,b] \to \mathbb{R}$ is strictly convex, then there is a unique minimiser of f on $[a,b]$.

Suppose $\hat{X}_1, \hat{X}_2$ are two distinct minimisers.

$$f(\hat{x}_1) = f(\hat{x}_2) = \alpha.$$

$$f\left(\frac{\hat{x}_1 + \hat{x}_2}{2}\right) < \frac{f(\hat{x}_1) + f(\hat{x}_2)}{2} = \alpha.$$

Contradiction with the definition of α.

- $f: [a,b] \to \mathbb{R}$ continuous and convex.

i). (Weak max principle).

$$\beta(f, I) = \max\{f(a), f(b)\}.$$

ii) (Strong max. principle).

If there is a point $c \in (a,b)$ that maximises f on $[a,b]$ then $f(x)$ is constant on $[a,b]$.

~~$D_+ f(x) \geq 0 \Rightarrow$ f increasing?~~

$$\cancel{\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \geq 0.}$$

Assume $\exists \hat{x} \in (a,b)$ s.t. $f(\hat{x}) = \beta$.

$$\begin{cases} f[x, \hat{x}] \geq 0 & a < x < \hat{x} \\ f[\hat{x}, x] \leq 0 & \hat{x} < x < b. \end{cases}$$

On the other hand, if $a < x_1 < \hat{x} < x_2 < b$.

$$0 \leq f[x_1, \hat{x}] \leq f[x_1, x_2] \leq f[\hat{x}, x_2] \leq 0. \quad (***)$$

because $(***)$ it implies

$$f[x_1, x_2] = 0 \Rightarrow f(x_1) = f(x_2)$$

And since x_1, x_2 are arbitrary points.

f is constant on $[a,b]$.

Inequalities

$p \in (1,\infty)$, $\frac{1}{p^*} = \frac{p-1}{p}$.

Let $Y \in \mathbb{R}$.

$$\frac{1}{p}|x|^p - xY \geq -\frac{1}{p^*}|Y|^{p^*} \quad \forall x \in \mathbb{R}.$$

If $Y \neq 0$ equality holds above iff

$$Y = |x|^{p-2}x.$$

- Young's inequality in $\mathbb{R}^n$

$p \in (1,\infty)$, $p^* = \frac{p}{p-1}$.

$$|\langle x, y\rangle| \leq \frac{1}{p}\|x\|_p^p + \frac{1}{p^*}\|y\|_{p^*}^{p^*} \quad x, y \in \mathbb{R}^n.$$

"non-zero" x, y, equality holds iff

$$y_j = |x_j|^{p-2}x_j$$

$$\|x\|_p = \left\{\sum_{j=1}^n |x_j|^p\right\}^{1/p}.$$

Proof. Use the 1-D case for x_j, y_j and add to get

$$|\langle x, y\rangle| \le \sum_{j=1}^{n} |x_j y_j| \le \sum_{j=1}^{n}\left(\frac{|x_j|^p}{p} + \frac{|y_j|^{p^*}}{p^*}\right).$$

- Holder's inequality.

$p \in (1, P)$, $\quad p^* = \frac{p-1}{p}$.

$$|\langle x, y\rangle| \le \|x_p\| \cdot \|y\|_{p^*}), \quad x, y \in \mathbb{R}^n$$

when $x \neq 0$ equality holds here when $y_j = c|x_j|^{p-2} x_j$ for some $c \in \mathbb{R}$ and all j such that $x_j \neq 0$.

Proof. $u = \frac{x}{\|x\|_p}$, $v = \frac{y}{\|y\|_{p^*}}$. Young's inequality.

$$|\langle u, v\rangle| = \sum_{j=1}^{n} \frac{|x_j y_j|}{\|x\|_p \|y\|_{p^*}} \le \frac{1}{p} + \frac{1}{p^*}.$$

Equality similar as for Young's inequality.

- If $p \in (1,\infty)$, $x \in \mathbb{R}^n$

$$\|x\|_p = \sup_{\|y\|_{p^*} \le 1} \langle x, y \rangle .$$

<u>Proof</u>

$p \in (1,\infty)$ - it follows then from Holder inequality.

- Note that triangle inequality for p norms does not hold when $p \in (0,1)$.

- Minkowski's inequality.

$p \in (1,\infty)$, $x, y \in \mathbb{R}^n$

$$\|x+y\|_p \le \|x\|_p + \|y\|_p .$$

$x \neq 0$, equality hold iff $y = cx$, $c \geq 0$.

$|u|^p$ convex on $\mathbb{R}$ $\Rightarrow$ $|(1-t)u + tv|^p \le (1-t)|u|^p + t|v|^p$, $0 \le t \le 1$

(II)

$x \neq 0$, $y \neq 0$ let

$$u = \frac{x_j}{\|x\|_p}, \quad v = \frac{y_j}{\|y\|_p}, \quad t = \frac{\|y\|_p}{\|x\|_p + \|y\|_p}$$

$$(1-t)\,u + t\,v = \frac{x_j + y_j}{\|x\|_p + \|y\|_p}$$

~~II~~ From (II) for each j we obtain

$$\left(\|x\|_p + \|y\|_p\right)^{1-p} \sum_{j=1}^{n} |x_j + y_j|^p \leq \|x\|_p^{1-p} \sum_{j=1}^{n} |x_j|^p + \|y\|_p^{1-p} \sum_{j=1}^{n} |y_j|^p$$

Qed.