Convex sets in $\mathbb{R}^n$ [redacted]

$C \subset \mathbb{R}^n$ is convex

$$x, y \in C \Rightarrow (1-t)x + ty \in C \quad t \in [0,1]$$

A convex set is non-trivial if it contains at least two points.

Convex $C \subset \mathbb{R}^n$ is called convex cone if $cx \in C$ $\forall c \geq 0$ and $x \in C$

- The only convex sets of $\mathbb{R}$ are intervals and ϕ with the points. (singletons)
- V subspace of $\mathbb{R}^n$, V is convex and closed.

If $a \notin V$ then $\{a+v \,;\, v \in V\}$ is called affine subspace of $\mathbb{R}^n$ and is convex

- $H = \{ x \in \mathbb{R}^n ; \langle a, x \rangle = c \}$ where a is a unit vector in $\mathbb{R}^n$ is called hyperplane in $\mathbb{R}^n$ and it is closed and convex.

- ~~R~~ Positive orthant in $\mathbb{R}^n$.

$\{ x \in \mathbb{R}^n ; \quad x_j \geq 0, \ j \in \{1, \dots, n\} \}$.

$\mathbb{R}^n_{++} = (0, \infty)^n$ — set of all ~~positive~~ strictly positive vectors.

$\Gamma = \{ a_j , 1 \leq j \leq J \}$ finite set of vectors in $\mathbb{R}^n$.

$$x = \sum_{j=1}^{J} t_j a_j , \quad t_j \geq 0, \quad \sum_{j=1}^{J} t_j = 1$$

↳ convex combination of vector in Γ

$Co(\Gamma)$ — the set of all convex combinations of points in Γ

If Γ is finite then $co(\Gamma)$ is called polyhedron. and it is a bounded closed convex set of $\mathbb{R}^n$.

When $J = n+1$. such a set is called simplex. whose vertices are Γ.

Ex. $n=2$, Triangle is a simplex

$n=3$, tetrahedron is a simplex.

Unit simplex of $\mathbb{R}^n$.

closed convex hull of $\{0, e_1, \ldots, e_n\}$.

where $\{e_i\}_i$ are the canonical vectors.

$\Delta_n = co\{0, e_1, \ldots, e_n\}$.

The set of vectors in $\mathbb{R}^n$ whose components x_j satisfy

$$x_j \geq 0, \quad \sum_{j=1}^{n} x_j \leq 1$$

Δ_n has $n+1$ vertices and $n+1$ faces.

<u>Probability vectors</u> in $\mathbb{R}^n$.

$x \in co\{e_1, \dots, e_n\} = \Delta_n^1$

$$x_j \geq 0, \quad \sum_{j=1}^{n} x_j = 1$$

If $x \in \Delta_n^1$, $\quad |x| := (|x_1|, |x_2|, \dots, |x_n|)$

↳ notation

C_1, C_2 non-empty convex subset of $\mathbb{R}^n$.

$\lambda \in \mathbb{R}$

$\lambda C_1, \; C_1 \cap C_2, \; C_1 + C_2, \; C_1 - C_2$ are

convex subsets of $\mathbb{R}^n$.

$C_1 \cup C_2$ – in general not convex.

$\{C_k, k \in K\}$, $\quad \bigcap_{k \in K} C_k$ – is convex if C_k are convex

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear, affine transform. of C then $L(C)$ is convex if C is convex.

- $C \neq \emptyset$, convex, $0 \in C$.
  dimension of C is the maximal number of lin. independent vectors in C.

  If $0 \notin C$. choose $a \in C$ and let

  $C_1 = C - \{a\}$.

  dimension of C is the dimension of C_1

  dimension of C — $\dim(C)$

- $B = \{ a_0 + \sum_{j=1}^{m} t_j d_j \ , \ 0 \leq t_j \leq 1, \ 1 \leq j \leq m \}$

  $\hookrightarrow$ m-dimensional parallelogram of $\mathbb{R}^n$
  if $\{d_j\}_j$ are lin. independent.

  If $\{d_j\}$ are orthogonal B is m-dimensional box of $\mathbb{R}^n$

<u>Volume</u> $\quad |B|_m = \prod_{j=1}^{m} |d_j| \ , \quad |\Delta_n| = \frac{1}{n!} \det[d_1 \cdots d_n]$

Convex Functions in $\mathbb{R}^n$

$C \neq \phi$ convex set in $\mathbb{R}^n$.

$f : C \longrightarrow \overline{\mathbb{R}}$

$\text{dom } f = \{ x \in C \,,\, f(x) \in \mathbb{R} \}$ esential domain

$\text{epi}(f) = \{ (x,z) \in C \times \mathbb{R} \,,\, z \geq f(x) \}$.

f is convex if $\text{epi}(f)$ is convex set in $\mathbb{R}^{n+1}$

f is concave if $-f$ is convex

f is affine if $f(x+y) - f(0) = f(x) + f(y)$. $\forall x, y \in \mathbb{R}^n$.

f is real valued on C if $|f(x)|$ is finite for $\forall x \in C$. $\text{dom}(f) = C$.

f is proper if $\text{dom}(f) \neq \phi$ and $f(C) \subset (-\infty, \infty]$

<u>Note</u> This definition applies to extend real valued functions.

$f: C \to \overline{\mathbb{R}}$ may be extended to $\mathbb{R}^n$

$$f_e(x) = \begin{cases} f(x) & x \in C \\ \infty & x \notin C. \end{cases}$$

$\operatorname{epi}(f_e) = \operatorname{epi} f.$

f convex on C iff f_e convex on $\mathbb{R}^n$.

If f is proper convex function then $\operatorname{dom}(f)$ is nonempty convex set in $\mathbb{R}^n$.

* $C \neq \phi$, convex subset of $\mathbb{R}^n$.

$f: C \to [-\infty, \infty]$, proper is convex iff

$f((1-t)x + ty) \leq (1-t) f(x) + t f(y)$, $\forall t \in [0,1]$
$\forall x \neq y \in \operatorname{dom}(f)$

f is strictly proper convex if the inequality holds with ($<$) for $t \in (0,1)$.

- $f: C \longrightarrow \mathbb{R}$ convex

$\{a_j : 1 \le j \le J\}$ finite subset of C.

$$f\left(\sum_{j=1}^{J} t_j a_j\right) \le \sum_{j=1}^{J} t_j f(a_j)$$

$\forall \quad (t_1, t_2, \dots, t_J) \ge 0$ in $\mathbb{R}^J$ with

$$\sum_{j=1}^{J} t_j = 1$$

<u>Ex</u>. 1. $l(x) = \langle a, x \rangle + c = c + \sum_{j=1}^{n} a_j x_j$ convex in $\mathbb{R}^n$.

$a \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{R}$

2. $f(x) = \begin{cases} b & x \in C \\ \infty & x \notin C. \end{cases}$ - convex.

$b \in \bar{\mathbb{R}}$, $C \subset \mathbb{R}^n$

$b = 0$ f is indicator function, I_C.

Operations on Convex Fct

$f_1, f_2 : C \to \mathbb{R}$ convex.

$c_1 f_1 + c_2 f_2$ convex if $c_1, c_2 \geq 0$.

$f(x) = \sum_{j=1}^{m} c_j f_j(x)$ convex whenever $c_1, c_2, \dots, c_m \geq 0$

$\hookrightarrow$ f is a positive linear combination of $\{f_1, f_2, \dots f_m\}$.

Product in general not convex

Ex. $g, f : [-1, 1] \to \mathbb{R}$, $f(x) = x^2$, $g(x) = x$.

- $\{f_k, k \in K\}$. Collection of proper convex functions on $C \subset \mathbb{R}^n$.

$F : C \to (-\infty, \infty]$

$$F(x) = \sup_{k \in K} f_k(x)$$

F is proper convex function on C

- If k is finite and f_k continuous so is F
- k infinite. then F is l.s.c on C.

$L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transformation.

$L(C) \subset C_1$, $g: C_1 \longrightarrow \mathbb{R}$ convex.

Then $f = g \circ L$ is convex. | $f: C \to \mathbb{R}^m$

- $C \neq \emptyset$, $C \subset \mathbb{R}^n$ convex.

$I \subset \mathbb{R}$ interval $f: C \longrightarrow \mathbb{R}$ convex.

$f(C) \subset I$. If $\varphi: I \longrightarrow \mathbb{R}$ convex and increasing then $g(x) = \varphi(f(x))$ is convex.

Proof. $x, y \in C$

$\cancel{xt}$ $z = (1-t)x + ty$.

$$g(x(t)) \leq \varphi((1-t)f(x) + tf(y)).$$

Since f is convex.. Qed.

Ex. $f : C \to [0, \infty)$ convex fct.

$f^m : C \to [0, \infty)$. $\forall m \in \mathbb{N}$.

$f^m(x) = f(x)^m$ convex

$g : C \to \overline{\mathbb{R}}$) convex minorant of $f : C \to \overline{\mathbb{R}}$

$g(x) \leq f(x) \quad \forall x \in C. \iff epi(g) \supset epi(f)$

If g affine, g is called an affine minorant if $g(x) \leq f(x) \quad \forall x \in C$

If f is bounded below on I

$g(x) = c \leq \inf_{x \in I} f(x)$ is a convex minorant of f.

Let $\Gamma(f)$ - the set of all convex minorants of $f : C \longrightarrow \overline{\mathbb{R}}$.

Define $\overline{f}(x) : C \longrightarrow \mathbb{R}$

$$\overline{f}(x) = \sup \{ g(x) : g \in \Gamma(f) \}$$

$\overline{f}$ - convex hull, (envelope) of f on C and is convex on C.

$$\alpha(\overline{f}, C) = \alpha(f, C)$$

$$\overline{f}(x) \leq f(x) \quad \forall x \in C$$

$\overline{f}$ - largest such convex fct on I.

$f: C \longrightarrow \overline{\mathbb{R}}$ is said to be quasi-convex if $S_c(f) = \{ x \in C : f(x) \leq c \}$ are convex for all $c \in \mathbb{R}$.